Lower bound for local oscillations of Hermite processes

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Organization of the talk

1 Introduction and motivation

2 Proof of the main result

Hermite processes are a very classical class of non-Gaussian chaotic stochastic processes. In contrast with stable stochastic processes they have finite moment of any order. They play important roles in probability and statistics. Indeed, several decades ago, the seminal articles (Taqqu 1975 and 1979) and (Dobrushin and Major 1979) drew fundamental connections between them and Non-Central Limit Theorems. First, we will briefly present some of these connections. To this end, we need to begin with some recalls on Hermite polynomials and the notion of Hermite rank.

The sequence $\{H_n(x)\}_{n\in\mathbb{Z}_+}$ of the Hermite polynomials over $\mathbb{R}$ can be defined as: $H_0(x) = 1$ and, for all $n \geq 1$,

$$(-1)^n e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2}). \tag{1.1}$$

For instance, one has:

$$H_0(x) = 1\,, \quad H_1(x) = x\,, \quad H_2(x) = x^2 - 1\,, \quad H_3(x) = x^3 - 3x\,.$$

The sequence $\{H_n(x)\}_{n\in\mathbb{Z}_+}$ is an orthogonal basis of $L^2\big(\mathbb{R}, e^{-x^2/2}dx\big)$. For any deterministic nonconstant function G belonging to $L^2\big(\mathbb{R}, e^{-x^2/2}dx\big)$ the Hermite rank $N = N(G) \geq 1$ is the positive integer defined as:

$$N := \min\Big\{n \geq 1 : \int_{\mathbb{R}} G(x)H_n(x)e^{-x^2/2}\,dx \neq 0\Big\}. \tag{1.2}$$

The main motivation of Dobrushin, Major and Taqqu was to obtain limit theorems associated with a time series of the form $\{G(Y_n)\}_{n\in\mathbb{N}}$, where $\{Y_n\}_{n\in\mathbb{N}}$ is a centred stationary time series sequence of Gaussian random variables having the long range dependence property: for some exponent $H \in (1/2, 1)$, and for all $n, q \in \mathbb{N}$,

$$\big|\mathrm{cov}(Y_{n+q}, Y_n)\big| = q^{2H-2}L(q)\,; \tag{1.3}$$

the function $L : (0, +\infty) \to (0, +\infty)$ is a slowly varying function at infinity i.e.

$$\lim_{x\to+\infty} \frac{L(ax)}{L(x)} = 1\,, \quad \text{for all fixed } a \in (0, +\infty). \tag{1.4}$$

Theorem 1.1 (Dobrushin, Major and Taqqu)

For all fixed $m \in \mathbb{N}$, let $\{S_m(t)\}_{t\in\mathbb{R}_+}$ be the stochastic process defined as:

$$S_m(t) := \frac{\kappa(G)}{m^{N(H-1)+1}L^{N/2}(m)} \sum_{n=1}^{[mt]} \Big(G(Y_n) - \mathbb{E}\big(G(Y_n)\big)\Big), \quad \text{for each } t \in \mathbb{R}_+, \tag{1.5}$$

where $[\,\cdot\,]$ is the integer part function and $\kappa(G)$ a well-chosen constant not depending on m and t. Then, under the condition that $H \in \big(1 - 1/(2N), 1\big)$, when m goes to $+\infty$, the process $\{S_m(t)\}_{t\in\mathbb{R}_+}$ converges in the sense of the finite-dimensional distributions to $\{X^{N,H}(t)\}_{t\in\mathbb{R}}$; the Hermite process of rank N and parameter H, defined, for all $t \in \mathbb{R}_+$, through the following multiple Wiener integral with respect to a fixed real-valued Brownian motion $\{B(x)\}_{x\in\mathbb{R}}$:

$$X^{N,H}(t) := \int'_{\mathbb{R}^N} \Big(\int_0^t \prod_{p=1}^{N} (s - x_p)_+^{H-3/2}\, ds \Big)\, dB(x_1)\ldots dB(x_N)\,, \tag{1.6}$$

where, for each $(y,\alpha) \in \mathbb{R}^2$, $y_+^\alpha := y^\alpha$ if $y > 0$, and $y_+^\alpha := 0$ else. Notice that $X^{1,H}$ is the fractional Brownian motion and $X^{2,H}$ the Rosenblatt process.

