A network evolution model based on 2- and 3-interactions

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Modern Stochastics: Theory and Applications V, Kyiv, June 1-4, 2021

This work was supported by the construction EFOP-3.6.3-VEKOP-16-2017-00002. The project was supported by the European Union, co-financed by the European Social Fund.

The basic structure of our model

The basic units are edges and triangles.
At the initial time t = 0 we start with a single object, it can be either an edge or a triangle.

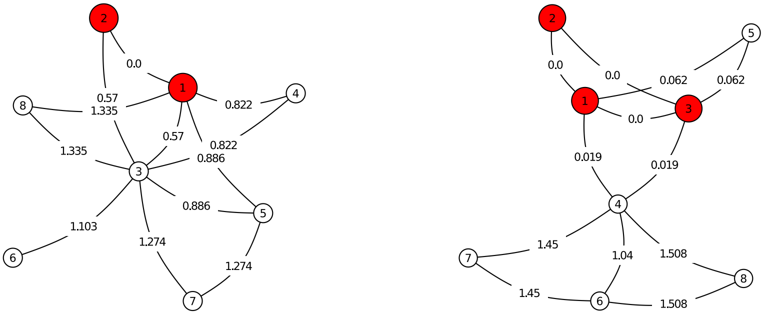


Figure: Possible type of ancestor objects (red)

This ancestor object produces offspring objects which can be also edges or triangles.

- (1-2-3): is a triangle, the ancestor with birth time $t = 0$,
- (1-2-3-4): represents 3 triangles, i.e. the offspring of (1-2-3) at its first reproduction time $t = 0.571$,
- (1-5): an edge, offspring of (1-2-3) with birth time $t = 0.847$,
- (1-5-6): a triangle, offspring of (1-5) with birth time $t = 1.06$.

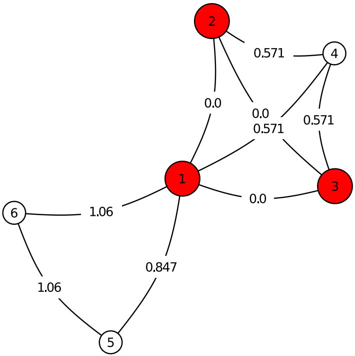
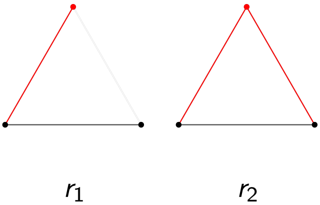


Figure: A short example of the graph evolution model

The reproduction steps of a fixed edge

The generic edge $\rightarrow$ Poisson process $\Pi_2(t)$
When the Poisson process jumps, then a new vertex appears an it is connected to our generic edge.

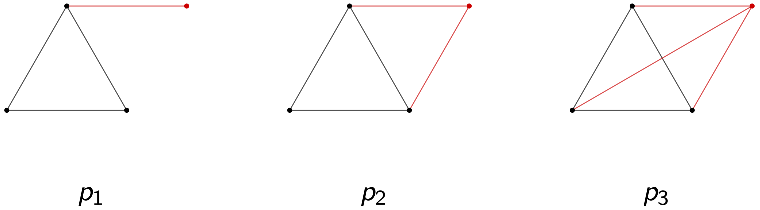


- connected to 1 vertex $\rightarrow$ r_1: probability that a new edge is born
- connected to 2 vertices $\rightarrow$ r_2: probability that a new triangle is born

At any reproduction time a new vertex is born and it will be connected to 1 or 2 vertices of the given edge.

The reproduction steps of a fixed triangle

The generic triangle $\rightarrow$ Poisson process $\Pi_3(t)$
When the Poisson process jumps, then a new vertex appears an it is connected to our generic triangle.



- connected to 1 vertex $\rightarrow p_1$: probability that a new edge is born
- connected to 2 vertices $\rightarrow p_2$: probability that a new triangle is born
- connected to 3 vertices $\rightarrow p_3$: probability that 3 new triangles are born

At any reproduction time a new vertex is born and it will be connected to 1, 2 or 3 vertices of the given triangle.

Basic denotions

Let us denote by $\xi_{i,j}(t)$ the number of type j offspring of the type i generic object up to time t $(i, j = 2, 3)$.

