Construction and limit theorems for supCAR fields

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In memory of P. Brockwell (1937-2023) and O.E. Barndorff-Nielsen (1935-2022)

The talk is based on the work with A. Olenko (La Trobe) and N. Leonenko (Cardiff)

Gaussian fields vs infinitely-divisible fields

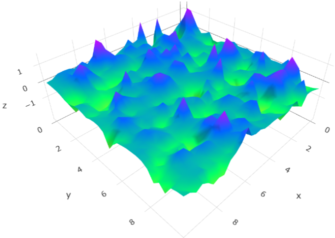

Figure: Gaussian field

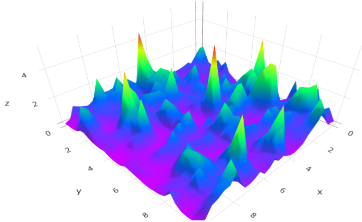

Figure: Gamma field

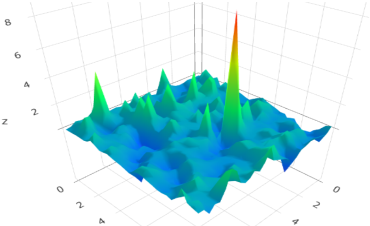

Figure: Student field

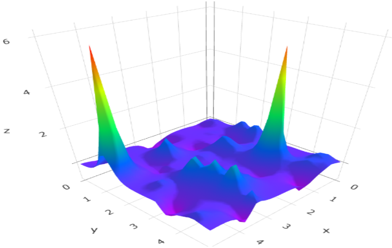

Figure: Cauchy field

Superpositions of OU-processes

Ornstein-Uhlenbeck (OU) type process $dX(t) = -\lambda X(t)dt + dL(\lambda t)$

$$X(t) = \int_{\mathbb{R}} e^{-\lambda t+s} \mathbb{1}_{[0,\infty)}(\lambda t - s)dL(s)$$

where $\{L(t)\}$ is background Lévy process (BDLP) and $\lambda > 0$.

- for any self-decomposable distribution $\mathcal{D}$ there corresponds a BDLP $L(\cdot)$ such that OU process $X(t) \stackrel{\mathrm{d}}{=} \mathcal{D}$
- acf: $r(t) = e^{-\lambda t}, \; t \geq 0$,
- how to obtain a more flexible dependence structure?

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Superpositions of OU (supOU) type process
"Randomize" the parameter λ (Barndorff-Nielsen (2001))

$$X(t) = \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{-\lambda t+s} \mathbb{1}_{[0,\infty)}(\lambda t - s) dL(s) \pi(d\lambda)$$

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$$r(t) = \int_{\mathbb{R}^+} e^{-\lambda t} \pi(d\lambda), \ t \geq 0.$$

See (Barndorff-Nielsen and Leonenko (2005)) for examples.

Lévy-driven CAR fields

Theorem 1 (Brockwell, Matsuda [2017])

If $L(\cdot)$ *is the second-order Lévy field and* $\lambda > 0$. *Then, we refer to*

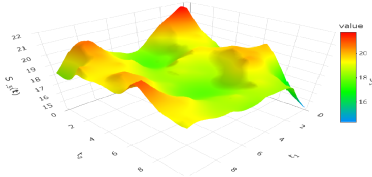
$$S_\lambda(\mathbf{t}) = -\int_{\mathbb{R}^d} \frac{1}{2\lambda} e^{-\lambda||\mathbf{t}-\mathbf{u}||} dL(\mathbf{u})$$

as CAR(1) field, with the acf

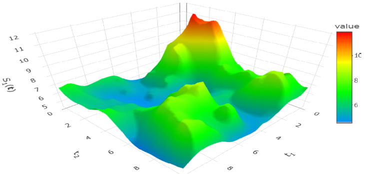
$$r(\mathbf{t}) = C_d \left(\frac{\pi}{2}\right)^{\frac{d}{2}} \frac{||\lambda \mathbf{t}||^{d/2+1}}{\lambda^{d+2}\Gamma(d+1)} K_{d/2+1}(||\lambda \mathbf{t}||),\ \mathbf{t} \in \mathbb{R}^d,\ \text{(Matern covariance)},$$

$K_{d/2+1}(\cdot)$ *denotes the modified Bessel function of the second kind of order* $d/2+1$.

