

The adventures of a probabilist in homogeneous vector bundles

International Workshop “Modern Trends in Probability Theory and Mathematical Statistics IV” in honor of Professor Mykola Leonenko

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Time line

- **1959** Mikhaïlo Ĭosipovych Yadrenko [Yadrenko, 1959] published the Karhunen–Loève expansion of an isotropic random field on the sphere S^n in **real-valued** spherical harmonics:

$$X(\theta, \varphi) = \sum_{\ell=0}^{\infty} c_{\ell} \sum_{m=1}^{h(\ell, n)} X_{\ell}^m Y_{\ell}^m(\theta, \varphi)$$

with

$$\sum_{\ell=0}^{\infty} h(\ell, n) c_{\ell}^2 < \infty.$$

X_{ℓ}^m s are random, centred, **uncorrelated**, with unit variance.

The Bell Telephone Laboratories constructed a large microwave antenna known as the **Holmdel Horn Antenna**, for the purpose of satellite communications.

Time line

- **1964** Using this antenna, Arno Penzias and Robert Woodrow Wilson serendipitously discovered the Cosmic Microwave Background, which they describe as

This excess temperature is, within the limits of our observations, isotropic, unpolarized, and free from seasonal variations (July, 1964–April, 1965).

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- **1982** Philipp James Edwin Peebles introduced the expansion of the CMB temperature in **complex-valued** spherical harmonics

$$T(\theta, \phi) = T_b(1 + \sum a_l^m Y_l^m),$$

a_l^m s are random, centred, but **highly correlated**: $a_l^{-m} = (-1)^m \overline{a_l^m}$.

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- **2011–2014** Three teams of researchers: Daryl Geller and Domenico Marinucci; myself; Paolo Baldi and Maurizia Rossi, published a rigorous mathematical theory of **isotropic random cross-sections of homogeneous vector bundles**.

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- **2011–2014** Three teams of researchers: Daryl Geller and Domenico Marinucci; myself; Paolo Baldi and Maurizia Rossi, published a rigorous mathematical theory of **isotropic random cross-sections of homogeneous vector bundles**.
- **2024** Tianshi Lu [Lu, 2024] published the Karhunen–Loève expansions of isotropic and reflexive tangential random vector fields on a sphere.

A general question

[Lu, 2024] defines a tangential random vector field on a sphere as a family $\{\mathbf{J}(\mathbf{x}): \mathbf{x} \in S^n\}$ such that every random vector $\mathbf{J}(\mathbf{x})$ belongs to $T_{\mathbf{x}}S^n$, the tangent space to the sphere at the point $\mathbf{x}$.

He calls such a field **isotropic** (resp., **reflexive**) if

$$\mathbb{E}[\mathbf{J}(g\mathbf{x})] = \mathbb{E}[g\mathbf{J}(\mathbf{x})], \quad \mathbb{E}[\mathbf{J}(g\mathbf{x})\mathbf{J}^\top(g\mathbf{y})] = \mathbb{E}[g\mathbf{J}(\mathbf{x})\mathbf{J}^\top(\mathbf{y})g^\top]$$

for any $\mathbf{x}, \mathbf{y} \in S^n$, and $g \in \text{SO}(n+1)$ (resp., $g \in \text{O}(n+1)$). A little bit mysterious, isn't it?

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He obtains simple Karhunen–Loève expansions for such random fields using the embedding $S^n \subset \mathbb{R}^{n+1}$ as the centred unit sphere.

Is his theory a part of the general theory mentioned above? If yes, what is a natural class of objects, for which a generalisation of notion of an isotropic tangential random vector field can be elaborated and a simple Karhunen–Loève expansion can be proved?

An alternative construction of the tangent bundle to S^n

- Introduce an equivalence relation in the Cartesian product $SO(n+1) \times \mathbb{R}^n$:
 $(g, \mathbf{x}) \sim (gh, h^{-1}\mathbf{x})$ for all $h \in SO(n)$.

