On the Maximum of the Symmetric Telegraph Process

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The telegraph process

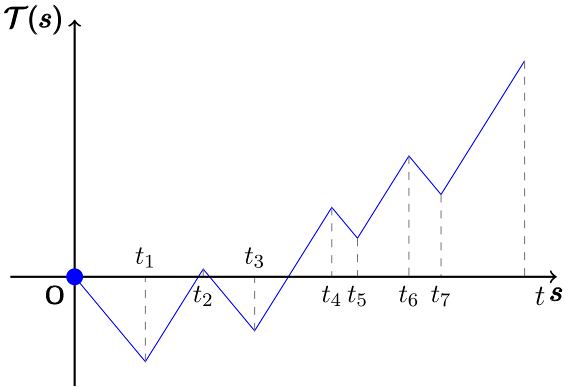
The symmetric telegraph process $\mathcal{T} = \{\mathcal{T}(t), t \geq 0\}$ has the form

$$\mathcal{T}(t) = \int_0^t V(0)(-1)^{N(s)}\,\mathrm{d}s \tag{1}$$

where $N = \{N(t), t \geq 0\}$ is a homogeneous Poisson process with rate $\lambda > 0$ and $V(0)$ is a two valued symmetric random variable taking values $\pm c$, $c > 0$, independent of N.
We interpret the telegraph process as the position if a particle moving on the line with velocity c and with Poisson paced reversals of motion.

Sample paths

A typical sample path of $\mathcal{T}$ is



where $V(0,\omega) = -c$ and $N(t,\omega) = 7$. $T_j(\omega) = t_j,\ j \geq 1$ are the instants where the changes of direction take place on the path ω.

Conditional probability densities (1/2)

For the telegraph process are known all the following, $|x| < ct$,

$$P\{\mathcal{T}(t) \in \mathrm{d}x \,|\, N(t) = 2k+1, V(0) = c\} = \\ = P\{\mathcal{T}(t) \in \mathrm{d}x \,|\, N(t) = 2k+1, V(0) = -c\} \\ = \frac{(2k+1)!}{k!^2} \frac{(c^2t^2 - x^2)^k}{(2ct)^{2k+1}} \,\mathrm{d}x, \quad k \geq 0,$$

$$P\{\mathcal{T}(t) \in \mathrm{d}x \,|\, N(t) = 2k, V(0) = c\} = \\ = \frac{(2k)!}{(k-1)!k!} \frac{(ct+x)^k (ct-x)^{k-1}}{(2ct)^{2k}} \,\mathrm{d}x, \quad k \geq 1,$$

$$P\{\mathcal{T}(t) \in \mathrm{d}x \,|\, N(t) = 2k, V(0) = -c\} = \\ = \frac{(2k)!}{(k-1)!k!} \frac{(ct+x)^{k-1} (ct-x)^k}{(2ct)^{2k}} \,\mathrm{d}x, \quad k \geq 1.$$

Conditional probability densities (2/2)

$$
\begin{aligned}
P\{\mathcal{T}(t) \in \mathrm{d}x \,|\, N(t) = 2k+1\} = \\
&= P\{\mathcal{T}(t) \in \mathrm{d}x \,|\, N(t) = 2k+2\} \\
&= \frac{(2k+1)!}{k!^2} \frac{(c^2t^2 - x^2)^k}{(2ct)^{2k+1}} \,\mathrm{d}x, \quad k \geq 0,
\end{aligned}
$$

All these distributions can be obtained by means of order statistics and considering the exchangeability of the random displacements between Poisson events.

