On the sample paths properties of φ-sub-Gaussian processes related to the heat equation with random initial conditions

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Basic concepts

Let us consider the definitions of φ-sub-Gaussian random variable ([1, 2]).

Definition

Let φ be an N-function satisfying condition:
$\liminf_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0$, *and* $\{\Omega, L, \mathrm{P}\}$ *be a standard probability space. The random variable ζ belongs to the space $Sub_\varphi(\Omega)$, if* $\mathrm{E}\zeta = 0$, $\mathrm{E}\exp\{\lambda\zeta\}$ *exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$:* $E\exp\{\lambda\zeta\} \leq \exp\{\varphi(\lambda a)\}$. *Stochastic process $\xi(t)$, $t \in T$ belongs to the space $Sub_\varphi(\Omega)$ if for for all $t \in T$ $\xi(t)$ is in $Sub_\varphi(\Omega)$.*

To derive the main results we will need additional notions and statements ([3]).

Lemma

Let $Z(u), u \geq 0$ be a continuous monotonically increasing function such that $Z(u) > 0$ and $\frac{u}{Z(u)}$ is nondecreasing for $u \geq u_0$, where $u_0 \geq 0$ is a constant. Then for $u > 0, v > 0$

$$\min(\frac{u}{v}, 1) < \frac{Z(u+u_0)}{Z(v+u_0)}.$$

Definition

The function $Z(u), u \geq 0$, is called admissible for the space $Sub_\phi(\Omega)$, if for Z the conditions of Lemma 1 hold and for some $\varepsilon > 0$

$$\int_0^\varepsilon \Psi\Big(\ln\Big(Z^{(-1)}\Big(\frac{1}{s}\Big) - u_0\Big)\Big) ds < \infty,$$

where $\Psi(v) = \frac{v}{\phi^{(-1)}(v)}, v > 0.$

Consider the Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial^2 x}, t > 0, x \in \mathbb{R}, \mu > 0, \quad (1)$$

subject to the random initial condition

$$u(0, x) = \eta(x), x \in \mathbb{R}, \quad (2)$$

where $\eta(x), x \in \mathbb{R}$, is a stochastic process. We suppose that η is φ-sub-Gaussian process satisfying the next assumption.

H.1 $\eta(x), x \in \mathbb{R}$ is a real, measurable, mean-square continuous stationary (in wide sense) stochastic process, which is strictly φ-sub-Gaussian with the determining constant c_η.

Let for η the spectral representation holds

$$\eta(x) = \int_{\mathbb{R}} e^{i\lambda x} Z(d\lambda). \tag{3}$$

Consider the process $u(t,x), t > 0, x \in \mathbb{R}$, defined by

$$u(t,x) = \int_{\mathbb{R}} g(t, x - y)\eta(y)dy, \tag{4}$$

where $g(t,x) = \frac{1}{(4\pi\mu t)^{1/2}} \exp\left\{ -\frac{x^2}{4\mu t} \right\}, t > 0, x \in \mathbb{R}$, is the fundamental solution to the heat equation (1).

Taking into account (3), we can write the following representation of the process given by (4):

$$u(t,x) = \int_{\mathbb{R}} \exp\left\{ i\lambda x - \mu t \lambda^2 \right\} Z(d\lambda). \tag{5}$$

The process (5) can be interpreted as the mean-square or $L_2(\Omega)$ solution to the Cauchy problem (1)–(2).

Assume that $Z(u), u \geq 0$, is a function satisfying conditions of Lemma 1 and the following integral converges:

$$C_Z^2 = \int_{\mathbb{R}} \left(Z^2 \left(\mu\lambda^2 + u_0 \right) + 4Z^2 \left(\frac{|\lambda|}{2} + u_0 \right) \right) F(d\lambda) < \infty. \quad (6)$$

Denote

$$\hat{I}_{\varphi}(\sigma) = \int_0^{\sigma} \Psi \left(\ln \left[\left(\frac{b-a}{2} \left(Z^{(-1)} \left(\frac{c_\eta C_Z}{s} \right) - u_0 \right) + 1 \right) \times \right. \right.$$

$$\left. \left. \times \left(\frac{d-c}{2} \left(Z^{(-1)} \left(\frac{c_\eta C_Z}{s} \right) - u_0 \right) + 1 \right) \right] \right) ds; \quad \Psi(u) = \frac{u}{\varphi^{(-1)}(u)}, \quad (7)$$

where $F(\lambda)$ is a spectral measure.

Theorem

Let $u(t,x), a \le t \le b, c \le x \le d$, be a separable modification of the stochastic process given by (5) and assumption H.1 hold. Assume that $Z(u), u \ge 0$, is an admissible function for the space $Sub_\phi(\Omega)$, and the integral (6) converges. Then for $0 < \theta < 1$ and $u > \frac{2\hat{I}_\varphi(\min(\theta\Gamma,\gamma_0))}{\theta(1-\theta)}$ the following inequality holds true:

$$\mathrm{P}\Big\{ \sup_{\substack{a \le t \le b; \\ c \le x \le d}} |u(t,x)| > u \Big\} \le 2A(u,\theta), \quad (8)$$

where $A(u,\theta) = \exp\Big\{ -\varphi^*\Big(\frac{1}{\Gamma}(u(1-\theta) - \frac{2}{\theta}\hat{I}_\varphi(\min(\theta\Gamma,\gamma_0)))\Big)\Big\}$;
$\Gamma = c_\eta \Big(\int_{\mathbb{R}} F(d\lambda) \Big)^{1/2}, \gamma_0 = \frac{c_\eta C_Z}{Z(\frac{1}{\varkappa}+u_0)}$, $\varkappa = \max(b-a, d-c)$; C_Z *is defined in (6),* $\hat{I}$ *and* $\Psi(u)$ *is defined in (7).*

Consider now the process $u(t,x), (x,t) \in V$ defined on an unbounded domain of the following form: $V = [-A, A] \times [0, \infty)$, $A > 0$.