Recalls on multiple Wiener integral and Wiener chaoses

Let $[u_l, v_l]$, $l = 1, \ldots, N$, be N arbitrary nonempty compact intervals of $\mathbb{R}$ such that for some $\sigma \in \mathbb{S}_N$ (the set of the permutations of $\{1, \ldots, N\}$) one has

$$u_{\sigma(1)} < v_{\sigma(1)} < u_{\sigma(2)} < v_{\sigma(2)} < \ldots < u_{\sigma(N)} < v_{\sigma(N)} \,; \tag{1.7}$$

which is equivalent to say that these intervals are disjoint. Then, the multiple Wiener integral of the indicator function $\mathbf{1}_{\prod_{l=1}^N [u_l, v_l]}$ is defined as:

$$\begin{aligned} \mathbb{I}_N(\mathbf{1}_{\prod_{l=1}^N [u_l, v_l]}) &= \int'_{\mathbb{R}^N} \mathbf{1}_{\prod_{l=1}^N [u_l, v_l]}(x_1, \ldots, x_N)\, dB(x_1) \ldots dB(x_N) \\ &:= \prod_{l=1}^N \big(B(v_l) - B(u_l)\big) = \prod_{l=1}^N \big(B(v_{\sigma(l)}) - B(u_{\sigma(l)})\big) \,. \end{aligned} \tag{1.8}$$

The definition is extended in the natural way to the elementary functions, that is to the linear combinations of indicator functions satisfying (1.7). Finally, for any $f \in L^2(\mathbb{R}^N)$, the random variable $\mathbb{I}_N(f)$ is defined as the limit in $L^2(\Omega)$ of the multiple Wiener integrals of elementary functions which approximate f in $L^2(\mathbb{R}^N)$.

One always has that $\mathbb{I}_N(f) := \mathbb{I}_N(\widetilde{f})$, where $\widetilde{f}$ denotes the symmetrization of f:

$$\widetilde{f}(x_1,\dots,x_N) := \frac{1}{N!}\sum_{\sigma\in\mathbb{S}_N} f(x_{\sigma(1)},\dots,x_{\sigma(N)}), \quad \text{for all } (x_1,\dots,x_N)\in\mathbb{R}^N. \tag{1.9}$$

Moreover, the following isometry type property holds:

$$\mathbb{E}\Big(\big|\mathbb{I}_N(f)\big|^2\Big) = N!\int_{\mathbb{R}^N}\big|\widetilde{f}(x_1,\dots,x_N)\big|^2\,dx_1\dots dx_N\,. \tag{1.10}$$

The random variable $\mathbb{I}_N(f)$ belongs to $\mathcal{H}_N$ the homogeneous Wiener chaos of order N that we are now going to precisely define. Our presentation of it is inspired by the one in the book *Janson (1997)*. $\mathcal{G}$ denotes the Gaussian subspace of $L^2(\Omega)$ defined as: $\mathcal{G} := \Big\{\mathbb{I}_1(g), g\in L^2(\mathbb{R})\Big\}$.

The (inhomogeneous) Wiener chaos of an arbitrary order $N\in\mathbb{Z}_+$ is denoted by $\overline{\mathcal{P}}_N$. The space $\overline{\mathcal{P}}_0$ is defined to be the closed subspace of $L^2(\Omega)$ consisting of all the constant random variables. When $N\ge 1$, the space $\overline{\mathcal{P}}_N$ is defined as the closed subspace of $L^2(\Omega)$ spanned by the following set of random variables:

$$\left\{\prod_{l=1}^N \xi_l^{m_l}\ :\ (\xi_1,\dots,\xi_N)\in\mathcal{G}^N \text{ and } (m_1,\dots,m_N)\in\mathbb{Z}_+^N \text{ with } \sum_{l=1}^N m_l\le N\right\}.$$

One clearly has $\overline{\mathcal{P}}_{N-1} \subseteq \overline{\mathcal{P}}_N$, for all $N \in \mathbb{N}$. One sets $\mathcal{H}_0 := \overline{\mathcal{P}}_0$. If $N \geq 1$, the homogeneous Wiener chaos $\mathcal{H}_N$ is defined as: $\overline{\mathcal{P}}_N = \overline{\mathcal{P}}_{N-1} \oplus^{\perp} \mathcal{H}_N$,

Some nice fundamental properties:

(a) Let $\mathcal{F}$ be the smallest σ-algebra for which the underlying Brownian motion B is measurable. Then, for all fixed $p \in (0, +\infty)$, the space $\overline{\mathcal{P}}_* := \bigcup_{N \in \mathbb{Z}_+} \overline{\mathcal{P}}_N$ is dense in $L^p(\Omega, \mathcal{F})$. Hence: $L^2(\Omega, \mathcal{F}) = \bigoplus_{N \in \mathbb{Z}_+}^{\perp} \mathcal{H}_N$.

(b) For all $N \in \mathbb{Z}_+$ and $p \in (0, +\infty)$, the $L^p(\Omega)$-(quasi-)norms are equivalent on $\overline{\mathcal{P}}_N$. Moreover, for each sequence of random variables in $\overline{\mathcal{P}}_N$ convergence in probability is equivalent to convergence in any $L^p(\Omega)$-(quasi-)norm.

(c) For all $N \geq 1$, there is an universal constant $c(N) > 0$, depending only on N, such that, for every random variable $\chi \in \overline{\mathcal{P}}_N$ and $y \in [2, +\infty)$, one has

$$\mathbb{P}\Big(|\chi| > y\|\chi\|_{L^2(\Omega)}\Big) \leq \exp\big(-c(N)y^{2/N}\big) . \tag{1.11}$$

A bad news: In contrast with Gaussian and stable random variables, there is no explicit and easy exploitable formula for the characteristic function of a chaotic random variable in $\mathcal{H}_N$, even in the simplest non-Gaussian case $N = 2$.