Unconditional probability distribution

The unconditional distribution of $\mathcal{T}(t),\ t \geq 0$, has support in $[-ct, ct]$ with an absolutely continuous component

$$P\{\mathcal{T}(t) \in \mathrm{d}x \,|\, V(0) = c\}$$
$$= \frac{e^{-\lambda t}}{c}\left[\lambda I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\Big) + \frac{\partial}{\partial t} I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2 - x^2}\Big)\right]\mathrm{d}x,$$

where $I_0(x) = \sum_{k=0}^{\infty} x^{2k}/(2^{2k}k!^2)$ is the modified Bessel function. The singular part of the distribution of $\mathcal{T}(t)$ is concentrated at the endpoints $\pm ct$ and has probability

$$P\{\mathcal{T}(t) = ct\} = P\{\mathcal{T}(t) = -ct\} = \frac{e^{-\lambda t}}{2}.$$

Characteristic function

The characteristic function of $\mathcal{T}(t),\ t \geq 0$ reads

$$\mathbb{E}e^{i\gamma\mathcal{T}(t)} = \frac{e^{-\lambda t}}{2}\Bigg[\Bigg(1+\frac{\lambda}{\sqrt{\lambda^2-c^2\gamma^2}}\Bigg)e^{t\sqrt{\lambda^2-c^2\gamma^2}} + \Bigg(1-\frac{\lambda}{\sqrt{\lambda^2-c^2\gamma^2}}\Bigg)e^{-t\sqrt{\lambda^2-c^2\gamma^2}}\Bigg],$$

with $|\gamma| < \lambda/c$.

Maximum of the telegraph process: $V(0) = c$

If $V(0) = c, \; N(t) = 2k + 1$ with $k \geq 0$, the current position of the moving particle is

$$\mathcal{T}(t) = cT_1 - c(T_2 - T_1) + ... + c(T_{2k+1} - T_{2k}) - c(t - T_{2k+1}) \quad (2)$$

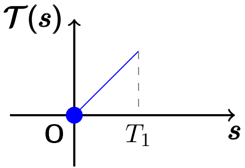
which consists of $k + 1$ upward displacements and $k + 1$ downward displacements (equivalently rightwards or backwards).
The maximum of $\mathcal{T}$ can be attained by its truncated displacements, truncation being performed at odd-order Poisson times $T_{2j+1}, \; j \geq 0$.

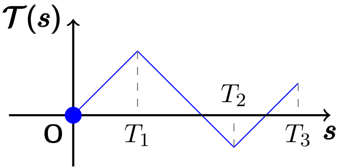
Maximum of the telegraph process: $V(0) = c$

The displacements involved are

$$\mathcal{T}(T_{2j+1}) = cT_1 - c(T_2 - T_1) + ... + c(T_{2j+1} - T_{2j})$$

for $0 \leq j \leq k$, with $j+1$ rightward displacements and j leftward ones.
Examples of truncated displacements with positive initial velocities are the following





Maximum of the telegraph process: $V(0) = c$

We evaluate the following probability, for $0 < \beta < ct$ and $k \geq 0$

$$P\{\max_{0\leq s\leq t} \mathcal{T}(s) \in \mathrm{d}\beta \mid N(t) = 2k+1, V(0) = c\}$$

$$= P\{\cap_{j=0}^{k}\{\mathcal{T}(T_{2j+1}) \in \mathrm{d}\beta\} \mid N(t) = 2k+1, V(0) = c\} \qquad (3)$$

Our first result is $0 < \beta < ct$ and $k \geq 0$

$$P\{\max_{0\leq s\leq t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = c, N(t) = 2k+1\} \qquad (4)$$
$$= 2\frac{(2k+1)!}{k!^2}\frac{(c^2t^2-\beta^2)^k}{(2ct)^{2k+1}}\,\mathrm{d}\beta$$
$$= 2P\{\mathcal{T}(t) \in \mathrm{d}\beta \mid V(0) = c, N(t) = 2k+1\}$$

Maximum of the telegraph process: $V(0) = c$

The idea behind result (4) emerges from the calculation of (3) for small values of k which shows a regularity in the structure of the distribution. On this basis, by an inductive procedure it is possible to obtain (4) for all numbers of changes of direction and to establish a reflection principle connecting the distribution of the maximum in the interval $[0, t]$ with the current position at time t.

Maximum of the telegraph process: $V(0) = c$

In order to obtain result (4) we must consider the following three cases.

(i) At time $T_1,\ cT_1 < \beta$, but at time T_2 the position reached by the particle $cT_1 - c(T_2 - T_1)$ permits us to overcome β in the residual time interval $(T_2, t]$.

(ii) At time T_1 we reach level β and move leftward so little that in $(T_2, t]$ we can overcome β again.