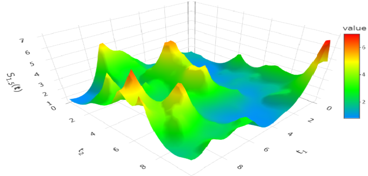
Lévy-driven CAR fields



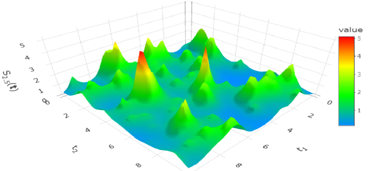
(a) $\lambda = 0.5$



(b) $\lambda = 1$



(c) $\lambda = 1.5$



(d) $\lambda = 2.5$

Figure: Realizations of CAR(1) fields $S_\lambda(\cdot)$ with marginal Gamma distributions

SupCAR fields

Let $\Lambda = \{\Lambda(A),\ A \in \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}^d)\}$ be a random measure such that

$$C(\zeta \ddagger \Lambda(A)) := \log E e^{i\zeta\Lambda(A)} = [\pi \times Leb](A)\mathcal{K}(\zeta),$$

where $\pi(\cdot)$ is a probability measure on $\mathbb{R}^+$ and $\mathcal{K}(\cdot)$ is the cumulant function of infinitely-divisible r.v. with Lévy-Khintchine representation

$$\mathcal{K}(\zeta) := i\zeta a - b\frac{\zeta^2}{2} + \int\limits_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta x \mathbb{1}_{[-1,1]}(x)\right) W(dx),$$

Definition 1

A *supCAR field* *is a field* $X = \{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}^d\}$ *with characteristic quadruple* (a, b, π, W) *is defined as*

$$X(\mathbf{t}) = -\int\limits_{\mathbb{R}^+}\int\limits_{\mathbb{R}^d} \frac{1}{2\lambda} e^{-\lambda||\mathbf{t}-\mathbf{u}||}\Lambda(d\mathbf{u}, d\lambda).$$

Existence of SupCAR fields

Theorem 2 (Exitence of supCAR fields)

The supCAR field

$$X(\mathbf{t}) = -\int_{\mathbb{R}^+}\int_{\mathbb{R}^d} \frac{1}{2\lambda} e^{-\lambda||\mathbf{t}-\mathbf{u}||}\Lambda(d\mathbf{u}, d\lambda),\ \mathbf{t} \in \mathbb{R}^d,$$

with Lévy-Khintchine representation of $\Lambda(\cdot)$

$$C(\zeta \ddagger \Lambda(A)) = [\pi \times Leb](A) \cdot \left(i\zeta a - \frac{\zeta^2}{2} b + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta x \mathbb{1}_{[-1,1]}(x) \right) W(dx) \right),$$

is well-defined if

$$\int_0^\infty \frac{|\ln(\lambda)|^d}{\lambda^{d+2}} \pi(d\lambda) < +\infty \quad \text{and} \quad \int_{1\le|x|\le\infty} |x|(\ln(|x|))^{d-1} W(dx) < +\infty.$$

Cumulant function of supCAR fields

Proposition 1

For supCAR field $X = \{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{R}^d\}$ with characteristic quadruple (a, b, π, W), the following holds true

$$C(\zeta_1, ..., \zeta_k \ddagger X(\mathbf{t}_1), ..., X(\mathbf{t}_k)) := \log E e^{i(\zeta_1 X(\mathbf{t}_1) + \ldots + \zeta_k X(\mathbf{t}_k))}$$

$$= \int_0^\infty \int_{\mathbb{R}^d} \mathcal{K}\left(-\sum_{i=1}^{k} \frac{\zeta_i}{2\lambda} e^{-\lambda ||\mathbf{t}_i - \mathbf{u}||}\right) d\mathbf{u} \pi(d\lambda),$$

where

$$\mathcal{K}(\zeta) := i\zeta a - b\frac{\zeta^2}{2} + \int_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta x \mathbb{1}_{[-1,1]}(x)\right) \mu(dx).$$