Uniform modulus of continuity of a Hermite process $X^{N,H}$

Using the definition of the multiple Wiener integral defining $\{X^{N,H}(t)\}_{t\in\mathbb{R}_+}$, one can show that this process satisfies the following two fundamental properties:

(i) It is self-similar with exponent $N(H-1)+1 \in (1/2,1)$, that is, for each fixed positive real number a, the two processes $\left\{X^{N,H}(at)\right\}_{t\in\mathbb{R}_+}$ and $\left\{a^{N(H-1)+1}X^{N,H}(t)\right\}_{t\in\mathbb{R}_+}$ have the same finite-dimensional distributions.

(ii) It has stationary increments, which means that, for every fixed $t_0 \in \mathbb{R}_+$, the two processes $\left\{X^{N,H}(t_0+t)-X^{N,H}(t_0)\right\}_{t\in\mathbb{R}_+}$ and $\left\{X^{N,H}(t)\right\}_{t\in\mathbb{R}_+}$ have the same finite-dimensional distributions.

Thus, for any fixed $p \in [1,+\infty)$, there is a constant $c = c(N,H,p)$ such that

$$\mathbb{E}\left(\left|X^{N,H}(t')-X^{N,H}(t'')\right|^p\right) = c|t'-t''|^{pN(H-1)+p}, \quad \text{for all } (t',t'') \in \mathbb{R}_+^2. \quad (1.12)$$

Therefore, a strong version of the Kolmogorov's continuity Theorem and the fact that p can be taken arbitrarily large imply that $X^{N,H}$ has a modification whose paths are, with probability 1, on each compact interval $\mathcal{K}$, Hölder functions of any arbitrary order $\gamma < N(H-1)+1$. This uniform modulus of continuity can be improved thanks to results of (Viens and Vizcarra 2007): one has almost surely

$$\sup_{(t',t'')\in\mathcal{K}^2,\, t'\neq t''} \left\{ \frac{\left|X^{N,H}(t') - X^{N,H}(t'')\right|}{|t'-t''|^{N(H-1)+1} \log^{N/2}\left(1+|t'-t''|^{-1}\right)} \right\} < +\infty\,. \tag{1.13}$$

For all $\omega \in \Omega$, $\tau \in (0,+\infty)$ and $r \in (0,\tau]$, the oscillation on the compact interval $[\tau - r, \tau + r]$ of the path $X^{N,H}(\omega)$ is defined as:

$$\mathrm{Osc}\big(X^{N,H}(\omega),\tau,r\big) := \sup\left\{\left|X^{N,H}(t',\omega) - X^{N,H}(t'',\omega)\right| \,:\, (t',t'') \in [\tau-r,\tau+r]^2\right\}. \tag{1.14}$$

It easily follows from (1.13) that, for almost all $\omega \in \Omega$,

$$\limsup_{r\to 0^+} \left\{ r^{-N(H-1)-1} \,|\log_2 r\,|^{-N/2} \sup_{\tau\in\widetilde{\mathcal{K}}} \mathrm{Osc}\big(X^{N,H}(\omega),\tau,r\big)\right\} < +\infty\,, \tag{1.15}$$

where $\widetilde{\mathcal{K}} \subset (0,+\infty)$ denotes an arbitrary deterministic compact interval.

It seems natural to look for a non-trivial almost sure lower bound for the asymptotic behavior of $\mathrm{Osc}\big(X^{N,H}(\omega),\tau,r\big)$, as r goes to 0. It is important that such a lower bound be valid on an event of probability 1 not depending on τ (nowhere differentiability of paths, monofractality of Hermite processes, ...).

A general and powerful strategy for dealing with this type of problems on everywhere irregularity of paths was first introduced in the early 70's by Berman in the Gaussian frame and was later extended by (Nolan 1989) to the frame of stable distributions. It relies on a very clever intuitive idea called the Berman's principle: "the more regular is a local time in the time variable, uniformly in the space variable, the more irregular is the associated stochastic process".

Many more or less recent important developments of this classical and powerful strategy relying on local times are due to Xiao.

Unfortunately, this strategy can hardly be used in the framework of the Hermite process $X^{N,H}$ since, in contrast with Gaussian and stable processes, there is no explicit and easy exploitable formulas for the characteristic functions of the finite-dimensional distributions of $X^{N,H}$, even in the most simple non-Gaussian case of the Rosenblatt process where $N=2$.

Fractional Brownian motion, Rosenblatt process and more generally Hermite processes are usually viewed as stochastic processes with strongly correlated increments. Our new strategy relies on a different and maybe new way to view their increments in the setting of the study of their path behavior: "many of the increments are independent random variables up to negligible remainders".

