LAMBERT-W FUNCTION IN MARKOV BRANCHING PROCESSES WITH GEOMETRIC BRANCHING MECHANISM

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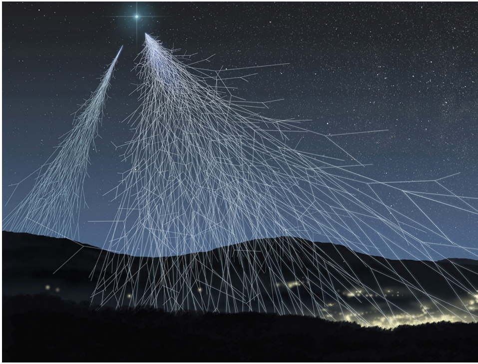
"Industrial scale" branching processes:

Cosmic rays showers, Colliders, etc.

- Flux variations are separate or joint of:
  - initial conditions
  - medium changes
- Randomised branching.
  - Usually following any specific distribution.
- After considering experimental errors or unknown factors:
  - difficulties in applying statistical inference based on classical branching theory.



Source:
https://scied.ucar.edu/image/cosmic-ray-air-shower

The solution based on branching theory could be searched by:

- Using a brute force through computer simulations.
- Analytical solution or approximations for particular parts of branching.

Our work is dedicated on the second option with aim to:

- introduce randomised branching by well know Geometric probability distribution.
- The solutions to be computationally optimised by using a special functions.

Let us consider a single particle branching process driven by 'simplified' randomised reproduction mechanism, such as:

- $X(t), t > 0$, is a time homogeneous MBP starting with one particle as initial condition
- The lifetime of particles is a exponentially distributed random variable with a constant parameter $K > 0$, enabling Markov property.
- The reproduction with mean m-number offsprings is as follows:

$$P(\eta = k) = \frac{m^k}{(1+m)^{k+1}}, \quad k = 0, 1, ..., \quad m > 0, \tag{1}$$

  where the integer-valued random η is the offspring number.
- The p.g.f.'s are denotes as:

$$h(s) = \frac{1}{1+m-ms}, \quad h'(s) = \frac{m}{(1+m-ms)^2}, \quad h''(s) = \frac{2m^2}{(1+m-ms)^3}.$$

The infinitesimal generating function $f(s)$ present in the Kolmogorov equations is defined by the constant K and p.g.f. $h(s)$ as

$$f(s) = K(h(s) - s) = K\left(\frac{1}{1+m-ms} - s\right) = \frac{K(1-s)(1-ms)}{1+m(1-s)},$$

and has the following derivatives, for $k = 1, 2, 3, \ldots,$

$$f'(s) = \frac{f'(1) - Km(1-s)\{2+m-ms\}}{(1+m-ms)^2}, \quad f^{(k)}(s) = \frac{Km^k k!}{(1+m-ms)^{k+1}}.$$

The p.g.f. of the number of particles alive at the time $t > 0$

$$F(t,s) = \sum_{k=0}^{\infty} s^k P(X(t) = k | X(0) = 1) \tag{2}$$

yields the backward Kolmogorov equation

$$\frac{\partial}{\partial t}(F(t,s)) = f(F(t,s)), \quad F(t,s) = s. \tag{3}$$

Our first results were published in [Tchorbadjieff and Mayster, MSTA2020].
The following applications are obtained:

- After applying Lagrange Inverse method in **subcritical case** are computed directly:
  - p.g.f.
  - factorial moments and yielded from them *p.m.f.*, *central moments* and *Variance-to-mean ratio (VMR) for* $0 < m < 1$
- The study of **critical case** is commenced.