(iii) At time T_1 the moving particle reaches the level β and then in $(T_1, T_2]$ goes leftward so much that in the remaining time interval will not be able to move over β again.

Maximum of the telegraph process: $V(0) = c$

A similar reasoning permits us to obtain, for $0 < \beta < ct$ and $k \geq 0$

$$P\{\max_{0 \leq s \leq t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = c, N(t) = 2k+2\} \tag{5}$$
$$= P\{\max_{0 \leq s \leq t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = c, N(t) = 2k+1\}$$
$$= 2\frac{(2k+1)!}{k!^2} \frac{(c^2t^2 - \beta^2)^k}{(2ct)^{2k+1}} \,\mathrm{d}\beta.$$

Maximum of the telegraph process: $V(0) = c$

The combination of results (4) and (5) permits us to obtain three new formulas for the distribution of the maximum where the initial velocity is positive. For $0 < \beta < ct$

$$P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta, N(t) \text{ even} |\ V(0) = c\} \tag{6}$$

$$= \mathrm{d}\beta \frac{e^{-\lambda t}\lambda t}{\sqrt{c^2t^2-\beta^2}} I_1\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big) = \frac{e^{-\lambda t}}{c}\frac{\partial}{\partial t} I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)\,\mathrm{d}\beta,$$

and

$$P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta, N(t) \text{ odd} |\ V(0) = c\} \tag{7}$$

$$= \frac{\lambda e^{-\lambda t}}{c} I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)\,\mathrm{d}\beta.$$

Maximum of the telegraph process: $V(0) = c$

Finally $0 < \beta < ct$

$$
\begin{aligned}
&P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = c\} \qquad (8)\\
&= \frac{e^{-\lambda t}}{c}\left[\lambda I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big) + \frac{\partial}{\partial t} I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)\right]\mathrm{d}\beta\\
&= 2P\{\mathcal{T}(t) \in \mathrm{d}\beta\}
\end{aligned}
$$

Maximum of the telegraph process: $V(0) = c$

Result (8) shows that if $V(0) = c$, the distribution of the maximum is obtained by folding the telegraph process because a form of reflection principle holds.
Clearly at $\beta = ct$, the distribution of the maximum has a singularity of probability $e^{-\lambda t}$ because if $N(t) = 0$, the particle reaches $x = ct$ since no reversals of the motion occurs.
It must be pointed out that in the last relationship of (8) the conditional event $V(0) = c$ exerts no influence on the absolutely continuous component (but only on the singular part).

Maximum of the telegraph process: $V(0) = -c$

The main difference of the maximum distribution for the case where $V(0) = -c$ with respect to $V(0) = c$ is that the maximum can be zero with positive probability.
Apparently this makes the situation more complicated than in the previous case, but, profiting of the results obtained above, the analysis is simplyfied and two possible situation can occur.

(i) The particle has moved initially leftward so little that during the remaining time interval $(T_1, t]$ it can overpass the level β, that is $\beta - (-cT_1) \leq c(t - T_1)$.

(ii) The particle has initially performed a leftward oriented displacement that it will not be able to overcome the level β. In this case the maximum will be zero.

Maximum of the telegraph process: $V(0) = -c$

The results obtained are summarized here. For $k \geq 1$ and $0 < \beta < ct$

$$P\{\max_{0\leq s\leq t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = -c, N(t) = 2k\} = \frac{2(2k)!}{k!(k-1)!} \frac{(c^2t^2 - \beta^2)^{k-1}(ct-\beta)}{(2ct)^{2k}} \mathrm{d}\beta = 2P\{\mathcal{T}(t) \in \mathrm{d}\beta \mid V(0) = -c, N(t) = 2k\} \tag{9}$$

and

$$P\{\max_{0\leq s\leq t} \mathcal{T}(s) = 0 \mid V(0) = -c, N(t) = 2k\} = \binom{2k}{k} \frac{1}{2^{2k}}. \tag{10}$$

We note that the singularity (10) decreases as $\frac{1}{\sqrt{\pi k}}$.