ACF and spectral density of supCAR fields

What is the dependence structure of supCAR fields?

$$cov(X(\mathbf{0}), X(\mathbf{h})) = \frac{\partial^2}{\partial\zeta_1\partial\zeta_2} C(\zeta_1, \zeta_2 \ddagger X(\mathbf{0}), X(\mathbf{h}))\Big|_{\zeta_1=\zeta_2=0}$$

$$= \mathcal{K}''(0) \int_0^\infty \int_{\mathbb{R}^d} \frac{1}{4\lambda^2} e^{-\lambda||\mathbf{u}||-\lambda||\mathbf{u}-\mathbf{h}||} d\mathbf{u}\pi(d\lambda).$$

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The covariance function

$$cov(X(\mathbf{0}), X(\mathbf{h})) = c_d \int_0^\infty \lambda^{-d-2} ||\lambda\mathbf{h}||^{d/2+1} K_{d/2+1}(||\lambda\mathbf{h}||)\pi(d\lambda)$$

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The spectral density

$$f(\boldsymbol{\omega}) = c_d \int_0^\infty \frac{1}{(||\boldsymbol{\omega}||^2 + \lambda^2)^{d+1}} \pi(d\lambda).$$

Examples

Example 1

Let $\pi(\cdot)$ *be the Gamma measure* $\pi(d\lambda) = \frac{1}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\lambda}d\lambda$. *Then,*

$$cov(X(\mathbf{0}), X(\mathbf{t})) = c||\boldsymbol{t}||^{d+2} {}_2F_1\left(\frac{\alpha+1}{2}, \frac{\alpha}{2}; \alpha - \frac{d}{2} - \frac{1}{2}; 1 - ||\boldsymbol{t}||^2\right),$$

where ${}_2F_1(\cdot)$ *is the hypergeometric function.*

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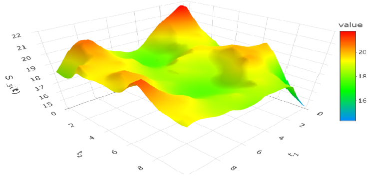
where ${}_2F_1(\cdot)$ *is the hypergeometric function.*

As for fixed values a, b *and* c*,*

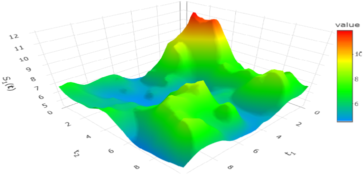
$${}_2F_1(a, b; c; x) \sim (-x)^{-\min\{a,b\}},\ x \to -\infty,$$

it holds

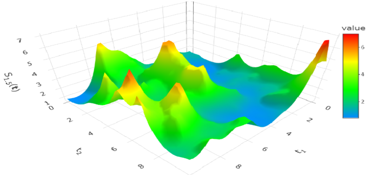
$$\int\limits_{\mathbb{R}^d} cov(X(\mathbf{0}), X(\mathbf{t}))d\mathbf{t} = \begin{cases} \infty,\ \alpha \leq 2d+2,\ \text{LRD}, \\ < \infty,\ \alpha > 2d+2,\ \text{SRD}. \end{cases}$$