- $\xi_2(t) = \xi_{2,2}(t) + \xi_{2,3}(t)$
- $\xi_3(t) = \xi_{3,2}(t) + \xi_{3,3}(t)$

- λ_2: non-negative random variable
  $\rightarrow$ the life-length of the generic edge
- λ_3: non-negative random variable
  $\rightarrow$ the life-length of the generic triangle

Triangle survival

Let $L_3(t)$ denote the distribution function of λ_3. Then the survival function of a triangle's life-length is

$$1 - L_3(t) = \mathbb{P}(\lambda_3 > t) = \exp\left(-\int_0^t l_3(u)\,du\right),$$

where $l_3(t)$ is the hazard rate of the life-length λ_3. We assume that the hazard rate depends on the total number of offspring, so that

$$l_3(t) = b + c\xi_3(t)$$

with positive constants b and c.

Edge survival

Let λ_2 be the life-length of the generic edge. Then $\xi_2(t) = \xi_2(\lambda_2)$ for $t > \lambda_2$. As the edge always gives birth to one offspring (which can be an edge or a triangle), so the total number of offspring of the generic edge is

$$\xi_2(t) = \Pi_2(t \wedge \lambda_2),$$

where $\Pi_2(t)$ is the Poisson process.
Let $L_2(t)$ denote the distribution function of λ_2. Then the survival function of the life-length of an edge is

$$1 - L_2(t) = \exp\left(-\int_0^t l_2(u)\,du\right),$$

where we assume that the hazard rate of the life-length λ_2 is of the form $l_2(t) = b + c\xi_2(t)$.

Survival results

Theorem

The survival function for a triangle is

$$\mathbb{P}(\lambda_3 > t) = e^{-t(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-ct}\right)+p_3\left(1-e^{-3ct}\right)}{3c}}.$$

The survival function for an edge is

$$\mathbb{P}(\lambda_2 > t) = e^{-t(b+1)} e^{\frac{1-e^{-ct}}{c}}.$$

The mean offspring number of an edge

Let us denote by $m_{i,j}(t) = \mathbb{E}\xi_{i,j}(t)$ the expectation of the number of type j offspring of a type i mother until time t. Let

$$m^*_{i,j}(\kappa) = \int_0^\infty e^{-\kappa t} m_{i,j}(dt), \qquad i,j = 2,3,$$

be the Laplace transform of $m_{i,j}$.

Theorem

For any $\kappa \geq 0$ we have

$$m^*_{2,2}(\kappa) = r_1 A(\kappa), \qquad m^*_{2,3}(\kappa) = r_2 A(\kappa),$$

where

$$A(\kappa) = \int_0^\infty e^{-\kappa s} e^{-(b+1)s} e^{\frac{1-e^{-cs}}{c}} ds = \frac{1}{c}\int_0^1 (1-u)^{\frac{\kappa+b+1}{c}-1} e^{\frac{u}{c}} du.$$

The mean offspring number of a triangle

Theorem

For any $\kappa \geq 0$ we have

$$m^*_{3,2}(\kappa) = p_1 B(\kappa), \qquad m^*_{3,3}(\kappa) = (p_2 + 3p_3) B(\kappa),$$

where

$$B(\kappa) = \int_0^\infty e^{-\kappa s} e^{-s(b+1)} e^{\frac{3(p_1+p_2)\left(1-e^{-cs}\right)+p_3\left(1-e^{-3cs}\right)}{3c}} ds =$$

$$= \frac{1}{c} \int_0^1 (1-u)^{\frac{\kappa+b+1}{c}-1} e^{\frac{u}{3c}\left(p_3 u^2 - 3p_3 u + 3\right)} du.$$

The Perron root

Let

$$M(\kappa) = \begin{pmatrix} m^*_{2,2}(\kappa) & m^*_{2,3}(\kappa) \\ m^*_{3,2}(\kappa) & m^*_{3,3}(\kappa) \end{pmatrix}$$

be the matrix of the Laplace transforms. The greater characteristic root of $M(\kappa)$ is called the Perron root, that is

$$\frac{(p_2 + 3p_3)B(\kappa) + r_1 A(\kappa) + \sqrt{((p_2 + 3p_3)B(\kappa) - r_1 A(\kappa))^2 + 4p_1 B(\kappa) r_2 A(\kappa)}}{2}.$$

The Malthusian parameter

That value of κ for which the Perron root is equal to 1 is called the Malthusian parameter. α is the Malthusian parameter if $\varrho(\alpha) = 1$. We assume the existence of the Mathusian parameter. From relation $\varrho(\alpha) = 1$ we obtain, that for the Malthusian α we have

$$r_1 A(\alpha)(p_2 + 3p_3)B(\alpha) - (r_1 A(\alpha) + (p_2 + p_3)B(\alpha)) = r_2 A(\alpha) p_1 B(\alpha) - 1.$$