Maximum of the telegraph process: $V(0) = -c$

The situation changes again if $V(0) = -c$ and the number of Poisson events is odd. Also in this case the maximum can be zero with positive probability. We have therefore, for $0 < \beta < ct$ and $k \geq 0$

$$P\{\max_{0\leq s\leq t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = -c, N(t) = 2k+1\}/\,\mathrm{d}\beta$$

$$= \frac{(2k+1)!}{(k-1)!(k+1)!} \frac{(c^2t^2 - \beta^2)^{k-1}(ct-\beta)}{(2ct)^{2k}} + \frac{(2k+1)!}{k!(k+1)!} \frac{(c^2t^2-\beta^2)^k}{(2ct)^{2k+1}} \tag{11}$$

and

$$P\{\max_{0\leq s\leq t} \mathcal{T}(s) = 0 \mid V(0) = -c, N(t) = 2k+1\} = \binom{2k+1}{k} \frac{1}{2^{2k+1}}. \tag{12}$$

Maximum of the telegraph process: $V(0) = -c$

Formulas (10) and (12) show that the probabilities of the singularity, $\beta = 0$, do not depend on time t and the velocity c, but only on the number of Poisson events. Furthermore, these probabilities display a sort of periodicity in the sense that

$$P\{\max_{0\le s\le t} \mathcal{T}(s) = 0 \mid V(0) = -c, N(t) = 2k+1\}$$
$$= P\{\max_{0\le s\le t} \mathcal{T}(s) = 0 \mid V(0) = -c, N(t) = 2k+2\}$$

and

$$P\{\max_{0\le s\le t} \mathcal{T}(s) = 0 \mid V(0) = -c, N(t) = 2k\}$$
$$> P\{\max_{0\le s\le t} \mathcal{T}(s) = 0 \mid V(0) = -c, N(t) = 2k+1\}.$$

Maximum of the telegraph process: $V(0) = -c$

The above presented maximal distributions are interlaced by the following sursprising relationship

$$\begin{aligned} P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = -c, N(t) = 2k+1\} \\ = \frac{2k+1}{2k+2} P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = -c, N(t) = 2k\} \\ + \frac{1}{2k+2} P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = c, N(t) = 2k+1\}. \end{aligned} \tag{13}$$

Thus, probability (11) is a weighted sum of two conditional maximal distributions, the last one involving the initially right-oriented motion. This second part becomes negligible as k increases.

Maximum of the telegraph process: $V(0) = -c$

It is now relatively easy to obtain the maximal distributions under the condition that $V(0) = -c$. The simplest one is, $0 < \beta < ct$

$$P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta, N(t) \text{ even} |\, V(0) = -c\} \tag{14}$$

$$= e^{-\lambda t} \frac{\lambda(ct-\beta)}{c\sqrt{c^2t^2-\beta^2}} I_1\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)\,\mathrm{d}\beta$$

$$= \frac{e^{-\lambda t}}{c}\Big(1-\frac{\beta}{ct}\Big)\frac{\partial}{\partial t} I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)\,\mathrm{d}\beta.$$

Maximum of the telegraph process: $V(0)=-c$

Much more work requires the derivation of

$$P\{\max_{0\le s\le t}\mathcal{T}(s)\in \mathrm{d}\beta, N(t)\text{ odd}|\ V(0)=-c\} \tag{15}$$

$$=\frac{e^{-\lambda t}}{ct+\beta}\left[\lambda t I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)-\sqrt{\frac{ct-\beta}{ct+\beta}}I_1\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)\right]\mathrm{d}\beta=$$

$$=\frac{ct}{ct+\beta}P\{\max_{0\le s\le t}\mathcal{T}(s)\in \mathrm{d}\beta, N(t)\text{ odd}|\ V(0)=c\}$$

$$-\frac{c}{\lambda(ct+\beta)}P\{\max_{0\le s\le t}\mathcal{T}(s)\in \mathrm{d}\beta, N(t)\text{ even}|\ V(0)=-c\}.$$

The last formula is a consequence of (13) and connects distributions with positive and negative initial velocities, with even and odd number of velocity reversals.