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- The quotient space, $SO(n+1) \times \mathbb{R}^n / \sim$, is a **smooth manifold**, call it E .
- The sphere is a **homogeneous space** of the left cosets:

$$S^n = M = G/H = \{gH : g \in G\}, \quad G = SO(n+1), \quad H = SO(n).$$

- Define $\pi: E \rightarrow M$, $(g, \mathbf{x}) \mapsto gH$. The triple (E, π, M) is **exactly the tangent bundle**, whose **fibres**, $\pi^{-1}(p)$, are tangent planes!

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- Define $\pi: E \rightarrow M$, $(g, \mathbf{x}) \mapsto gH$. The triple (E, π, M) is **exactly the tangent bundle**, whose **fibres**, $\pi^{-1}(p)$, are tangent planes!
- For most values of n , E is **not** isomorphic to $S^n \times \mathbb{R}^n$, but is **“twisted”** instead.
- The tangential vector fields $\{\mathbf{X}(p) : p \in M\}$, both deterministic or random, are the **cross-sections** of that bundle. That is, $M \xrightarrow{\mathbf{X}} E \xrightarrow{\pi} M$ is the identity on M .

The reason for a mysterious definition, or where my adventures take place

- The group $G = \mathrm{SO}(n+1)$ acts on $G \times \mathbb{R}^n$ by $g_1 \cdot (g, p) = (g_1 g, p)$. This action respects the equivalence relation: the equivalence classes are not destroyed. By this reason, the bundle is called **homogeneous**.

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- **Before** $g \in G$ acted, the initial point p of the vector $(g \cdot J)(p)$ had been located at $g^{-1}p$.

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- **Under** the action of g , the vector $J(g^{-1}p)$ becomes $g \cdot J(g^{-1}p)$.
- The group G acts on tangential vector fields, **both random and deterministic**, by

$$(g \cdot J)(p) = g \cdot J(g^{-1}p), \quad g \in \text{SO}(n+1), \quad p \in S^n.$$

- Calculate the cross-correlation of both hand sides. You obtain Lu's definition!

An important link

- We constructed the tangent bundle to S^n , using a **representation** μ of H given by $\mu(h) = h$. A homogeneous vector bundle is constructed by replacing this μ by any other finite-dimensional representation, which can be real, complex, or even quaternionic!

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- Now, you can easily write down the definition of a centred isotropic random cross-section of a homogeneous vector bundle:

$$E[g \cdot \mathbf{X}(g^{-1}p) \otimes g \cdot \mathbf{X}(g^{-1}q)] = E[\mathbf{X}(p) \otimes \mathbf{X}(q)], \quad g \in G, \quad p, q \in M.$$

Replace G with the group $G = I(M)$ of all isometries of M . You obtain the definition of a **reflexive** random cross-section.

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Replace G with the group $G = I(M)$ of all isometries of M . You obtain the definition of a **reflexive** random cross-section.

- **A link**: if $\{\mathbf{X}(p) : p \in M\}$ is a **deterministic** square-integrable cross-section, then the action $\mathbf{X}(p) \mapsto g \cdot \mathbf{X}(g^{-1}p)$ determines what is called the representation of G **induced** by μ . Denote it by $\text{Ind}_H^G(\mu)$.

The above link helps to prove an important result

- Decompose $\text{Ind}_H^G(\mu)$ into **irreducible pieces** λ_i with $i \in \mathcal{I}$. If this list does not contain duplicates, we say that $\text{Ind}_H^G(\mu)$ is **multiplicity free**.

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- Choose an orthonormal basis $\{\mathbf{e}_{im}(p) : 1 \leq m \leq \dim \lambda_i\}$ in each space where λ_i acts.

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- Choose an orthonormal basis $\{\mathbf{e}_{im}(p) : 1 \leq m \leq \dim \lambda_i\}$ in each space where λ_i acts.
- The Karhunen–Loève expansion of an isotropic random cross-section has the form

$$\mathbf{X}(p) = \sum_{i \in \mathcal{I}} c_i \sum_{m=1}^{\dim \lambda_i} \mathbf{e}_{im}(p) X_{ij} \quad c_i \geq 0, \quad \mathbb{E}[X_{im}] = 0, \quad \mathbb{E}[|X_{im}|^2] = 1$$

with

$$\sum_{i \in \mathcal{I}} \dim \lambda_i c_i^2 < \infty.$$

New questions arise

- How to find multiplicity-free induced representations? *Answer:* [Pezzini and van Pruijssen, 2023] contains a lot.

