Parameter estimation in the generalized Cox–Ingersoll–Ross model

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We consider a stochastic differential equation of the form

$$dr_t = (a - br_t)\,dt + \sigma r_t^{\beta}\,dW_t, \quad r_t\Big|_{t=0} = r_0 > 0,$$

where W is a Wiener process, a, b and σ are positive constants.

This equation is known as the CKLS model; the particular case $\beta = \frac{1}{2}$ corresponds to the famous Cox–Ingersoll–Ross process.

Maximum likelihood estimation of drift parameters

Let $\frac{1}{2} < \beta < 1$. Then the maximum likelihood estimator for the couple (a, b) constructed by the continuous observations of r over the interval $[0, T]$, has the following form

$$\hat{a}_T = \frac{\int_0^T \frac{dr_t}{r_t^{2\beta}} \cdot \int_0^T r_t^{2-2\beta} dt - \int_0^T r_t^{1-2\beta} dr_t \cdot \int_0^T r_t^{1-2\beta} dt}{\int_0^T \frac{dt}{r_t^{2\beta}} \cdot \int_0^T r_t^{2-2\beta} dt - \left(\int_0^T r_t^{1-2\beta} dt \right)^2}; \tag{1}$$

$$\hat{b}_T = \frac{\int_0^T \frac{dr_t}{r_t^{2\beta}} \cdot \int_0^T r_t^{1-2\beta} dt - \int_0^T r_t^{1-2\beta} dr_t \cdot \int_0^T \frac{dt}{r_t^{2\beta}}}{\int_0^T \frac{dt}{r_t^{2\beta}} \cdot \int_0^T r_t^{2-2\beta} dt - \left(\int_0^T r_t^{1-2\beta} dt \right)^2}. \tag{2}$$

Theorem (1.1)

Assume that $\frac{1}{2} < \beta < 1$, $a > 0$, $b > 0$, $\sigma > 0$. *Then the estimator* $(\hat{a}_T, \hat{b}_T)$ *is strongly consistent.*

Theorem (1.2)

Let $a > 0$, $b > 0$, $\sigma > 0$, $\frac{1}{2} < \beta < 1$. *Then*

$$\sqrt{T}\begin{pmatrix} \hat{a}_T - a \\ \hat{b}_T - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \sigma^2\Sigma^{-1}), \quad T \to \infty,$$

where

$$\Sigma = \begin{pmatrix} \int_0^\infty x^{-2\beta} p_\infty(x)dx & -\int_0^\infty x^{1-2\beta} p_\infty(x)dx \\ -\int_0^\infty x^{1-2\beta} p_\infty(x)dx & \int_0^\infty x^{2-2\beta} p_\infty(x)dx \end{pmatrix}.$$

An alternative approach to drift parameters estimation

We start with two auxiliary lemmas:

Lemma (2.1)

Let $a > 0$, $b > 0$, $\sigma > 0$, $\frac{1}{2} < \beta < 1$. *Then*

$$\frac{1}{T}\int_0^T r_t dt \to \frac{a}{b} \quad a.s., \text{ as } T \to \infty.$$

Lemma (2.2)

Let $a > 0$, $b > 0$, $\sigma > 0$, $\frac{1}{2} < \beta < 1$. *Then*

$$\frac{1}{T}\int_0^T r_t^{3-2\beta} dt - \frac{a}{bT}\int_0^T r_t^{2-2\beta} dt \to \frac{\sigma^2(1-\beta)a}{b^2} \quad a.s., \text{ as } T \to \infty.$$

The previous convergences enable us to construct the estimators of the drift parameters:

$$\tilde{a_T} = \frac{\sigma^2(1-\beta)\left(\int_0^T r_t dt\right)^2}{T\int_0^T r_t^{3-2\beta} dt - \int_0^T r_t dt \cdot \int_0^T r_t^{2-2\beta} dt}; \qquad (3)$$

$$\tilde{b_T} = \frac{\sigma^2(1-\beta) T \int_0^T r_t dt}{T\int_0^T r_t^{3-2\beta} dt - \int_0^T r_t dt \cdot \int_0^T r_t^{2-2\beta} dt}. \qquad (4)$$

Theorem (2.1)

Let $a > 0$, $b > 0$, $\sigma > 0$, $\frac{1}{2} < \beta < 1$. Then $(\tilde{a_T}, \tilde{b_T})$ is a strongly consistent estimator of the parameter (a, b)

An identification of the diffusion parameters σ and β

Let σ be known.

Proposition (3.1)

For any $t \in [0, T]$,

$$\beta = \lim_{h \to 0} \frac{\log\left(\frac{[r]_{t+h} - [r]_t}{\sigma^2 h}\right)}{2 \log |r_t|} \quad a.\,s.,$$

where

$$[r]_t = \lim_{n \to \infty} \sum_{k=1}^{2^n} \left([r]_{\frac{kt}{2^n}} - [r]_{\frac{(k-1)t}{2^n}} \right)^2.$$

Let σ be unknown.

Proposition (3.2)

For any s,t $\in [0, T], s \neq t$,

$$\beta = \lim_{h\to 0} \frac{\log\left(\frac{[r]_{t+h}-[r]_t}{[r]_{s+h}-[r]_s}\right)}{2\log\left|\frac{r_t}{r_s}\right|} \quad a.s.$$

Let β be known.

Proposition (3.3)

For any s,t $\in [0, T], s \neq t$,

$$\sigma^2 = \frac{[r_t]}{\int_0^t {r_s}^{2\beta} ds} = \lim_{h\to 0} \frac{[r]_{t+h} - [r]_t}{h{r_t}^{2\beta}} \quad a.s.$$

Thank you for your attention!

If you have any questions, please email `dehtiar.olena@knu.ua`