Maximum of the telegraph process: $V(0) = -c$

From (14) and (15) we arrive at

$$P\{\max_{0\le s\le t} \mathcal{T}(s) \in \mathrm{d}\beta \mid V(0) = -c\} \tag{16}$$

$$= e^{-\lambda t}\left[\frac{\lambda t}{ct+\beta} I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big) + \sqrt{\frac{ct-\beta}{ct+\beta}}\Big(\frac{\lambda}{c} - \frac{1}{ct+\beta}\Big) I_1\Big(\frac{\lambda}{c}\sqrt{c^2t^2-\beta^2}\Big)\right]\mathrm{d}\beta. \tag{17}$$

The singularity at $\beta = 0$ as the fine form

$$P\{\max_{0\le s\le t} \mathcal{T}(s) = 0 \mid V(0) = -c\} = e^{-\lambda t}\Big[I_0\Big(\lambda t\Big) + I_1\Big(\lambda t\Big)\Big].$$

Cumulative distribution function

So far we derived and discussed the probability density functions of the maximum under different conditions. The initial velocity is extremely important and makes the distribution quite different if $V(0) = c > 0$ (relatively more simple) and $V(0) = -c$ (much more complicated). In some cases the cumulative distributions take a handsome form as the next result shows.

Cumulative distribution function: $V(0) = c$

For $0 < \beta < ct$ and $k \geq 0$

$$P\{\max_{0\leq s\leq t} \mathcal{T}(s) < \beta \mid V(0) = c, N(t) = 2k+2\} \tag{18}$$

$$= P\{\max_{0\leq s\leq t} \mathcal{T}(s) < \beta \mid V(0) = c, N(t) = 2k+1\}$$

$$= \frac{\beta}{ct} \sum_{j=0}^{k} \binom{2j}{j} \Big(\frac{\sqrt{c^2t^2 - \beta^2}}{2ct}\Big)^{2j}.$$

Formula (18) immediately shows that for $\beta = 0$ we have zero and for $\beta = ct$ we get one.
Special cases, for $N(t) = 1, 3, 5$ were obtained in Orsingher (1990), inspiring the general form of the densities (5) and indirectly of the cumulative distribution (18).

Cumulative distribution function: $V(0) = -c$

For $V(0) = -c$, we have

$$P\{\max_{0\le s\le t} \mathcal{T}(s) \le \beta \mid V(0) = -c, N(t) = 2k\}$$

$$= \frac{\beta}{ct}\sum_{j=0}^{k-1}\Big(\frac{c^2t^2-\beta^2}{c^2t^2}\Big)^j \frac{1}{2^{2j}}\binom{2j}{j} + \frac{(c^2t^2-\beta^2)^k}{(2ct)^{2k}}\binom{2k}{k} \qquad (19)$$

with $0 \le \beta < ct$ and $k \ge 1$. The main difference between (18) and (19) is due to the singularity at $\beta = 0$, which is the reason of the additional term in (20).

Also in this case for small values of k ($k = 2, 3$) the distributions of the maximum with negative initial velocity were obtained in Orsingher (1990), but in this case no idea for general case appeared.

Cumulative distribution function: $V(0) = -c$

Finally

$$P\{\max_{0\le s\le t} \mathcal{T}(s) \le \beta \mid V(0) = -c, N(t) = 2k+1\} \tag{20}$$
$$= \frac{\beta}{ct} \sum_{j=0}^{k-1} \Big(\frac{c^2t^2 - \beta^2}{c^2t^2}\Big)^j \frac{1}{2^{2j}} \binom{2j}{j}$$
$$+ \frac{(2k)!}{k!(k+1)!} \frac{(c^2t^2 - \beta^2)^k}{(2ct)^{2k+1}} \Big((2k+1)ct + \beta\Big).$$

The first term coincides with
$P\{\max_{0\le s\le t} \mathcal{T}(s) \le \beta \mid V(0) = c, N(t) = 2k-1\}$.