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- How to compute a basis $\{\mathbf{e}_{im}(p) : 1 \leq m \leq \dim \lambda_i\}$? *Answer:* Solved with big number of examples in [Malyarenko, 2024, Malyarenko and Ostoj-Starzewski, 2023].

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- How to find multiplicity-free induced representations? *Answer:* [Pezzini and van Pruijsen, 2023] contains a lot.
- How to compute a basis $\{\mathbf{e}_{im}(\rho): 1 \leq m \leq \dim \lambda_i\}$? *Answer:* Solved with big number of examples in [Malyarenko, 2024, Malyarenko and Ostoj-Starzewski, 2023].
- All representations of H are equal, but there is one, say ρ , which is more equal than others. The homogeneous vector bundle constructed from ρ is the tangent one. Do we have a hope that the corresponding induced representation is multiplicity free? *Answer:* next slides.

Strongly isotropy irreducible spaces

If the **real isotropy representation** ρ is **reducible**, we do not have any hope.

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Definition

A homogeneous space $M = G/H$ is called **isotropy irreducible** if ρ is irreducible.

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Definition

A homogeneous space $M = G/H$ is called **strongly isotropy irreducible** if the restriction of ρ to the connected component of identity H_0 is irreducible.

There are several tens of infinite series and several tens of *exceptional* irreducible (that is, not Cartesian products of other ones) compact strongly isotropy irreducible spaces.

- In a **symmetric space** M , for any $p \in M$, there is a **symmetry** s_p : a non-identical map $M \rightarrow M$ such that $s_p(p) = p$ and s_p^2 is the identical map.

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- A sphere is a symmetric space, where s_p is just a rotation by π around the axis that passes through p .
- Any symmetric space is strongly isotropy irreducible.
- To make calculations on a manifold, you need to use local charts and transition functions. On a symmetric space M , [Cacciatori et al, 2017] constructed a special chart which they called the **Euler coordinates**. It covers an open dense subset of M . With some precaution, you may work in this chart as if it is a region in Euclidean space!

An example of Élie Cartan

- Let M be the **Graßmannian** of oriented 4-planes in $\mathbb{R}^8$. Exactly 100 years ago, Élie Cartan proved that $I(M)$ contains 24 connected components!

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- For this nice symmetric space, we have $G_0 = \mathrm{SO}(8)/\mathbb{Z}_2$,
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- Let $[0]$ be the trivial representation of H_0 which maps all its elements to 1. The induced representation $\mathrm{Ind}_{H_0}^{G_0}([0])$ is multiplicity free.
- **A good news.** The corresponding homogeneous vector bundle is trivial, that is, $E = M \times \mathbb{R}^1$. You can forget about bundles and cross-sections and consider just an isotropic random field on M . It has the form

$$X(\theta, \varphi) = \sum_{\ell_1, \ell_2, \ell_3, \ell_4=0}^{\infty} c_{\ell_1 \ell_2 \ell_3 \ell_4} \sum_{m=1}^{\dim \lambda_{\ell_1 \ell_2 \ell_3 \ell_4}} X_{\ell_1 \ell_2 \ell_3 \ell_4 m} Y_{\lambda_{\ell_1 \ell_2 \ell_3 \ell_4}^{\mathbb{R}} m}(\theta, \varphi).$$

- **Under which conditions, this field is reflexive?**

Some explanations

- The representation $\lambda_{\ell_1 \ell_2 \ell_3 \ell_4}$ looks at the **Dynkin diagram** as follows:

$$\lambda_{\ell_1 \ell_2 \ell_3 \ell_4} = 2\ell_1 \circ \begin{array}{c} \circ 2\ell_3 \\ | \\ \circ 2\ell_2 \\ | \\ \circ 2\ell_4 \end{array} \quad (1)$$

- A graduate student Eugene Dynkin invented this method in 1944 while he has been preparing a talk at the seminar by Israel Gel'fand!