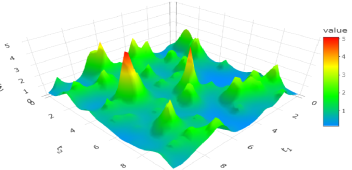
(a) $S_{0.5}(\cdot)$



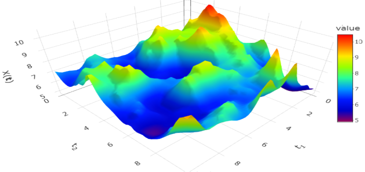
(b) $S_1(\cdot)$



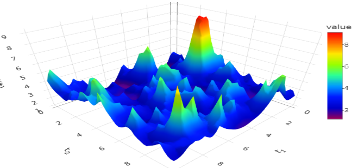
(c) $S_{1.5}(\cdot)$



(d) $S_{2.5}(\cdot)$



(a) $X(\cdot)$ for $\alpha = 5$



(b) $X(\cdot)$ for $\alpha = 8$

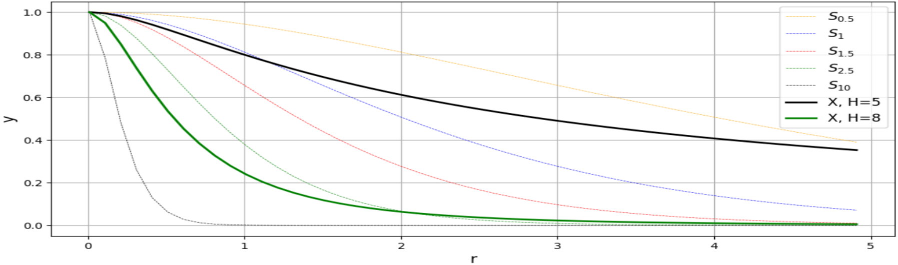


Figure: Normalized covariances of CAR (dotted lines) and supCAR (solid lines) fields

Integrated supCAR process

We will consider limiting behaviour of the integrated supCAR field

$$X^*(t) = \int_{\Delta(t^{\frac{1}{d}}T)} X(\mathbf{s})d\mathbf{s}, \;\; t \in [0,1], \;\; T \to \infty,$$

when

- Gaussian part is present $b > 0$,
- Gaussian part is not present $b = 0$,
- SRD and LRD cases.

We start with the short-range dependence

$$\int_{\mathbb{R}^d} cov(X(\mathbf{0}), X(\mathbf{h}))d\mathbf{h} < \infty$$

Theorem 3

Let $\int_0^\infty \frac{1}{\lambda^{2d+2}} \pi(d\lambda) < \infty$, *then*

$$X_T^*(t) := \frac{1}{c_3 T^{d/2}} \int\limits_{\Delta(t^{1/d} T)} X(\mathbf{s}) d\mathbf{s} \underset{T\to\infty}{\overset{fdd}{\to}} B(t), \ \ t \in [0, 1].$$

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Proof Method of cumulant functions.

$$C(\zeta_1, ..., \zeta_k \ddagger X_T^*(t_1), ..., X_T^*(t_k))$$

$$= \int\limits_{\mathbb{R}^+\times\mathbb{R}^d} \mathcal{K}\left(-\sum_{i=1}^k \frac{\zeta_i}{2c_3\lambda T^{d/2}} \int\limits_{\Delta(t_i^{1/d}T)} e^{-\lambda||\mathbf{s}-\mathbf{u}||}d\mathbf{s}\right)d\mathbf{u}\pi(d\lambda).$$

$$\underset{T\to\infty}{\to} -\frac{1}{2}\sum_{i=1}^k\sum_{j=1}^k \zeta_i\zeta_j \min(t_i, t_j).$$

We proceed with long-range dependence

$$\int_{\mathbb{R}^d} cov(X(\mathbf{0}), X(\mathbf{h}))d\mathbf{h} = \infty$$

Theorem 4

Let $b > 0$, (the Gaussian part is present) and $\pi(\cdot)$ has a density $p(\lambda) = \lambda^{\alpha-1}L(1/\lambda)$, $\alpha \in (d+1, 2d+2)$. Then,

$$X_T^*(t) := \frac{1}{c_5 T^{(3d+2-\alpha)/2} L^{1/2}(t)} \int_{\Delta(t^{1/d}T)} X(\mathbf{s})d\mathbf{s}, \ t \in [0,1],$$

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converges to generalized Brownian motion $Y_{d,1}(t)$

$$\int_{\mathbb{R}^d}' |\lambda|^{(\alpha-d)/2} \int_{\Delta(t^{1/d})} e^{i(\lambda,x)} dx W(d\lambda),\ \ t \in [0,1],$$

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Proof *Use the relation $X(\boldsymbol{s}) = X_1(\boldsymbol{s}) + X_2(\boldsymbol{s})$, where X_1 is Gaussian and X_2 has marginals corresponding to the Lévy measure W. $\int X_1(\boldsymbol{s})d\boldsymbol{s} \underset{T\to\infty}{\overset{fdd}{\to}}$ g.B.m., and $\int X_2(\boldsymbol{s})d\boldsymbol{s} \underset{T\to\infty}{\overset{fdd}{\to}} 0$.*