Cumulative distribution function: $V(0) = -c$

Also the cumulative function (20) is related to the previous ones by means of the weighted sum

$$P\{\max_{0\le s\le t} \mathcal{T}(s) \le \beta \mid V(0) = -c, N(t) = 2k+1\}$$
$$= \frac{2k+1}{2k+2} P\{\max_{0\le s\le t} \mathcal{T}(s) \le \beta \mid V(0) = -c, N(t) = 2k\}$$
$$+ \frac{1}{2k+2} P\{\max_{0\le s\le t} \mathcal{T}(s) < \beta \mid V(0) = c, N(t) = 2k+1\}.$$

Moments: $V(0) = c$

We have the general expressions for the conditional and unconditional moments of the maximum for $V(0) = c$.

$$\mathbb{E}\left[\left(\max_{0\le s\le t} \mathcal{T}(s)\right)^m \mid V(0) = c, N(t) = 2k+1\right] \tag{21}$$

$$= \mathbb{E}\left[\left(\max_{0\le s\le t} \mathcal{T}(s)\right)^m \mid V(0) = c, N(t) = 2k+2\right]$$

$$= \frac{(2k+1)!(ct)^m}{2^{2k+1}k!} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(k+1+\frac{m+1}{2}\right)}. \tag{22}$$

Moments: $V(0) = c$

Formula (21) specializes for $m = 1, 2$ as

$$\mathbb{E}\left[\max_{0\le s\le t} \mathcal{T}(s) \mid V(0) = c, N(t) = 2k{+}1\right] = \binom{2k+1}{k} \frac{ct}{2^{2k+1}} \quad (23)$$

and

$$\mathbb{E}\left[\left(\max_{0\le s\le t} \mathcal{T}(s)\right)^2 \mid V(0) = c, N(t) = 2k + 1\right] = \frac{c^2t^2}{2k+3}. \quad (24)$$

The last result immediately shows that the maximum of the telegraph process converges in mean square to zero as $k \longrightarrow \infty$. This is because the increasing number of changes of direction constraints the moving particle in a neighborhood of the starting point.

Moments: $V(0) = c$

The unconditional m-th moment reads

$$\mathbb{E}\left[\left(\max_{0\le s\le t} \mathcal{T}(s)\right)^m \mid V(0) = c\right] \tag{25}$$

$$= e^{-\lambda t}(ct)^m\left(\frac{2}{\lambda t}\right)^{\frac{m-1}{2}}\Gamma\left(\frac{m+1}{2}\right)\left[I_{\frac{m-1}{2}}(\lambda t) + I_{\frac{m+1}{2}}(\lambda t)\right].$$

Moments: $V(0) = c$

In particular the first two moments read

$$\mathbb{E}\Big[\max_{0\le s\le t} \mathcal{T}(s) \mid V(0) = c\Big] = e^{-\lambda t} ct \Big[I_0(\lambda t) + I_1(\lambda t) \Big] \qquad (26)$$

and

$$\mathbb{E}\left[\Big(\max_{0\le s\le t} \mathcal{T}(s)\Big)^2 \mid V(0) = c\right] = \frac{c^2 t}{\lambda} + \frac{c^2}{2\lambda^2}\big(e^{-2\lambda t} - 1\big). \qquad (27)$$

Moments: $V(0) = -c$

The condition that $V(0) = -c$ clearly modifies all the values of the moments.

$$\mathbb{E}\left[\left(\max_{0\le s\le t} \mathcal{T}(s)\right)^m \mid V(0) = -c, N(t) = 2k\right] \tag{28}$$

$$= \frac{(2k)!(ct)^m}{2^{2k}k!}\left[\frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(k+\frac{m+1}{2}\right)} - \frac{\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(k+1+\frac{m}{2}\right)}\right]$$

Moments: $V(0) = -c$

The first moment read

$$\mathbb{E}\left[\max_{0\le s\le t}\mathcal{T}(s)\mid V(0)=-c, N(t)=2k\right] = ct\frac{(2k)!}{k!^2 2^{2k}} - \frac{ct}{2k+1} = \tag{29}$$

$$= ctP\{\max_{0\le s\le t}\mathcal{T}(s)=0\mid V(0)=-c, N(t)=2k\} - \frac{ct}{2k+1}.$$

The mean value increases initially with k, attains the maximum value for $2k = 4$ and then decreases. For sufficiently large values of k, the mean of the maximal displacement is a multiple of the probability of the singularity (which itself decreases as $\frac{1}{\sqrt{\pi k}}$).

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*** Thank You ***