Critical process

- Reproduction of particles of mean $m = 1 <=> q = 1$
- The lifetime of particles is a random variable defined by the **exponential distribution** with mean $1/K > 0$.
- The p.g.f.

$$h(s) = \frac{1}{2-s}, \quad f(s) = \frac{K(1-s)^2}{2-s}, \quad \frac{K}{f(s)} = \frac{1}{(1-s)^2} + \frac{1}{1-s}.$$

- The backward Kolmogorov equation is the same as (3), but $f'(1) = 0$.
- The implicit solution of the Kolmogorov equation is

$$\frac{1}{1-F(t,s)} - \log(1-F(t,s)) = \frac{1}{1-s} - \log(1-s) + Kt, \quad |s| < 1.$$

In aim to apply the Lagrange inversion method we introduce $\mathcal{C}(s) = V(G(s))$:

- For $s \neq 1$:

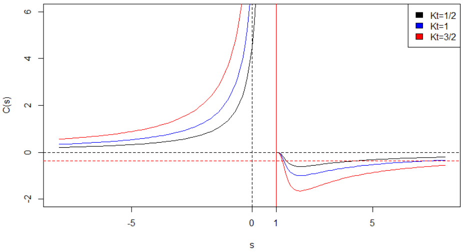
$$\mathcal{C}(s) = \frac{1}{1-s}\exp\left(\frac{1}{1-s}\right)$$

$$\frac{\mathcal{C}'(s)}{\mathcal{C}(s)} = \frac{K}{f(s)}$$

$$G(s) = \frac{1}{1-s}.$$

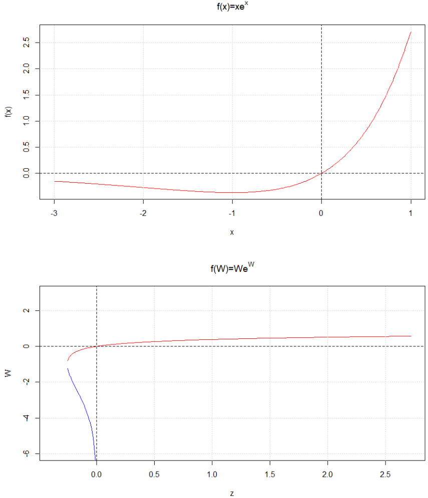
- $V(x) = xe^x$ implies connection to Lambert W function
- Then for the solution of Kolmogorov equation can be yielded from:

$$\mathcal{C}(F(t,s)) = e^{Kt}\mathcal{C}(s), \quad |s| < 1.$$



Lambert W-function W is the inverse function of $f(W) = We^W$

- *Principal branch* W_0:
  W is real for any $x > -1/e$
  (red branch in graphics below).
- $W(-\pi/2) = i\pi/2$
- $min\{x\} : W(-1/e) = -1$
- $W(0) = 0$
- $W(e) = 1$
- $exp(-W(1)) = W(1)$
- There is also a second solution
  W_{-1} for $-1/e \le x < 0$
  (blue branch in graphics below)



After applying Lagrange inversion theorem the Taylor series expansion is:

$$W(z) = \sum_{n-1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n$$

It is slowly convergent series for large n and $z > 0.4(\approx 1/e)$ and thus not useful for direct implementation.
However, for large z a good approximation for $W_0(x)$ is:

$$W(z) \approx ln(z) - lnlnz + \frac{lnlnz}{lnz}, \quad |z| < 1/e.$$

For small z Newton-Raphson iteration yields good approximations.
Many easy to use computational tools and libraries in *R*, *Matlab*, *etc.*
For more details [Corless et.al 1996, Corless et.al 1997].