What happens in the pure jump scenario (no Gaussian component)?

$$\mathcal{K}(\zeta) = \int\limits_{\mathbb{R}} \left(e^{i\zeta x} - 1 - i\zeta x \mathbb{1}_{[-1,1]}(x) \right) W(dx)$$

Blumental-Getoor index of Lévy measure $W(\cdot)$ is

$$\beta = \inf\{\gamma > 0, \int_{|x|<1} |x|^{\gamma} W(dx) < \infty\}.$$

If the probability measure $\pi(\cdot)$ has a density $p(\lambda) = \lambda^{\alpha-1} L(1/\lambda)$, then the supCAR field is well-defined if

$$\alpha > \max\{d + \beta, d + 1\}.$$

Theorem 5 (Stable Lévy process case)

Let $\pi(\cdot)$ *has a density* $p(\lambda) = \lambda^{\alpha-1} L(1/\lambda)$, *and* $W(\cdot)$ *has a B.-G. index* β *such that* $\alpha \in (d+1, 2d+2),\ 0 < \beta < \frac{\alpha}{d+1}$. *Then*

$$X_T^*(t) := \frac{1}{c_6 T^{\frac{d(d+1)}{\alpha}} L^*(T)^{\frac{d}{d+1}}} \int\limits_{\Delta(t^{1/d} T)} X(\mathbf{s}) d\mathbf{s} \xrightarrow[T \to \infty]{fdd} L_{\frac{\alpha}{d+1}}(t),\ \ t \in [0,1],$$

where $L_{\frac{\alpha}{d+1}}$ *is* $\frac{\alpha}{d+1}$*-stable Lévy process.*

Theorem 6 ($Z_{\alpha,\beta}$)

Let $\pi(\cdot)$ has a density $p(\lambda) = \lambda^{\alpha-1}L(1/\lambda)$, and W has a B.-G. index β such that $\alpha \in (d+\beta, 2d+2)$, $\frac{\alpha}{d+1} < \beta < 2$. Then

$$X_T^*(t) := \frac{1}{c_7 T^{d+1-\frac{\alpha-d}{\beta}} L^{1/\beta}(T)} \int_{\Delta(t^{1/d}T)} X(\mathbf{s})d\mathbf{s}, \xrightarrow[T\to\infty]{fdd} Z_{\alpha,\beta}(t), \ t \in [0,1],$$

$$Z_{\alpha,\beta}(t) = \int_0^\infty \int_{\mathbb{R}^d} \int_{\Delta(t^{1/d})} e^{-\lambda||\mathbf{s}-\mathbf{u}||} d\mathbf{s} K(d\lambda, d\mathbf{u}),$$

where K is a stable Lévy basis on $\mathbb{R}^+ \times \mathbb{R}^d$ with control measure $\alpha\lambda^{\alpha-1}d\lambda d\mathbf{u}$.

The process $Z_{\alpha,\beta}$ was first obtained by Puplinskaite and Surgailis (2010)

- $H = 1 - \alpha/\beta$ self-similar
- has stationary increments
- and has continuous paths a.s.

Quick summary

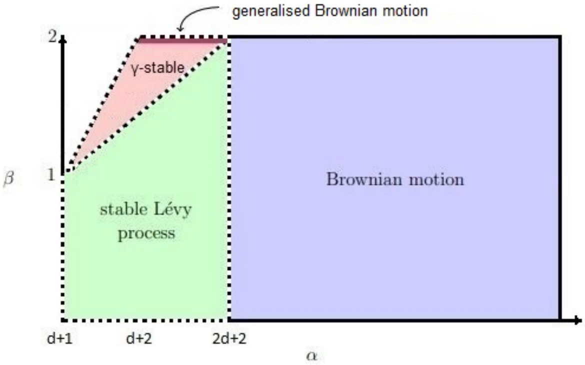

Figure: Parameter space diagram of possible limit scenarios

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Thank you!

This research was supported by the Australian Research Council's Discovery Projects funding scheme (principal investigator Prof. A. Olenko)