Approximate solution of the integral equations involving kernel with additional singularity

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Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space that supports all the stochastic processes presented below, and it is assumed that they are all adapted to this filtration.
Fix $T > 0$ and let $\mathbb{P}_1$ be another probability measure on $\mathcal{F}_T$.
According to [3], pp. 121-130, the relative entropy of a probability measure $\mathbb{P}_1$ with respect to $\mathbb{P}$ is defined as

$$\mathrm{H}(\mathbb{P}_1|\mathbb{P}) := \begin{cases} \mathrm{E}\left[\frac{d\mathbb{P}_1}{d\mathbb{P}} \log \frac{d\mathbb{P}_1}{d\mathbb{P}}\right] & \text{if } \mathbb{P}_1 \ll \mathbb{P}, \\ +\infty & \text{otherwise.} \end{cases}$$

Minimization of entropy functional

Now, introduce two independent stochastic processes, namely, the Wiener process $W = \{W(t), t \in [0, T]\}$ and the fractional Brownian motion (fBm) with Hurst index $H \in (0, 1)$, $B^H = \left\{B^H(t), t \in [0, T]\right\}$, that is the Gaussian process with zero mean and the covariance function

$$\mathrm{E}B^H(t)B^H(s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), t, s \in [0, T].$$

Minimization of entropy functional

More precisely, we consider the the mixed fractional Brownian motion with the drift, i.e., the process of the form

$$Z(t) = B^H(t) + W(t) + \int_0^t f(s)ds, \quad t \in [0, T], \tag{1}$$

where $f \in \Lambda_H$ is a non-random function. Consider the following problem: to annihilate the drift by the change of the probability measure. More precisely, to choose the other probability measure $\widetilde{\mathbb{Q}}$ such that

$$Z(t) = \widetilde{B}^H(t) + \widetilde{W}(t), \quad t \in [0, T].$$

Minimization of entropy functional

The main idea of the solution is to apply Girsanov theorem to fractional Brownian motion and Wiener process with drifts. Since B^H and W are independent, we can write $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} = \frac{dQ_W}{d\mathbb{P}} \frac{dQ_{B^H}}{d\mathbb{P}}$, where

$$\frac{dQ_W}{d\mathbb{P}} = \exp\left\{ -\int_0^T f_1(t)dW(t) - \frac{1}{2}\|f_1\|^2_{L_2([0,T])} \right\}, \tag{2}$$

according to standard Girsanov theorem, and

$$\frac{dQ_{B^H}}{d\mathbb{P}} = \exp\left\{ -\int_0^T (K_0^{H,*} f_2)(t)dB(t) - \frac{1}{2}\|K_0^{H,*} f_2\|^2_{L_2([0,T])} \right\}. \tag{3}$$

Minimization of entropy functional

Now consider the minimization of entropy functional, which can be formulated as follows: define the functions f_1 and f_2 in (2) and (3), which minimize the entropy-type functional

$$\mathrm{H}_1(\mathbb{P}, Q_W, Q_{B^H}) = \mathrm{E}\left[\left(\frac{dQ_W}{d\mathbb{P}}\log\frac{dQ_W}{d\mathbb{P}}\right)\right] + \mathrm{E}\left[\left(\frac{dQ_{B^H}}{d\mathbb{P}}\log\frac{dQ_{B^H}}{d\mathbb{P}}\right)\right]. \tag{4}$$

Integral equation with singularity

Entropy functional $\mathrm{H}_1(\mathbb{P}, Q_W, Q_{B^H})$ could be represented as

$$\mathrm{H}_1(\mathbb{P}, Q_W, Q_{B^H}) = \frac{1}{2}\|f_1\|^2_{L_2([0,T])} + \frac{1}{2}\|(K_0^{H,*} f_2)\|^2_{L_2([0,T])}. \quad (5)$$

$$\|f - x\|^2_{L_2([0,T])} + \|K_0^{H,*} x\|^2_{L_2([0,T])} \xrightarrow{x \in \Lambda_H} \min \quad (6)$$

It was shown in [1] for $H \in (0, 1/2)$ and in [3] for $H \in (1/2, 1)$, that the minimization in (6) is a solution of the following fractional integral/differential equation

$$x(t) + \left[K_T^{H,*} K_0^{H,*} x\right](t) = f(t), \quad t \in [0, T]. \quad (7)$$

Integral equation with singularity

Thus, the optimization problem reduces the solution of the Fredholm integral equation of the second kind, which can be represented as

$$x(t) + C_2(H)\int_0^T \varkappa_H(t,v)x(v)dv = f(t), \qquad (8)$$

where

$$\varkappa_H(t,v) = \begin{cases} (tv)^{\frac{1}{2}-H}\int_{t\vee v}^{T}(z-t)^{-\frac{1}{2}H}z^{2H-1}(z-v)^{-\frac{1}{2}-H}dz \\ (tv)^{H-\frac{1}{2}}\int_0^{t\wedge v}(t-z)^{H-\frac{3}{2}}z^{1-2H}(v-z)^{H-\frac{3}{2}}dz \end{cases} \qquad (9)$$

and $C_2(H)$ – some constant.