Theorem 1.2

There exist Ω^**, an universal event of probability* 1 *not depending on* τ*, and* $c^*_{N,H}$ *a (strictly) positive deterministic finite constant only depending* (N,H)*, such that, for all* $\omega \in \Omega^*$ *and for every* $\tau \in (0,+\infty)$*, one has*

$$\liminf_{r\to 0^+} \left\{ \Big(r^{-1} \,|\log_2 r\,|\, S\big(|\log_2 r|\big)\Big)^{N(H-1)+1} \mathrm{Osc}\big(X^{N,H}(\omega),\tau,r\big)\right\} \geq c^*_{N,H} > 0\,, \tag{1.16}$$

where $S:\mathbb{R}_+ \to [2,+\infty)$ *is any increasing continuous function satisfying, for all fixed* $\varepsilon > 0$ *and* $\alpha > 0$*,*

$$\lim_{r\to+\infty} \frac{z^{\frac{N}{2(1-H)}}}{S(z)} = 0\,, \quad \lim_{r\to+\infty} \frac{z^{\frac{N}{2(1-H)}+\varepsilon}}{S(z)} = +\infty\,, \quad \sup_{z\in\mathbb{R}_+} \frac{S\big(z+\alpha\log_2(2+z)\big)}{S(z)} < +\infty\,. \tag{1.17}$$

Remarks 1.1

(i) *One can assume that* S *is with values in* $(0, +\infty)$ *instead of* $[2, +\infty)$.

(ii) *There are many classes of examples of increasing continuous functions* S *on* $\mathbb{R}_+$ *which satisfy the condition of the previous theorem; a natural one of them is*

$$S(z) := (2+z)^{\frac{N}{2(1-H)}} \left(\log_2(3+z)\right)^{\beta}, \quad \text{for all } z \in \mathbb{R}_+, \tag{1.18}$$

where the positive real number β *is arbitrary and fixed.*

(iii) *The lower estimate of* $\mathrm{Osc}\big(X^{N,H}(\omega), \tau, r\big)$, *provided by Theorem 1.2, is quasi-optimal since the quantity* r *in it is raised at the same power as the one in the upper estimate of* $\mathrm{Osc}\big(X^{N,H}(\omega), \tau, r\big)$, *namely* $N(H-1)+1$.

(iv) *Last but not least, in the proof of Theorem 1.2, it is implicitly shown that on any compact interval the variation of* $X^{N,H}$ *of order* γ *is almost surely infinite as soon as* $1/\gamma > N(H-1)+1$. *The latter inequality is valid when* $\gamma = 1$ *for instance.*

An important straightforward consequence of Theorem 1.2 is that there is no point in $(0,+\infty)$ at which a typical path of the Hermite process $X^{N,H}$ satisfies a pointwise Hölder condition of order strictly larger than $N(H-1)+1$. More precisely:

Corollary 1.1

Let Ω^* be the same event of probability 1 as in Theorem 1.2. Let an arbitrary real number $\mu \in \big(N(H-1)+1,1\big)$. Then, for all $\omega \in \Omega^*$ and for each $\tau \in (0,+\infty)$, one has that

$$\limsup_{t\to\tau} \frac{\big|X^{N,H}(t,\omega)-X^{N,H}(\tau,\omega)\big|}{|t-\tau|^{\mu}} = +\infty\,.$$

This clearly implies that, for any $\omega \in \Omega^*$, the path $X^{N,H}(\omega)$ is nowhere differentiable on the interval $(0,+\infty)$.

Organization of the talk

1 Introduction and motivation

2 Proof of the main result

For the sake of simplicity, one assumes that τ is arbitrary but such that $\tau \in (0,1)$.

For any integers $j \geq 1$ and $k \in \{0, \dots, 2^j - 1\}$, one denotes by $\Delta(j,k)$ the increment of the process $X^{N,H}$ such that

$$\Delta(j,k) := X^{N,H}(d_{j,k+1}) - X^{N,H}(d_{j,k}), \tag{2.1}$$

where $d_{j,k+1}$ and $d_{j,k}$ are the two dyadic numbers in the interval $[0,1]$ defined as:

$$d_{j,k+1} := (k+1)/2^j \quad \text{and} \quad d_{j,k} := k/2^j. \tag{2.2}$$

Roughly speaking, the proof consists in showing that, almost surely, for all $\tau \in (0,1)$ and for each j large enough, the amplitudes of "many" of these dyadic increments are "rather large"; more precisely there is a deterministic constant $c > 0$, not depending on (j,k), such that, for every j big enough, the following inequality holds:

$$\big|\Delta(j,k)\big| \geq c\, 2^{-j(N(H-1)+1)}, \quad \text{for many } d_{j,k}\text{'s "near to } \tau\text{"}. \tag{2.3}$$

To this end, it would be very nice to have, for each fixed j, the independence of the random variables $\Delta(j,k)$, $0 \leq k < 2^j$. Unfortunately, this is not the case. How can one overcome this difficulty?