Finally, for the composite function $V(G(s))$ and its relation to W-function we have:

$$\mathcal{C}(s) = V(G(s)), \quad s < 1, \quad \mathcal{C}^{-1}(x) = G^{-1}(V^{-1}(x)) = G^{-1}(W(x)), \quad x > 0.$$

The solution is for interval of $x > 0$ where G^{-1} has a vertical asymptote at zero.
Then, the solution $F(t,s)$ of the backward Kolmogorov equation is given by,

$$\frac{1}{1 - F(t,s)} = W\left(\frac{e^{Kt}}{(1-s)} \exp\left(\frac{1}{1-s}\right)\right), \quad |s| < 1.$$

In particular, the extinction probability is computed by:

$$F(t,0) = 1 - \frac{1}{W(e^{Kt+1})} < F(t,s) < 1, \quad 0 < s < 1, \quad t > 0. \tag{4}$$

The Taylor series expansion of the function $C(s)$ in the neighbourhood of 0 is expressed by the Lah numbers as follows:

$$\mathcal{C}(s) = e + \sum_{k=1}^{\infty} \frac{c_k s^k}{k!}, \quad c_n = e \sum_{k=1}^{n} (k+1) L(n,k), \tag{5}$$

where $L(n,k)$ are defined by the partial Bell polynomials over the sequence

$$(\bullet!) = (n!, n = 1, 2, \ldots), \quad L(n,k) = B_{n,k}(\bullet!) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{n!}{k!}.$$

The equivalent results can be obtained if the coefficient c_k in the Taylor series expansion (5) of the function $\mathcal{C}(s)$ are known, we can apply the Lagrange inversion method in its general setting to obtain the representation

$$\mathcal{C}_0^{-1}(x) = \sum_{k=1}^{\infty} \frac{d_k x^k}{k!}, \quad \mathcal{C}^{-1}(x) = \mathcal{C}_0^{-1}(x-e), \quad x > 0,$$

where $d_1 c_1 = 1$ and the coefficients d_n for $n = 2, 3, \ldots,$ are given by

$$(2e)^n d_n = \sum_{k=1}^{n-1} (-1)^k [n]_{k\uparrow} B_{n-1,k}(q_\bullet), \quad (q_\bullet) = (q_n = c_{n+1}/(n+1)c_1). \qquad (6)$$

Applying Lambert-W function

However, this approach require:
- enormous iterative computations for high precision.
- it works only for a small Kt.
- after some additional work we obtain:

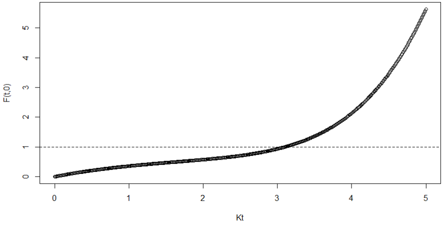
$$F(t,0) = \sum_{j=1}^{\infty} \frac{r_j (Kt)^j}{j!},$$

$$r_j = \sum_{n=1}^{j} S(j,n) e^n d_n,$$

where $S(j,n)$ are Stirling numbers of second kind.

For example, $F(t,0) =$

$$\frac{1}{2}.Kt - \frac{3}{2^3}.\frac{(Kt)^2}{2!} + \frac{11}{2^5}.\frac{(Kt)^3}{3!} - \frac{45}{2^7}.\frac{(Kt)^4}{4!} + \frac{193}{2^9}.\frac{(Kt)^5}{5!} + \ldots.$$



Thus, the solution is obtained by:

$$P(X(t)=n)=\frac{1}{n!}F_s^{(n)}(t,0),\quad n=0,1,2,\ldots.$$

For critical time-homogeneous MBP with geometric branching mechanism the consecutive derivatives of the p.g.f. $F(t,s)$ at zero are:

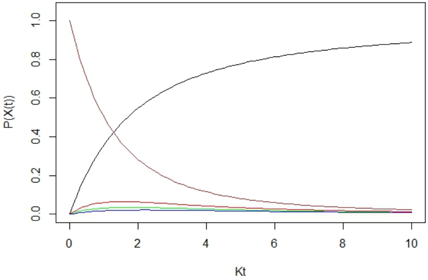
$$F_s^{(n)}(t,0)=\sum_{k=1}^{n}(e^{Kt})^k B_{n,k}(c_\bullet)\left(\sum_{j=1}^{k}\frac{(-1)^{j-1}j!B_{k,j}(W_\bullet)}{(W(e^{Kt+1}))^{j+1}}\right).$$

where the sequence $(W_\bullet)$ is given by the derivatives of the Lambert-W function at the point e^{Kt+1}.

