Explosion of Crump-Mode-Jagers Processes with critical immediate offspring

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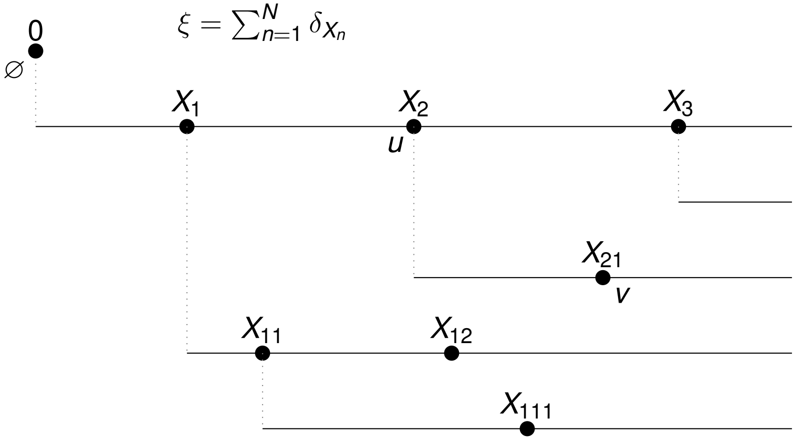
Outline

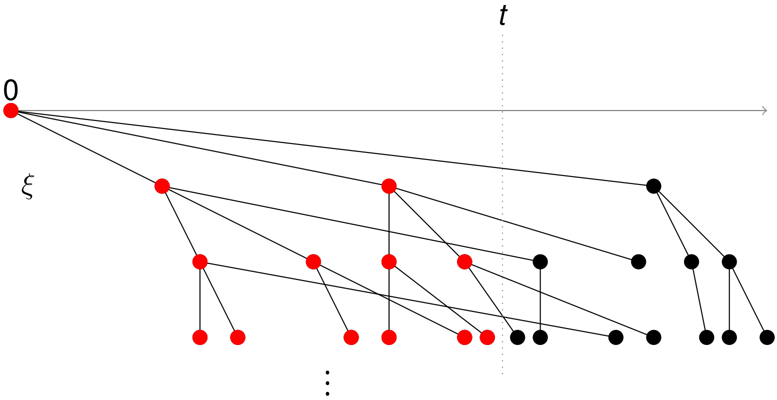
1. CMJ process definition, examples
2. Explosion problem, known results
3. New results: sufficient explosion criteria and characterization for Poisson point processes

Crump-Mode-Jagers Processes



$$S(v) = S(u) + X_{21} = S(\varnothing) + X_2 + X_{21} = X_2 + X_{21}$$

Crump-Mode-Jagers Processes



$\mathcal{Z}_t := \sum_{u \in \mathcal{I}} \mathbb{1}_{\{S(u) \leq t\}}$

The general branching process: origins

- The general branching process was introduced in the late 1960s independently by
  - Ryan (1968),
  - by Crump and Mode (1968),
  - and Jagers (1969).
- The process was designed to incorporate more features and unify earlier models of branching processes (which I am not going to list here).
- General branching processes are by now classical, but still have various applications.

The general branching process: applications

- General branching processes serve as models of
  - biological populations such as humans, cells or plants, (Kimmel & Axelrod 2015, Haccou et al 2007, Jagers 1975)
  - tumor growth (Durrett 2015, Kimmel & Axelrod 2015),
  - neutron chain reactions (Asmussen & Hering 1983)
  - fragmentation (after a change of time) (Janson & Neininger 2008),
  - the initial phase of epidemics (Ball et al 2014, Britton et al 2019, Britton & Scalia Tomba 2019).
- They figure in the study of asymptotic properties of random graph growth models driven by preferential attachment dynamics (Athreya et al 2008, Bhamidi et al 2015) or in first-passage percolation in the configuration model (Bhamidi, van der Hofstad, Hooghiemstra 2010).

Examples

1. $\xi = N\delta_1$ Galton-Watson process (N $\mathbb{N}_0$-valued)

$$Z_n = \sum_{j=1}^{Z_{n-1}} N_{n,j}, \quad (N_{n,j})_{n,j\in\mathbb{N}} \sim N \text{ i.i.d.}$$

$$\mathbb{P}(\forall n \in \mathbb{N} : \; Z_n > 0) > 0 \; \Leftrightarrow \mathbb{E}[N] > 1$$

2. $\xi = Z\delta_W$ Bellman-Harris process ($Z \in \mathbb{N}_0, W \geq 0$)
3. $\xi \sim PPP(\mu)$ Poisson point process (μ measure on $[0,\infty)$)

$$\xi(B) \sim \mathrm{Poi}(\mu(B)), \quad B_1 \cap B_2 = \varnothing \Rightarrow \xi(B_1) \perp\!\!\!\perp \xi(B_2)$$

Explosion

Definition

$$\xi \text{ explosive} \quad :\Leftrightarrow \quad \exists t \geq 0 : \ \mathbb{P}(\mathcal{Z}_t = \infty) > 0$$