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- A graduate student Eugene Dynkin invented this method in 1944 while he has been preparing a talk at the seminar by Israel Gel'fand!
- The group H is an **extension** of H_0 by the 24-elements group S_4 of permutations of 4 objects. See [Ceccherini-Silberstein et al, 2022].

$$1 \longrightarrow S_4 \longrightarrow H = H_0 \rtimes S_4 \longrightarrow H_0 \longrightarrow 1.$$

- To solve the puzzle, we will draw a **commutative diagram**.

A commutative diagram

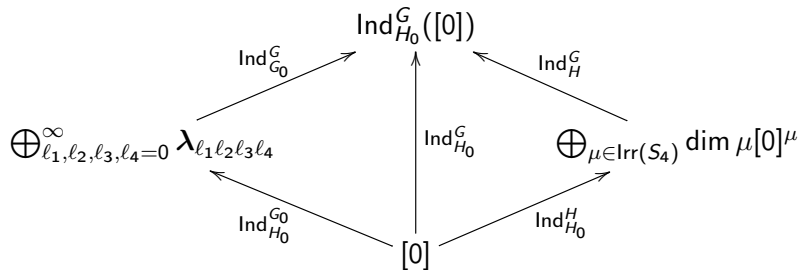


Figure: The structure of the induced representation $\text{Ind}_{H_0}^G([0])$.

The **Inducing-in-Stage Theorem** guarantees that the diagram commutes.

The lower-right arrow

The group S_4 **does not move** $[0]$ and has 5 irreducible representations: two 1-dimensional A_1 and A_2 ; one 2-dimensional E ; and two 3-dimensional T_1 and T_2 . The 24-dimensional representation $\text{Ind}_{H_0}^H([0])$ has the following structure.

$$\text{Ind}_{H_0}^H([0]) = [0]^{A_1} \oplus [0]^{A_2} \oplus 2[0]^E \oplus 3[0]^{T_1} \oplus 3[0]^{T_2}.$$

In fact, this is just the celebrated **Peter–Weyl Theorem**.

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There are 2 reflexive random fields and 3 reflexive random cross-sections (which I promised to not consider).

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The representations $[0]^{A_1}$ and $[0]^{A_2}$ are **extensions** of $[0]$, the others are not.

The upper-right arrow

The induced representation $\text{Ind}_H^G \left(\bigoplus_{\mu \in \text{Irr}(S_4)} \dim \mu [0]^\mu \right)$ has 10 components.

$$\text{Ind}_H^G \left(\bigoplus_{\mu \in \text{Irr}(S_4)} \dim \mu [0]^\mu \right) = \text{Ind}_H^G([0]^{A_1}) \oplus \text{Ind}_H^G([0]^{A_2}) \oplus \dots$$

The first two ones:

- correspond to the reflexive random **fields**;
- act in the same Hilbert space $L^2(M)$ where $\text{Ind}_{H_0}^{G_0}([0])$ acts;
- but partition this space into bigger irreducible components.

The upper-left arrow

- The group S_4 acts on the Dynkin diagram (1) by leaving the central node fixed and permuting the other three branches.

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- The group S_4 **acts** on the Dynkin diagram (1) by leaving the central node fixed and permuting the other three branches.

- In the **fully symmetric case** $2\ell \circ \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} \begin{array}{c} 2\ell \\ 2\ell_2 \\ 2\ell \end{array}$, the orbit of the representation $\lambda_{\ell\ell_2\ell\ell}$ under the action of S_4 is a singleton. As before, we obtain

$$\text{Ind}_{G_0}^G(\lambda_{\ell\ell_2\ell\ell}) = \lambda_{\ell\ell_2\ell\ell}^{A_1} \oplus \lambda_{\ell\ell_2\ell\ell}^{B_1} \oplus 2\lambda_{\ell\ell_2\ell\ell}^E \oplus 3\lambda_{\ell\ell_2\ell\ell}^{T_1} \oplus 3\lambda_{\ell\ell_2\ell\ell}^{T_2}.$$