Using the definitions of $\Delta(j,k)$ and $X^{N,H}$ one has

$$\Delta(j,k) = \int_{\mathbb{R}^N}' \Big(\int_{d_{j,k}}^{d_{j,k+1}} \prod_{p=1}^{N} (s - x_p)_+^{H-3/2}\, ds \Big)\, dB(x_1)\dots dB(x_N) \tag{2.4}$$

$$= \int_{\mathbb{R}^N}' \Big(\mathbf{1}_{\mathcal{I}_{j,k}}(x_1,\dots,x_N) \int_{d_{j,k}}^{d_{j,k+1}} \prod_{p=1}^{N} (s - x_p)_+^{H-3/2}\, ds \Big)\, dB(x_1)\dots dB(x_N)\,,$$

where $\mathbf{1}_{\mathcal{I}_{j,k}}$ is the indicator function of the unbounded rectangle of $\mathbb{R}^N$:

$$\mathcal{I}_{j,k} := (-\infty, d_{j,k+1}]^N. \tag{2.5}$$

The last equality in (2.4) follows from the fact that

$$\int_{d_{j,k}}^{d_{j,k+1}} \prod_{p=1}^{N} (s - x_p)_+^{H-3/2}\, ds = 0, \quad \text{if } x_p > d_{j,k+1} \text{ for some } p \in \{1,\dots,N\}. \tag{2.6}$$

In order to explain the idea which will allow us to get independent increments, let us assume for a while that $N = 1$. Then, one clearly has that

$$\begin{aligned}\Delta(j,k) &= \int_{-\infty}^{d_{j,k+1}} \Big(\int_{d_{j,k}}^{d_{j,k+1}} (s-x)_+^{H-3/2}\, ds \Big)\, dB(x) \\ &= \int_{d_{j,k}}^{d_{j,k+1}} \Big(\int_{d_{j,k}}^{d_{j,k+1}} (s-x)_+^{H-3/2}\, ds \Big)\, dB(x) \\ &\quad + \int_{-\infty}^{d_{j,k}} \Big(\int_{d_{j,k}}^{d_{j,k+1}} (s-x)^{H-3/2}\, ds \Big)\, dB(x).\end{aligned} \tag{2.7}$$

It is tempting to work with the independent random variables

$$\int_{d_{j,k}}^{d_{j,k+1}} \Big(\int_{d_{j,k}}^{d_{j,k+1}} (s-x)_+^{H-3/2}\, ds \Big)\, dB(x), \quad 0 \le k < 2^j \tag{2.8}$$

instead of the random variables $\Delta(j,k)$, $0 \le k < 2^j$. This is our initial idea for proving the theorem.

Unfortunately, one cannot neglect the other parts of the $\Delta(j,k)$'s, that is the random variables

$$\int_{-\infty}^{d_{j,k}} \Big(\int_{d_{j,k}}^{d_{j,k+1}} (s-x)^{H-3/2}\, ds \Big)\, dB(x), \quad 0 \le k < 2^j. \tag{2.9}$$

One cannot neglect them, mainly because of the fact that on the interval $(-\infty, d_{j,k}]$ the kernel function $x \mapsto \int_{d_{j,k}}^{d_{j,k+1}} (s-x)^{H-3/2}\, ds$ has a singularity in $d_{j,k}$. In order to avoid it, $(-\infty, d_{j,k}]$ has to be replaced by $\big(-\infty, d_{j,k+1} - e_j 2^{-j}\big]$, where $(e_j)_{j\in\mathbb{N}}$ is some increasing sequence of integers (bigger than 2) which goes to $+\infty$ at a "slow" rate. The optimal choice for e_j is

$$e_j := \lfloor S(j) \rfloor . \tag{2.10}$$

Having explained the idea for getting independent increments, from now on one drops our previous convient hypothesis that $N = 1$, and one assumes as usual that the integer $N \ge 2$ is arbitrary.

One focuses on the indices k which are multiple of e_j that is of the form $k = le_j$, where

$$l \in \mathcal{L}^j := \mathbb{N} \cap \left[1, (2^j/e_j) - 1\right]. \tag{2.11}$$

For each $l \in \mathcal{L}^j$, one denotes by $\left\{\mathcal{D}_{j,le_j}, \overline{\mathcal{D}}_{j,le_j}\right\}$ the partition of the integration domain $\mathcal{I}_{j,le_j} := (-\infty, d_{j,le_j+1}]^N$ defined as:

$$\mathcal{D}_{j,le_j} := [d_{j,(l-1)e_j+1}, d_{j,le_j+1}]^N \tag{2.12}$$

and

$$\overline{\mathcal{D}}_{j,le_j} := \mathcal{I}_{j,le_j} \setminus \mathcal{D}_{j,le_j} = \left\{x \in \mathcal{I}_{j,le_j} : x \notin \mathcal{D}_{j,le_j}\right\}. \tag{2.13}$$