Integral equation with singularity

Let $B(\alpha, \beta)$ be the Beta function and $B(x, \alpha, \beta)$ be an incomplete beta function defined for $x \in [0, 1]$, given by

$$B(x, \alpha, \beta) = \int_0^x y^{\alpha-1}(1-y)^{\beta-1} dy, \alpha, \beta > 0.$$

Integral equation with singularity

Lemma 1. The kernel (9) equals
a) for $H \in (0, 1/2)$

$$\varkappa_H(t,v) = \frac{1}{|t-v|^{2H}} B\left(\frac{T/(t \vee v) - 1}{T/(t \wedge v) - 1}, \frac{1}{2} - H, 2H\right),$$

where numerator is bounded on $[0, T]^2$, meanwhile, has no limit at points $(0, 0)$ and (T, T).
b)for $H \in (1/2, 1)$

$$\varkappa_H(t,v) = \frac{1}{|t-v|^{2-2H}} B\left(H - \frac{1}{2}, 2 - 2H\right), t, v \in [0, T]$$

Integral equation with singularity

We called such a kernel as having an additional singularity. Both cases, $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$ were considered and as the result, the problem was reduced to the couple of the same integral equations as in the case where the minimization of small deviations was studied.
In the case $H \in (0, \frac{1}{2})$ the integral equation contains the kernel with additional singularity whereas in the case $H \in (\frac{1}{2}, 1))$ the kernel is simply weakly singular.

Integral equation with singularity

In order to deal with the kernel with additional singularity applying well-known methods for weakly singular kernels [1-2], we prove the theorem on the approximation of solution of integral equation with the kernel containing additional singularity by the solutions of the integral equations whose kernels are weakly singular but the numerator is continuous.

Approximation theorem

Let us consider the integral operator A defined by its kernel function $K(t,s)$ via formula

$$Ax)(t) = \int_0^T K(t,s)x(s)ds, 0 \leq t \leq T, \qquad (10)$$

where x is taken from some space of functions defined on $[0,T]$, and consider the Fredholm integral equation of the second kind

$$x(t) + (Ax)(t) = g(t), t \in [0,T], \qquad (11)$$

where $g(t) \in L([0,T])$ is a given function.

Approximation theorem

Theorem 1. Let $K(t, s)$ be the kernel defined above, where the numerator $L(t, s)$ has the following properties

(i) L is bounded and symmetric.
(ii) L is continuous, except finite number of points.
(iii) L is a positively definite kernel.

Approximation theorem

Let $x \in L_2([0, T])$ be a unique solution of equation (11), $L_n(t,s)$ can be chosen in such a way that the respective integral operators are self-adjoint, for sufficiently large $n \geq 1$ the equation

$$x_n(t) + (A_n x)(t) = g(t),$$

$$(A_n x)(t) = \int_0^T K_n(t,s)x(s)ds, \quad 0 \leq t \leq T,$$

$$K_n(t,s) = \frac{L_n(t,s)}{|t-s|^\nu}, \quad \nu \in (0,1),$$

has a unique solution $x_n \in L_2([0, T])$, and

$$\|x_n - x\|_{L_2([0,T])} \to 0, n \to \infty.$$

We compare the approximate solution of equation (8) with the exact solution given by

$$x_2(t) = \begin{cases} 8I_{\{x\leq 0.5\}} - I_{\{0.5<x\leq 0.75\}} + 4I_{\{0.75<x\leq 1\}}, H \in (0, \frac{1}{2} \\ 6 - 7t, H \in (\frac{1}{2}, 1). \end{cases}$$

The right-hand side of (8) is computed by $f_2(t) = x_2(t) + C_2(H)\int_0^1 \varkappa_H(t,s)x_2(s)ds$. The graphs of numerical solution and function f are presented in Figure 18. One can mention that approximate solution visually indistinguishable with the exact solution.

Numerical illustration

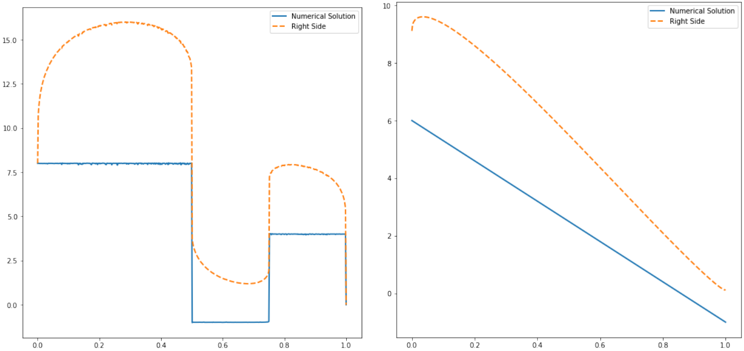

Figure: Numerical approximations of x_2 with $N = 500$ for $H = 0.25$ (left) and $H = 0.75$.

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