- In the **partially symmetric case**, say, $2\ell \circ \begin{array}{c} \circ \\ \diagup \quad \diagdown \\ \circ \quad \circ \end{array} \begin{array}{c} 2\ell \\ 2\ell_2 \\ 2\ell_4 \end{array}$, the subgroup which **does not move** $\lambda_{\ell\ell_2\ell\ell_4}$, is equal to the **dihedral group** D_4 , which has 5 irreducible representations. First, we induce from G_0 to the extension $G_0 \rtimes D_4$:

$$\text{Ind}_{G_0}^{G_0 \rtimes D_4}(\lambda_{\ell\ell_2\ell\ell_4}) = \lambda_{\ell\ell_2\ell\ell_4}^{A_1} \oplus \lambda_{\ell\ell_2\ell\ell_4}^{A_2} \oplus \lambda_{\ell\ell_2\ell\ell_4}^{B_1} \oplus \lambda_{\ell\ell_2\ell\ell_4}^{B_2} \oplus 2\lambda_{\ell\ell_2\ell\ell_4}^E.$$

The upper-left arrow (cont.)

- The quotient group for the remaining step is $S_4/D_4 = \mathbb{Z}_3$. Because the stabiliser is already exhausted, the only valid representation for the second index is the trivial one of $\mathbb{Z}_3$, that is, A :

$$\text{Ind}_{G_0}^G(\lambda_{\ell\ell_2\ell\ell_4}) = \lambda_{\ell\ell_2\ell\ell_4}^{A_1A} \oplus \lambda_{\ell\ell_2\ell\ell_4}^{A_2A} \oplus \lambda_{\ell\ell_2\ell\ell_4}^{B_1A} \oplus \lambda_{\ell\ell_2\ell\ell_4}^{B_2A} \oplus 2\lambda_{\ell\ell_2\ell\ell_4}^{EA},$$

and similarly for the cases of $2\ell \circ \begin{array}{c} \circ 2\ell_3 \\ \circ 2\ell_2 \\ \circ 2\ell \end{array}$ and $2\ell_1 \circ \begin{array}{c} \circ 2\ell \\ \circ 2\ell_2 \\ \circ 2\ell \end{array}$.

The upper-left arrow (cont.)

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and similarly for the cases of $2\ell \circ \begin{array}{c} \circ 2\ell_3 \\ \circ 2\ell_2 \\ \circ 2\ell \end{array}$ and $2\ell_1 \circ \begin{array}{c} \circ 2\ell \\ \circ 2\ell_2 \\ \circ 2\ell \end{array}$.

- In the **fully asymmetric case** $2\ell_1 \circ \begin{array}{c} \circ 2\ell_3 \\ \circ 2\ell_2 \\ \circ 2\ell_4 \end{array}$, with $\ell_1 \neq \ell_3 \neq \ell_4 \neq \ell_1$, the subgroup which **does not move** $\lambda_{\ell_1\ell_2\ell_3\ell_4}$, is equal to the **Klein four-group** $\mathbb{Z}_2 \times \mathbb{Z}_2$, which has 4 irreducible representations. First, we induce from G_0 to the extension $G_0 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$:

$$\text{Ind}_{G_0}^{G_0 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)}(\lambda_{\ell_1\ell_2\ell_3\ell_4}) = \lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1} \oplus \lambda_{\ell_1\ell_2\ell_3\ell_4}^{B_1} \oplus \lambda_{\ell_1\ell_2\ell_3\ell_4}^{B_2} \oplus \lambda_{\ell_1\ell_2\ell_3\ell_4}^{B_3}.$$

The upper-left arrow (cont.)

The quotient group for the remaining step is $S_4/\mathbb{Z}_2 \times \mathbb{Z}_2 = S_3$. When we induce the 4 extensions up to G :

- The trivial extension $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1}$ is invariant under S_3 , so it induces to a unique block $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1A_1}$.

The upper-left arrow (cont.)

The quotient group for the remaining step is $S_4/\mathbb{Z}_2 \times \mathbb{Z}_2 = S_3$. When we induce the 4 extensions up to G :

- The trivial extension $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1}$ is invariant under S_3 , so it induces to a unique block $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1A_1}$.
- The 3 remaining non-trivial extensions are permuted as a single orbit by S_3 . They fuse together into a single global representation. Its second index corresponds to the 2-dimensional representation E of S_3 : $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{B_1E}$.

