A CLT for degenerate diffusions with periodic coefficients, and application to homogenization of linear PDEs

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Problem

$$\mathcal{L}^{\varepsilon} = 2^{-1}\mathrm{Tr}\big(a(\cdot/\varepsilon)\,\nabla\nabla^{\mathrm{T}}\big) + \big(\varepsilon^{-1}b(\cdot/\varepsilon) + c(\cdot/\varepsilon)\big)^{\mathrm{T}}\nabla$$ with degenerate diffusion coefficient $a(x)$.

- elliptic boundary-value problem

$$\mathcal{L}^{\varepsilon}u^{\varepsilon}(x) + e(x/\varepsilon)\,u^{\varepsilon}(x) + f(x) = 0\,, \qquad x \in \mathscr{D}\,,$$
$$u^{\varepsilon}(x) = g(x)\,, \qquad x \in \partial\mathscr{D}\,,$$

- parabolic initial-value problem

$$\partial_t u^{\varepsilon}(x,t) = \mathcal{L}^{\varepsilon}u^{\varepsilon}(x,t) + \big(\varepsilon^{-1}d(x/\varepsilon) + e(x/\varepsilon)\big)u^{\varepsilon}(x,t) + f(x)$$
$$u^{\varepsilon}(x,0) = g(x)\,, \qquad x \in \mathbb{R}^{n}\,,$$

Probability homogenization

infinitesimal generator $\mathcal{L}^{\varepsilon} \longrightarrow \mathcal{L}$

$\updownarrow \qquad \updownarrow$

diffusion process $X^{\varepsilon} \longrightarrow X$ (central limit theorem)

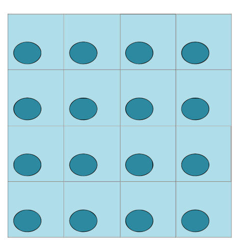
Degenerate diffusions

$$\mathrm{d}X^{\varepsilon}(x,t) = \big(\varepsilon^{-1}b\big(X^{\varepsilon}(x,t)/\varepsilon\big) + c\big(X^{\varepsilon}(x,t)/\varepsilon\big)\big)\,\mathrm{d}t + \sigma\big(X^{\varepsilon}(x,t)/\varepsilon\big)\,\mathrm{d}B(t)$$
$$X^{\varepsilon}(x,0) = x \in \mathbb{R}^{n}\,.$$

(i) there is $\sigma \in \mathcal{B}(\mathbb{R}^{n},\mathbb{R}^{n\times m})$ such that $a(x) = \sigma(x)\sigma(x)^{\mathrm{T}}$

(ii) $\sigma(x)$, $b(x)$, $c(x)$ Lipschitz continuous and τ-periodic

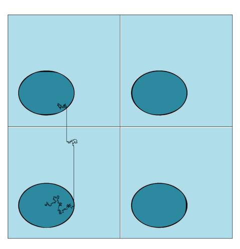
(iii) there is an open connected set $\mathscr{O} \subset [0,\tau]$ such that the matrix $a(x)$ is positive definite on $\overline{\mathscr{O}}$



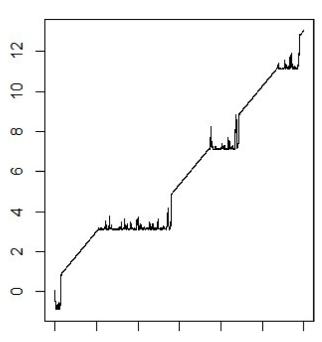
For $\varepsilon > 0$ let $\bar{X}^{\varepsilon}(x,t) := \varepsilon^{-1}X^{\varepsilon}(\varepsilon x, \varepsilon^{2}t)$, $t \ge 0$. For $\varepsilon \ge 0$, $x \in \mathbb{R}^{n}$ and $B \in \mathfrak{B}(\mathbb{R}^{n})$, let $\bar{\tau}^{\varepsilon,x}_{B} := \inf\{t \ge 0 : \bar{X}^{\varepsilon}(x,t) \in B\}$ be the first entry time of B by $\{\bar{X}^{\varepsilon}(x,t)\}_{t\ge 0}$.

(iv) there is $\varepsilon_0 > 0$ such that $\forall\,(\varepsilon, x)$

$$\mathbb{P}\big(\bar{\tau}^{\varepsilon,x}_{\mathscr{O}+\tau} < \infty\big) > 0.$$

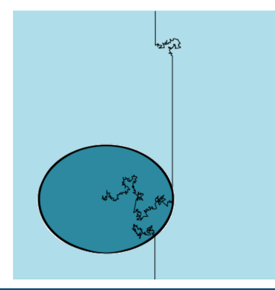


Example:



Results

Projection of the process on the cell of periodicity $\mathbb{R}^{d} \ni \bar{X}^{\varepsilon}(x,t) \longrightarrow \Pi\big(\bar{X}^{\varepsilon}(x,t)\big) = \bar{X}^{\varepsilon,\tau}(x,t) \in \mathbb{T}^{n}_{\tau}$, with transition kernel $\bar{p}^{\varepsilon,\tau}(t,x,B)$.



Proposition

There are $\gamma > 0$ and $\Gamma > 0$, such that $\forall\,(\varepsilon, t)$

$$\sup_{x\in\mathbb{T}^{n}_{\tau}} \|\bar{p}^{\varepsilon,\tau}(t,x,\mathrm{d}y) - \pi^{\varepsilon}(\mathrm{d}y)\|_{\mathrm{TV}} \le \Gamma e^{-\gamma t}.$$

CLT

$$\{X^{\varepsilon}(x,t) - \varepsilon^{-1}\pi^{0}(b)t\}_{t\ge 0} \xrightarrow[\varepsilon\to 0]{(\mathrm{d})} \{W^{\mathsf{a},\mathsf{b}}(x,t)\}_{t\ge 0}\,,$$

If $c \equiv 0$ CLT holds with $\mathsf{a} = 0$ and some b otherwise we need additional assumption

(v) $\sigma, b, c \in \mathcal{C}^{\infty}$, and

$$\inf_{t>0}\sup_{x\in\mathbb{R}^{n}} \mathbb{E}\big[\|J(x,t)\|_{\mathrm{HS}}\,\mathbb{1}_{[t,\infty]}(\bar{\tau}^{0,x}_{\mathscr{O}+\tau})\big] < 1\,,$$

where $\{J(x,t)\}_{t\ge 0}$ is the Jacobian of the stochastic flow associated to $\{\bar{X}^{0}(x,t)\}_{t\ge 0}$

and in this case CLT holds with

$$\mathsf{a} = \pi^{0}\big((\mathbb{I}_{n} - \mathrm{D}\beta)\,a\,(\mathbb{I}_{n} - \mathrm{D}\beta)^{\mathrm{T}}\big) \text{ and } \mathsf{b} = \pi^{0}\big((\mathbb{I}_{n} - \mathrm{D}\beta)\,c\big)\,,$$

$$\beta(x) = -\int_{0}^{\infty}\mathbb{E}\big[\big(b - \pi^{0}(b)\big)(\bar{X}^{0}(x,t))\big]\,\mathrm{d}t\,.$$

References

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