Then, one expresses the increment $\Delta(j, le_j)$ as:

$$\Delta(j, le_j) = \widetilde{\Delta}(j, le_j) + \check{\Delta}(j, le_j), \tag{2.14}$$

where

$$\widetilde{\Delta}(j, le_j) = \int'_{\mathbb{R}^N} \left(\mathbf{1}_{\mathcal{D}_{j,le_j}}(x_1, \ldots, x_N) \int_{d_{j,le_j}}^{d_{j,le_j+1}} \prod_{p=1}^{N} (s - x_p)_+^{H-3/2} \, ds \right) dB(x_1) \ldots dB(x_N) \tag{2.15}$$

and

$$\breve{\Delta}(j, le_j) = \int'_{\mathbb{R}^N} \left(\mathbf{1}_{\overline{\mathcal{D}}_{j,le_j}}(x_1, \ldots, x_N) \int_{d_{j,le_j}}^{d_{j,le_j+1}} \prod_{p=1}^{N} (s - x_p)_+^{H-3/2} \, ds \right) dB(x_1) \ldots dB(x_N) . \tag{2.16}$$

Remark 2.1

Using (2.15), the definition of multiple Wiener integral and the equality $\mathcal{D}_{j,le_j} := [d_{j,(l-1)e_j+1}, d_{j,le_j+1}]^N$, it can be shown that $\widetilde{\Delta}(j, le_j)$ is measurable with respect to the σ-algebra $\sigma\big(B(x) - B(y) \,:\, x, y \in (d_{j,(l-1)e_j+1}, d_{j,le_j+1})\big)$. Thus, for each fixed integer j, the random variables $\widetilde{\Delta}(j, le_j)$, $l \in \mathcal{L}^j$, are independent since the increments of the Brownian motion B are independent and the intervals $(d_{j,(l-1)e_j+1}, d_{j,le_j+1})$, $l \in \mathcal{L}^j$, are disjoint.

Basically, the following two lemmas show that, when j goes $+\infty$, the $L^2(\Omega)$-norm of the random variable $\breve{\Delta}(j, le_j)$ is negligible with respect to that of $\widetilde{\Delta}(j, le_j)$, uniformly in $l \in \mathcal{L}^j$.

Lemma 2.1

There is a (strictly) positive constant $\widetilde{c}_0$ such that, for all j large enough, one has

$$\inf_{l \in \mathcal{L}^j} \left\| \widetilde{\Delta}(j, le_j) \right\|_{L^2(\Omega)} \geq \widetilde{c}_0 \, 2^{-j(N(H-1)+1)} \,. \tag{2.17}$$

Lemma 2.2

There exists a positive (finite) constant $\breve{c}_0$ such that, for all j large enough, one has

$$\sup_{l \in \mathcal{L}^j} \left\| \breve{\Delta}(j, le_j) \right\|_{L^2(\Omega)} \leq \breve{c}_0 \, 2^{-j(N(H-1)+1)} \, e_j^{H-1} \,. \tag{2.18}$$

Notice that $\lim_{j \to +\infty} e_j^{H-1} = 0$ since $H < 1$ and $\lim_{j \to +\infty} e_j = +\infty$.

Proof of Lemma 2.1: The integrand associated with $\widetilde{\Delta}(j, le_j)$ being a symmetric function, using the "isometry" property of multiple Wiener integral one gets that

$$\big\|\widetilde{\Delta}(j, le_j)\big\|_{L^2(\Omega)}^2 = N! \int_{\mathcal{D}_{j,le_j}} \Big| \int_{d_{j,le_j}}^{d_{j,le_j+1}} \prod_{p=1}^{N} (s - x_p)_+^{H-3/2}\, ds \Big|^2 dx_1 \ldots dx_N\,. \qquad (2.19)$$

Next, making the change of variable $s = d_{j,le_j} + 2^{-j}u$, one obtains that

$$\big\|\widetilde{\Delta}(j, le_j)\big\|_{L^2(\Omega)}^2 = N!\, 2^{-2j} \int_{\mathcal{D}_{j,le_j}} \Big| \int_0^1 \prod_{p=1}^{N} (d_{j,le_j} + 2^{-j}u - x_p)_+^{H-3/2}\, du \Big|^2 dx_1 \ldots dx_N\,.$$

Then, the equalities

$$\mathcal{D}_{j,le_j} := [d_{j,(l-1)e_j+1}, d_{j,le_j+1}]^N = \big[d_{j,le_j} + (1-e_j)2^{-j}, d_{j,le_j} + 2^{-j}\big]^N \qquad (2.20)$$

and the change of variables $x_p = d_{j,le_j} + 2^{-j}y_p$, for all $p \in \{1, \ldots, N\}$, imply that

$$\begin{aligned} &\big\|\widetilde{\Delta}(j, le_j)\big\|_{L^2(\Omega)}^2 \\ &= N!\, 2^{-(N+2)j} \int_{[1-e_j,1]^N} \Big| \int_0^1 \prod_{p=1}^{N} (2^{-j}u - 2^{-j}y_p)_+^{H-3/2}\, du \Big|^2 dy_1 \ldots dy_N \\ &\geq \widetilde{c}^{\,2}\, 2^{-2j(N(H-1)+1)}\,. \end{aligned} \qquad \square \qquad (2.21)$$