Let α be the Malthusian parameter and let $(v_2, v_3)^\top$ be the right eigenvector of $M(\alpha)$ satisfying condition $v_2 + v_3 = 1$. Then

$$v_2 = \frac{(r_1 - 1)A(\alpha)}{(2r_1 - 1)A(\alpha) - 1}, \qquad v_3 = \frac{r_1 A(\alpha) - 1}{(2r_1 - 1)A(\alpha) - 1}.$$

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$$r_1 A(\alpha)(p_2 + 3p_3)B(\alpha) - (r_1 A(\alpha) + (p_2 + p_3)B(\alpha)) = r_2 A(\alpha) p_1 B(\alpha) - 1.$$

Again let α be the Malthusian parameter and let $(u_2, u_3)^\top$ be the left eigenvector of $M(\alpha)$ satisfying condition $u_2 v_2 + u_3 v_3 = 1$. Then

$$u_2 = \frac{p_1 B(\alpha)\left((2r_1 - 1)A(\alpha) - 1\right)}{p_1 B(\alpha)(r_1 - 1)A(\alpha) - (r_1 A(\alpha) - 1)^2},$$

$$u_3 = \frac{(1 - r_1 A(\alpha))\left((2r_1 - 1)A(\alpha) - 1\right)}{p_1 B(\alpha)(r_1 - 1)A(\alpha) - (r_1 A(\alpha) - 1)^2}.$$

Asymptotic theorems

Theorem

Assume that $\varrho(\alpha) = 1$ *has a finite positive solution* α. *Assume that* $0 \le r_1 < 1$, $0 < p_1 \le 1$ *and it is excluded, that both* $r_1 = 0$ *and* $p_1 = 1$ *are satisfied at the same time.*
Let ${}_2E(t)$ *and* ${}_3E(t)$ *denote the number of all edges being born up to time* t *if the ancestor of the population was an edge and triangle, respectively. Then*

$$\lim_{t\to\infty} e^{-\alpha t}\,{}_iE(t) = {}_iW \frac{v_i u_2}{\alpha D(\alpha)}$$

almost surely for $i = 2, 3$, *where*

$$D(\alpha) = \sum_{l,j=2}^{3} u_l v_j \left(-m^*_{l,j}(\alpha)\right)'.$$

${}_2W$ *and* ${}_3W$ *are a.s. non-negative,* $\mathbb{E}_2W = \mathbb{E}_3W = 1$, ${}_2W$ *and* ${}_3W$ *are a.s. positive on the event when the total number of offspring converges to infinity.*

Asymptotic theorems

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Assume that $\varrho(\alpha) = 1$ *has a finite positive solution* α. *Assume that* $0 \leq r_1 < 1$, $0 < p_1 \leq 1$ *and it is excluded, that both* $r_1 = 0$ *and* $p_1 = 1$ *are satisfied at the same time.*
Let ${}_2\tilde{E}(t)$ *and* ${}_3\tilde{E}(t)$ *denote the number of all edges alive at time* t *if the ancestor of the population was an edge and triangle, respectively. Then*

$$\lim_{t\to\infty} e^{-\alpha t}{}_i\tilde{E}(t) = {}_iW \frac{v_i u_2 A(\alpha)}{D(\alpha)}$$

almost surely for $i = 2, 3$.

Asymptotic theorems

Theorem

Assume that $\varrho(\alpha) = 1$ *has a finite positive solution* α. *Assume that* $0 \leq r_1 < 1$, $0 < p_1 \leq 1$ *and it is excluded, that both* $r_1 = 0$ *and* $p_1 = 1$ *are satisfied at the same time.*
Let ${}_2T(t)$ *and* ${}_3T(t)$ *denote the number of all triangles being born up to time* t *if the ancestor of the population was an edge and triangle, respectively. Then*

$$\lim_{t\to\infty} e^{-\alpha t}\,{}_iT(t) = {}_iW\,\frac{v_i u_3}{\alpha D(\alpha)}$$

almost surely for $i = 2, 3$.

Asymptotic theorems

Theorem

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Let ${}_2\tilde{T}(t)$ *and* ${}_3\tilde{T}(t)$ *denote the number of all triangles alive at time* t *if the ancestor of the population was an edge and triangle, respectively. Then*

$$\lim_{t\to\infty} e^{-\alpha t} {}_i\tilde{T}(t) = {}_iW \frac{v_i u_3 B(\alpha)}{D(\alpha)}$$

almost surely for $i = 2, 3$.

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