Example:

$$F_s'(t,0) = \frac{2}{W(1+W)}, \quad F_s''(t,0) == \frac{(W-1)(3W+1)}{W(1+W)^3},$$

$$F_s^{(3)}(t,0) = \frac{2(W-1)\{4W^3+2W^2+5W+1\}}{W(1+W)^5},$$



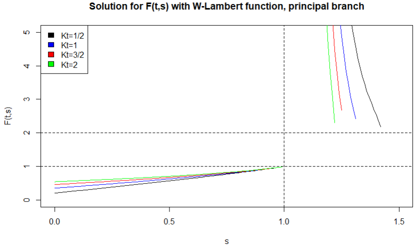
$n = 0$ is black
$n = 1$ is brown
$n = 2$ is red

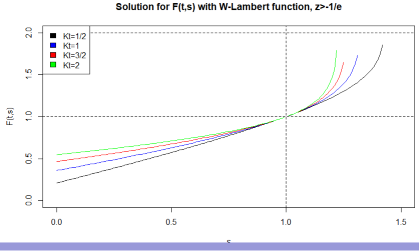
A problem.

The behaviour of the p.g.f. $F(t,s)$ in the neighbourhood of the point $s=1$, especially when $s>1$. The asymptotic behaviour of derivatives in the neighbourhood of the vertical asymptote, left and right, are for $k=2,3,\ldots$.

$$\lim_{s\to 1_-} \mathcal{C}^{(n)}(s) = +\infty, \quad \lim_{s\to 1_+} \mathcal{C}^{(n)}(s) = 0,$$

The vertical asymptote is a serious obstacle to solve this problem using the previous representations.





Denote the partial derivatives of the p.g.f. $F(t,s)$ as follows:

$$\frac{\partial^n F(t,s)}{\partial s^n} = F_s^{(n)}(t,s), \quad |s| < 1, \quad F_s^{(n)}(t,1) = \lim_{s \to 1_-} F_s^{(n)}(t,s), \quad s < 1.$$

The n-ths factorial moment is:

$$E[X(t)]_{n\downarrow} := E[X(t)(X(t)-1)...(X(t)-n+1)] = F_s^{(n)}(t,1).$$

If $f^{(n)}(1) < \infty$ then $F_s^{(n)}(t,1) < \infty$ [Sevastyanov, See Theorem 2, p.33]
Thus, the factorial moments $E[X(t)]_{n\downarrow}$ can be obtained by the consecutive derivatives of the forward Kolmogorov equations after applying the Leibniz formula for derivatives of the product of functions.

The expression for the n-ths moments is given by the Stirling numbers of the second kind $S(n,k)$ and factorial moments as

$$E[X(t)]^n = \sum_{k=1}^{n} S(n,k) E[X(t)]_{k\downarrow}, \quad n = 1, 2, \ldots.$$

In particular, the dispersion of particles when $E[X(t)] = 1$ is given by

$$Var[X(t)] = E[X(t)-1]^2 = EX(t)(X(t)-1) - E[X(t)-1] = F_s^{(2)}(t,1), \quad t > 0.$$

The measure of Shape as Skewness is defined by the factorial moments also.

$$E[(X-1)^3] = E[X(X-1)(x-2)] + E[X-1] = F_s^{(3)}(t,1),$$

and

$$Skew[X] = \frac{E[(X-1)^3]}{(E[(X-1)^2])^{3/2}} = \frac{F_s^{(3)}(t,1)}{(F_s^{(2)}(t,1))^{3/2}}.$$

The forward Kolmogorov equation is a linear differential equation in partial derivatives, i.e.