Proof of Lemma 2.2: The integrand associated with $\breve{\Delta}(j, le_j)$ being a symmetric function, using the "isometry" property of multiple Wiener integral, the equality

$$\overline{\mathcal{D}}_{j,le_j} := \Big\{(x_1, \ldots, x_N) \in (-\infty, d_{j,le_j+1}]^N : x_p < d_{j,(l-1)e_j+1} \text{ for some } p\Big\}, \quad (2.22)$$

and the fact that $z \mapsto z^{H-3/2}$ is a decreasing function on $(0, +\infty)$, one gets that

$$\begin{aligned}
&\big\|\breve{\Delta}(j, le_j)\big\|^2_{L^2(\Omega)} = N! \int_{\overline{\mathcal{D}}_{j,le_j}} \Big| \int_{d_{j,le_j}}^{d_{j,le_j+1}} \prod_{p=1}^{N} (s - x_p)_+^{H-3/2}\, ds\Big|^2 dx_1 \ldots dx_N \\
&\leq N \cdot N! \int_{-\infty}^{d_{j,(l-1)e_j+1}} \Bigg(\\
&\qquad \int_{\mathbb{R}^{N-1}} \Big| \int_{d_{j,le_j}}^{d_{j,le_j+1}} (s - x_N)^{H-3/2} \prod_{p=1}^{N-1} (s - x_p)_+^{H-3/2}\, ds\Big|^2 dx_1 \ldots dx_{N-1}\Bigg) dx_N \\
&\leq N \cdot N! \int_{-\infty}^{d_{j,(l-1)e_j+1}} (d_{j,le_j} - x_N)^{2H-3}\, dx_N \qquad (2.23) \\
&\qquad \times \int_{\mathbb{R}^{N-1}} \Big| \int_{d_{j,le_j}}^{d_{j,le_j+1}} \prod_{p=1}^{N-1} (s - x_p)_+^{H-3/2}\, ds\Big|^2 dx_1 \ldots dx_{N-1}\,.
\end{aligned}$$

It is clear that

$$\int_{-\infty}^{d_{j,(l-1)e_j+1}} (d_{j,le_j} - x_N)^{2H-3}\, dx_N = \frac{(d_{j,le_j} - d_{j,(l-1)e_j+1})^{2H-2}}{2-2H}$$
$$= \frac{(d_{j,e_j-1})^{2H-2}}{2-2H} \leq \frac{e_j^{2H-2}\, 2^{-2(j+1)(H-1)}}{2-2H}\,. \tag{2.24}$$

On the other hand, the "isometry" property of multiple Wiener integral imply that

$$(N-1)! \int_{\mathbb{R}^{N-1}} \Big| \int_{d_{j,le_j}}^{d_{j,le_j+1}} \prod_{p=1}^{N-1} (s - x_p)_+^{H-3/2}\, ds \Big|^2 dx_1 \ldots dx_{N-1} \tag{2.25}$$
$$= \mathbb{E}\Big(\big| X^{N-1,H}(d_{j,le_j+1}) - X^{N-1,H}(d_{j,le_j}) \big|^2 \Big) = c_{N-1,H}\, 2^{-2j((N-1)(H-1)+1)}\,.$$

Then combining (2.24) and (2.25) one obtains the lemma. □

In which ways the estimates for $\inf_{l\in\mathcal{L}^j}\left\|\widetilde{\Delta}(j,le_j)\right\|_{L^2(\Omega)}$ and $\sup_{l\in\mathcal{L}^j}\left\|\breve{\Delta}(j,le_j)\right\|_{L^2(\Omega)}$ (provided be the previous two lemmas) could be almost surely extended to the random variables $\inf_{l\in\mathcal{L}^j}\left|\widetilde{\Delta}(j,le_j)\right|$ and $\sup_{l\in\mathcal{L}^j}\left|\breve{\Delta}(j,le_j)\right|$?

For $\sup_{l\in\mathcal{L}^j}\left|\breve{\Delta}(j,le_j)\right|$ a logarithmic correction (that is the factor $j^{N/2}$) is needed:

Lemma 2.3

There are $\breve{c}$ a positive (finite) deterministic constant and $\breve{\Omega}$ an event of probability 1 such that on $\breve{\Omega}$, for every integer j large enough, one has

$$\sup_{l\in\mathcal{L}^j}\left|\breve{\Delta}(j,le_j)\right|\le \breve{c}\,2^{-j(N(H-1)+1)}\,e_j^{H-1}\,j^{N/2}\,. \tag{2.26}$$