Which representations from this zoo go to $\text{Ind}_H^G([0]^{A_1})$ and $\text{Ind}_H^G([0]^{A_2})$?

The irreducible components of $\text{Ind}_H^G([0]^{A_1})$

The celebrated **Frobenius Reciprocity** says:

The multiplicity of any λ in $\text{Ind}_H^G([0]^{A_i})$ is equal to the multiplicity of $[0]^{A_i}$ in the restriction of λ down to H .

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Upon restriction to H , the representation $[0]^{A_1}$ participates with multiplicity 1 in

- $\lambda_{\ell\ell_2\ell\ell}^{A_1}$. A singleton, does not glue anything.

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- $\lambda_{\ell\ell_2\ell\ell}^{A_1}$. A singleton, does not glue anything.
- $\lambda_{\ell\ell_2\ell\ell_4}^{A_1A}$, $\lambda_{\ell\ell_2\ell_3\ell}^{A_1A}$, $\lambda_{\ell_1\ell_2\ell\ell}^{A_1A}$. A 3-element orbit, glues the participants together, forces the conditions

$$C_{\ell\ell_2\ell\ell_4} = C_{\ell\ell_2\ell_4\ell} = C_{\ell_4\ell_2\ell\ell}, \quad C_{\ell\ell_2\ell_3\ell} = C_{\ell\ell_2\ell\ell_3} = C_{\ell_3\ell_2\ell\ell}, \quad C_{\ell_1\ell_2\ell\ell} = C_{\ell\ell_2\ell_1\ell} = C_{\ell\ell_2\ell\ell_1}.$$

- $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1A_1}$ and $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{B_1E}$ belong to a 6-elements orbit and force the condition

$$C_{\ell_1\ell_2\ell_3\ell_4} = C_{\ell_1\ell_2\ell_4\ell_3} = C_{\ell_3\ell_2\ell_1\ell_4} = C_{\ell_3\ell_2\ell_4\ell_1} = C_{\ell_4\ell_2\ell_1\ell_3} = C_{\ell_4\ell_2\ell_3\ell_1}.$$

The irreducible components of $\text{Ind}_H^G([0]^{A_2})$

Upon restriction to H , the representation $[0]^{A_2}$ participates with multiplicity 1 in $\lambda_{\ell\ell_2\ell\ell}^{A_2}$, $\lambda_{\ell\ell_2\ell\ell_4}^{B_1A}$, $\lambda_{\ell\ell_2\ell_3\ell}^{B_1A}$, $\lambda_{\ell_1\ell_2\ell\ell}^{B_1A}$, $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{B_1E}$, but not in $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1A_1}$.

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Because E is the 2-dimensional representation of S_3 , it cannot capture all 6 elements of the orbit by itself, but separates the even permutations from the odd ones:

$$c_{\ell_1\ell_2\ell_3\ell_4} = c_{\ell_3\ell_2\ell_4\ell_1} = c_{\ell_4\ell_2\ell_1\ell_3}, \quad c_{\ell_1\ell_2\ell_4\ell_3} = c_{\ell_4\ell_2\ell_3\ell_1} = c_{\ell_3\ell_2\ell_1\ell_4}.$$

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Upon restriction to H , the representation $[0]^{A_2}$ participates with multiplicity 1 in $\lambda_{\ell\ell_2\ell\ell}^{A_2}$, $\lambda_{\ell\ell_2\ell\ell_4}^{B_1A}$, $\lambda_{\ell\ell_2\ell_3\ell}^{B_1A}$, $\lambda_{\ell_1\ell_2\ell\ell}^{B_1A}$, $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{B_1E}$, but not in $\lambda_{\ell_1\ell_2\ell_3\ell_4}^{A_1A_1}$.

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Any $[0]^{A_1}$ -reflexive random field is $[0]^{A_2}$ -reflexive. Any $[0]^{A_2}$ -reflexive random field is isotropic. The converse is wrong.

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