$$\frac{\partial F(t,s)}{\partial t} = f(s)\frac{\partial F(t,s)}{\partial s}, \quad F(0,s) = s, \quad f(s) = \frac{K(s-1)^2}{2-s}. \qquad (7)$$

We found that all factorial moments exist and for $n = 2, 3, \ldots.$ they perform the following recurrence relation:

$$F_s^{(n+1)}(t,1) = (n+1)F_s^{(n)}(t,1) + Kn(n+1)\int_{x=0}^{t} F_s^{(n)}(x,1)dx,$$

In particular, $F(t,1) = 1$ and

$$F_s'(t,1) = 1, \quad F_s''(t,1) = 2Kt, \quad F_s^{(3)}(t,1) = 3!(Kt + (Kt)^2).$$

Obviously, the factorial moments can be expressed by polynomials as follows,

$$F_s^{(n)}(t,1) = n!Q_{n-1}(x), \quad x = Kt, \quad Q_0 = 1,$$

with the following recurrence relation for $n = 0, 1, 2, ...$

$$Q'_n(x) = Q'_{n-1}(x) + nQ_{n-1}(x), \quad Q_1 = x, \tag{8}$$

Consequently,

$$Q'_n(x) = 1 + 2Q_1(x) + ... + (n-1)Q_{n-2}(x) + nQ_{n-1}(x), \quad Q_1 = x.$$

The polynomials Q_n with their coefficients are denoted by

$$Q_n(x) = x + a_{n,2}x^2 + a_{n,3}x^3 + a_{n,4}x^4 + ...a_{n,n-1}x^{n-1} + x^n, \tag{9}$$

The coefficients $a_{n,k}$ can be obtained from following recurrent relation, derived from (8),

$$a_{n,k} = a_{n-1,k} + \binom{n}{k} a_{n-1,k-1}, \quad n = 1, 2, ..., \quad k = 1, 2, ..., \tag{10}$$

with initial and "boundaries" values of

$$a_{0,0} = 1, \quad a_{n,0} = a_{n,n+1} = 0, \quad n = 1, 2, ..., \quad a_{n,1} = 1.$$

Table: $Q'_n(x) = Q'_{n-1}(x) + nQ_{n-1}(x)$

polynomials Q_n	polynomials Q'_n
$Q_0(x) = 1$	$Q'_0(x) = 0$
$Q_1(x) = x$	$Q'_1(x) = 1$
$Q_2(x) = x + x^2$	$Q'_2(x) = 1 + 2x$
$Q_3(x) = x + \frac{5x^2}{2} + x^3$	$Q'_3(x) = 1 + 5x + 3x^2$
$Q_4(x) = x + \frac{9x^2}{2} + \frac{13x^3}{3} + x^4$	$Q'_4(x) = 1 + 9x + 13x^2 + 4x^3$
$Q_5(x) = x + \frac{14x^2}{2} + \frac{71x^3}{3!} + \frac{77x^4}{12} + x^5$	$Q'_5(x) = 1 + 14x + \frac{71x^2}{2} + \frac{77x^3}{3} + 5x^4$

Applying Taylor theorem for the p.g.f. $F(t,s)$ in the neighbourhood of the point $s=1$ yields:

$$F(t,s)=1+\sum_{n=1}^{\infty}F_s^{(n)}(t,1)\frac{(s-1)^n}{n!}=1+\sum_{n=1}^{\infty}Q_{n-1}(Kt)(s-1)^n, \qquad (11)$$

The series expansion (11) has a very little radius of convergence in the neighbourhood of 1 and we can not calculate the values of the p.g.f. $F(t,s)$. To overcome this disadvantage we proceed by the summation of diagonal terms. For this purpose, we use unsigned Stirling numbers of the first kind

$$|s(n,k)|=B_{n,k}((\bullet-1)!, \quad |s(n,1)|=(n-1)!,$$

to obtain the following useful form for $a_{n,n-j}$:

$$a_{n,n-j}=\frac{1}{(n-j)!}\sum_{k=1}^{j+1}(-1)^{j+1-k}|s(n+1,k)|, \quad n=j+1,j+2,\dots.$$

In particular, for $n=2,3,4...$, using $a_{n,n}=1$, and $H_n=\sum_{k=1}^{n}\frac{1}{k}$ we have:

$$a_{n,n-1}=n(H_n-1), \quad a_{n,n-2}=n(n-1)\left\{\frac{1}{2}(H_n^2-H_n^{(2)})-H_n+1\right\}$$

.