Introduce a family of segments $\{[b_k, b_{k+1}], k = 0, 1, \ldots\}$ such that $b_0 = 0, b_k > b_{k+1}, b_k \to \infty,\ k \to \infty;\ b_{k+1} - b_k \geq 2A$, define the sets $V_k = [-A, A] \times [b_k, b_{k+1}], k = 0, 1, \ldots$ and $V = \cup_{k=0}^{\infty} V_k$.

Let $\{a(t), t \geq 0\}$ be a continuous strictly increasing function such that $a(t) > 0, t \geq 0$ and $a(t) \to \infty$, as $t \to \infty$.

Denote:

$$a_k = \min_{t \in [b_k, b_{k+1}]} a(t); \quad \varepsilon_k = \sup_{(x,t) \in V_k} \tau_{\varphi}(u(t,x)).$$

Suppose that the following conditions hold:

1. $\sup_{k=\overline{0,\infty}} \frac{\varepsilon_k}{a_k} < \infty$.
2. For $\sigma > 0$

$$\hat{I}_\varphi(\sigma) = \int_0^\sigma \Psi\Big(\ln \Big[\Big(A(Z^{(-1)}(\frac{c_\eta C_Z}{s}) - u_0) + 1 \Big) \times \Big(\frac{b_{k+1} - b_k}{2} (Z^{(-1)}(\frac{c_\eta C_Z}{s}) - u_0) + 1 \Big) \Big] \Big) ds$$

and for some θ, $0 < \theta < 1$: $\sup_{k=\overline{0,\infty}} \frac{\hat{I}_{\varphi,k}(\theta\varepsilon_k)}{a_k} < \infty$.

3. The series $\sum_{k=0}^{\infty} \exp\Big\{ -\varphi^*\Big(\frac{s a_k(1-\theta)}{2\varepsilon_k} \Big) \Big\}$ converges for some s such that $\sup_{k=\overline{0,\infty}} \frac{4\varepsilon_k}{a_k(1-\theta)} < s < \frac{u}{2}$.

In the next theorem the rate of growth of $u(t,x), (x,t) \in V$, is evaluated.

Theorem

Let $u(t,x)$, $(t,x) \in V$, be a separable modification of the stochastic process given by (5) and assumption H.1 hold. Assume that $Z(u), u \geq 0$, is an admissible function for the space $Sub_{\varphi}(\Omega)$, and the integral (6) converges. Suppose that the above conditions hold.

Then for

$$u > \sup_{k=\overline{0,\infty}} \frac{\hat{I}_{\varphi,k}(\theta\varepsilon_k)}{a_k} \frac{4}{\theta(1-\theta)},$$

$$P\left\{ \sup_{(x,t)\in V} \frac{|u(t,x)|}{a(t)} > u \right\} \leq 2\exp\left\{ -\varphi^*\left(\frac{u}{s}\right)\right\} \times \sum_{k=0}^{\infty} \exp\left\{ -\varphi^*\left(\frac{sa_k(1-\theta)}{2\varepsilon_k}\right)\right\} =: 2A(u). \quad (9)$$

Corollary

Let the assumptions of above theorem hold true. Then there exists a random variable ξ *such that* $P\{\xi > u\} \leq 2A(u)$ *for* $u > \sup_{k=\overline{0,\infty}} \frac{\hat{I}_{\varphi,k}(\theta\varepsilon_k)}{a_k} \frac{4}{\theta(1-\theta)}$ *and* $|u(x,t)| < \xi a(t)$ *with probability one.*

So, in this talk, there are studied sample paths properties of stochastic processes representing solutions (in $L_2(\Omega)$ sense) of the heat equation with random initial conditions given by φ-sub-Gaussian stationary processes. The main results are the bounds for the distributions of the suprema for such stochastic processes considered over bounded and unbounded domains. For particular functions Z, the estimates can be calculated in the explicit form.

The talk is based on the results published in [4], [5].

- Buldygin V. V., Kozachenko Yu. V. Metric Characterization of Random Variables and Random Processes // AMS, Providence, RI. – 2000. – 257 p.
- Kozachenko Yu. V., Ostrovsky E. I. Banach spaces of random variables of sub-Gaussian type // Theor. Probab. Math. Stat. – 1985. – 32. – P.42–53.
- Kozachenko Yu., Orsingher E., Sakhno L., Vasylyk O. Estimates for functional of solution to Higher-Order Heat-Type equation with random initial condition // J. Stat. Phys. – 2018. – Vol. 72, No. 6. – P.1641–1662.
- Hopkalo O., Sakhno L. Investigation of sample paths properties for some classes of φ-sub-Gaussian stochastic processes // Modern Stochastics: Theory and Applications – 2021. – 8 (1). – P. 41–62.
- Hopkalo O., Sakhno L., Vasylyk O. Properties of φ-sub-Gaussian stochastic processes related to the heat equation with random initial conditions // Bulletin of Taras Shevchenko National University of Kyiv. Series: Physics & Mathematics. – Vol. 1-2. – 2020. – P. 17-24.