Sketch of the proof of Lemma 2.3: The main two ingredients of the proof are Borel-Cantelli's Lemma and the fact that there exists a positive, finite, deterministic, universal constant $c(N)$, depending only on N, such that, for every random variable χ in the Wiener chaos of order N and for each real number $y \geq 2$, one has

$$\mathbb{P}\Big(|\chi| > y\|\chi\|_{L^2(\Omega)}\Big) \leq \exp\big(-c(N)y^{2/N}\big)\,. \tag{2.27}$$

□

The issue becomes more tricky in the case of $\inf_{l\in\mathcal{L}^j}\big|\widetilde{\Delta}(j,le_j)\big|$ since, for almost all $\omega\in\Omega$, some of the realizations $\big|\widetilde{\Delta}(j,le_j,\omega)\big|$, $l\in\mathcal{L}^j$, can be much smaller than $\widetilde{c}_0\,2^{-j(N(H-1)+1)}$ while $\inf_{l\in\mathcal{L}^j}\big\|\widetilde{\Delta}(j,le_j)\big\|_{L^2(\Omega)}\geq\widetilde{c}_0\,2^{-j(N(H-1)+1)}$.

In order to overcome this difficulty the $\big|\widetilde{\Delta}(j,le_j)\big|$'s, $l\in\mathcal{L}^j$, have to be replaced by suprema of some them.

Namely, the $\left|\widetilde{\Delta}(j, le_j)\right|$'s, $l \in \mathcal{L}^j := \mathbb{N} \cap \left[1, (2^j/e_j) - 1\right]$, have to be replaced by the random variables $\widetilde{\Lambda}^j_m$, $m \in \{1, \ldots, M_j\}$, defined, for all m, as:

$$\widetilde{\Lambda}^j_m := \sup\left\{\left|\widetilde{\Delta}(j, le_j)\right| \, : \, l \in \mathcal{L}^j_m := \mathbb{N} \cap \left[U^j_{m-1}, U^j_m\right]\right\}, \tag{2.28}$$

where $(U^j_m)_{m \in \{0,1,\ldots,M_j\}}$ is the subdivision of the interval $\left[1, (2^j/e_j) - 1\right]$ by the $M_j + 1$ points such that:

$$U^j_{M_j} := (2^j/e_j) - 1 \quad \text{and} \quad U^j_m := 1 + m(n_0 j)\,, \text{ for all } m \in \{0, 1, \ldots, M_j - 1\}; \tag{2.29}$$

n_0 being a well-chosen fixed positive integer. Observe that

$$M_j = \mathcal{O}\left(2^j/(je_j)\right). \tag{2.30}$$

Lemma 2.4

There are $\widetilde{c}$ a (strictly) positive deterministic constant and $\widetilde{\Omega}$ an event of probability 1 such that on $\widetilde{\Omega}$ one has

$$\liminf_{j\to+\infty}\left\{2^{j(N(H-1)+1)}\inf_{1\le m\le M_j}\widetilde{\Lambda}_m^j\right\}\ge\widetilde{c}>0\,. \tag{2.31}$$

Sketch of the proof of Lemma 2.4: The main two ingredients of the proof are Borel-Cantelli's Lemma and the following lemma which, roughly speaking, provides an universal non-trivial upper bound for the probability that the absolute values of a centred chaotic random variable be significantly smaller than its standard deviation. □

Lemma 2.5

For any fixed integer $N\ge 1$, there exists an universal nonnegative deterministic constant γ_N strictly smaller than 1, such that, for each random variable χ belonging to the Wiener chaos of order N one has

$$\mathbb{P}\Big(|\chi|<2^{-1}\|\chi\|_{L^2(\Omega)}\Big)\le\gamma_N<1\,. \tag{2.32}$$

Conclusion

Recall that $\Delta(j, le_j) = \widetilde{\Delta}(j, le_j) + \breve{\Delta}(j, le_j)$. Roughly speaking, the following lemma shows almost surely that in any very small part of the interval $[0, 1]$ there are many dyadic increments $\Delta(j, k)$ such that $\left|\Delta(j, k)\right| \geq \widetilde{c}\, 2^{-j(N(H-1)+1)}$.

Lemma 2.6

One lets $(\Lambda_m^j)_{m\in\{1,\dots,M_j\}}$ be the sequence of the nonnegative finite random variables defined, for all $m \in \{1, \dots, M_j\}$, as:

$$\Lambda_m^j := \sup\left\{\left|\Delta(j, le_j)\right| \,:\, l \in \mathcal{L}_m^j\right\}. \tag{2.33}$$

One denotes by Ω^* the event of probability 1 defined as the intersection of the events of probability 1 $\breve{\Omega}$ and $\widetilde{\Omega}$ introduced in Lemmas 2.3 and 2.4. Then, assuming that $\widetilde{c}$ is the same strictly positive deterministic constant as in Lemma 2.4, one has on Ω^*

$$\liminf_{j\to+\infty}\left\{2^{j(N(H-1)+1)} \inf_{1\leq m\leq M_j} \Lambda_m^j\right\} \geq \widetilde{c} > 0\,. \tag{2.34}$$