Thus, for $F(t,s)$ we have two equivalent forms:

$$F(t,s) = 1 + \sum_{n=1}^{\infty} (s-1)^n \sum_{j=1}^{n-1} a_{n,j} (Kt)^j,$$

and after exchanging the order of summation:

$$F(t,s) = 1 + (s-1) \sum_{j=0}^{\infty} (s-1)^j \sum_{n=j}^{\infty} a_{n,n-j} (Kt(s-1))^{n-j}.$$

We remark that the generating function of the harmonic numbers is

$$\sum_{n=1}^{\infty} H_n z^n = \frac{-\log(1-z)}{1-z}, \quad H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Now, we can obtain on the main diagonal, for $j = 0$, where $a_{0,0} = a_{n,n} = 1$,

$$(s-1)\sum_{n=0}^{\infty}(Kt(s-1))^n = \frac{(s-1)}{1-Kt(s-1)} := A(t,s).$$

Thus, we can rewrite $F(t,s)$, such as:

$$F(t,s) = 1 + A(t,s) + (s-1)\sum_{j=1}^{\infty}(s-1)^j \sum_{n=j+1}^{\infty} a_{n,n-j}(Kt(s-1))^{n-j}.$$

The summation on the second diagonal for $|s-1| < 1/Kt$ is

$$(s-1)^2\sum_{n=2}^{\infty} a_{n,n-1}(Kt(s-1))^{n-1} = (s-1)^2\sum_{n=2}^{\infty} n(H_n-1)(Kt(s-1))^{n-1} = A^2(t,s)L(t,s),$$

where we have denoted

$$A(t,s) = \frac{(s-1)}{1-Kt(s-1)}, \quad L(t,s) = -\log(1-Kt(s-1)).$$

Finally, we can obtain a solution of the Kolmogorov equations by the diagonals summation, such as:

$$F(t,s) = 1 + \sum_{n=1}^{\infty} \left\{ \frac{(s-1)}{1-Kt(s-1)} \right\}^{n} Q_{n-1}(-\log(1-Kt(s-1))),$$

Using $A(t,s)$ and $L(t,s)$ we can rewrite it to:

$$F(t,s) = 1 + \sum_{n=1}^{\infty} A(t,s)^{n} Q_{n-1}(L(t,s))$$

The first approximation is a linear-fractional function

$$F_1(t,s) = 1 + \frac{(s-1)}{1-Kt(s-1)} = \frac{Kt - s(Kt-1)}{Kt+1-sKt},$$

- vertical asymptote at the point $s^* = 1 + 1/Kt$
- horizontal asymptote $s_* = 1 - 1/Kt$.

Let's consider Geometric branching with intensity of reproduction $K > 0$ and linear birth-death Birth-Death process with lifetime parameter $\beta > 0$ [Tchorbadjieff and Mayster, JAS 2020], i.e.

$$h_{GB}(s) = E[s^\eta] = \frac{1}{2-s}, \quad h_{BD}(s) = E[s^\eta] = \frac{1+s^2}{2},$$

The infinitesimal generating functions are respectively:

$$f_{GB}(s) = \frac{K(1-s)^2}{1-(s-1)}, \quad f_{BD}(s) = \frac{\beta(1-s)^2}{2}.$$

We recognise the p.g.f. of the critical linear birth-death processes (BD) with the lifetime parameter $\beta = 2K$.

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Thank You for Attention!