Let ξ be locally finite. Then

$$\begin{aligned} \xi \text{ explosive } &\Leftrightarrow \mathbb{P}(\exists \text{ infinite ray } (u_n)_{n\in\mathbb{N}} : \ \lim_{n\to\infty} S(u_n) < \infty) > 0 \\ &\Leftrightarrow \mathbb{P}(T < \infty) > 0 \text{ for } T := \lim_{n\to\infty} \min_{|u|=n} S(u) < \infty \end{aligned}$$

Properties of Explosion

- Let $\xi[0,t] \leq \xi'[0,t]\ \forall t \leq \varepsilon$. Then

$$\xi \text{ explosive } \Rightarrow \xi' \text{ explosive}$$

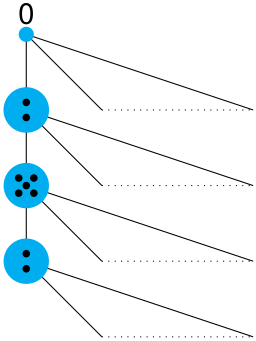
- ξ explosive $\Leftrightarrow \xi(\cdot \cap [0,\varepsilon])$ explosive
- ξ explosive $\Rightarrow \mathbb{P}(T < t) > 0\ \forall t > 0$
- $\overline{F}(t) := \mathbb{P}(T \geq t)$ is the (up to a shift) unique non-trivial solution to

$$\overline{F}(t) = \mathbb{E}\Big[\prod_{|u|=1} \overline{F}(t - S_u)\Big], \quad t \geq 0.$$

(This extends an earlier result by Jagers and Rösler (2004), where they covered the case $\xi[0,\infty) < \infty$ a.s.)

Explosion – easy criteria

Suppose $\mathbb{P}(\xi\{0\} > 0) > 0$.



Galton-Watson($\xi\{0\}$)

$\mathbb{E}[\xi\{0\}] > 1 \;\Rightarrow$ instant explosion!
$\mathbb{E}[\xi\{0\}] < 1 \;\Rightarrow$ no explosion at time 0.

Explosion

Theorem (Jagers 1975)

$\mathbb{E}[\xi[0,\varepsilon]] \leq 1$ *for some* $\varepsilon > 0 \Rightarrow \xi$ *not explosive*

In particular, sufficient for non-explosion:

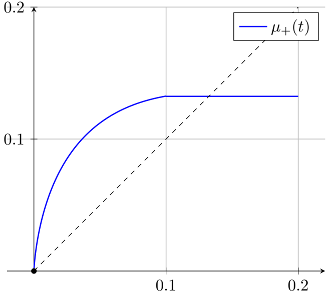
- $\mathbb{E}[\xi\{0\}] < 1$, $\mathbb{E}[\xi[0,\varepsilon]] < \infty$
- $\mathbb{E}[\xi\{0\}] = 1$, $\mathbb{E}[\xi(0,\varepsilon]] = 0$

Remaining cases:

1. What if $\mathbb{E}[\xi\{0\}] = 1$, $\mathbb{E}[\xi(0,t]] > 0$ for all $t > 0$?
2. What if $\mathbb{E}[\xi\{0\}] \leq 1$, $\mathbb{E}[\xi(0,t]] = \infty$ for all $t > 0$?

Sufficient explosion theorem 1

Let $\mathbb{E}[\xi\{0\}] = 1$, $\mu_+(t) := \mathbb{E}[\xi(0,t]] \in (0,\infty)$ for all small $t > 0$.



Theorem

Let $\mathbb{E}[\xi[0,\varepsilon]^2] < \infty$ *and assume*

$$\liminf_{t\downarrow 0} \frac{\mu_+(t)}{t|\log t|^{1+\delta}} > 0$$

Then ξ *is explosive.*

- $\xi = \xi\{0\} +$ homogeneous PPP $\Rightarrow \mu_+(t)$ linear.
- In this case, there is still explosion by an integral test for Markov branching processes in Asmussen's classical textbook.

Proof

Consider Galton-Watson process in varying environment (GWVE):

$$\mathcal{Z}_n = \sum_{j=1}^{\mathcal{Z}_{n-1}} Y_{n,j}, \quad (Y_{n,j})_{j\in\mathbb{N}} \text{ i.i.d.} \sim Y_n$$

Take $(a_n)_{n\in\mathbb{N}}$ with $\sum_{n\in\mathbb{N}} a_n < \infty$.

$$Y_n := \xi[0, a_n], \quad n \in \mathbb{N}_0.$$

GWVE with environment (Y_n) survives $\Rightarrow$ explosion!

GWVE survival

Theorem (Kersting 2020)

Let $(Z_n)_{n\in\mathbb{N}}$ be a GW with VE $(Y_n)_{n\in\mathbb{N}}$. Then, under some mild technical condition, $\mathbb{P}((Z_n)$ survives$) > 0$ iff

$$\sum_{n\in\mathbb{N}} \frac{\nu_n}{m_{n-1}} < \infty, \text{ and } \lim_{n\to\infty} m_n \in (0,\infty] \text{ exists,}$$

where

- $m_n := \mathbb{E}[Z_n] = \mathbb{E}[Y_1]\cdots\mathbb{E}[Y_n]$,
- $\nu_n := \mathbb{E}[Y_n(Y_n-1)]/\mathbb{E}[Y_n]^2$.

Sufficient explosion theorem 2

Let $\mathbb{E}[\xi\{0\}] = 1$, $\mu_+(t) := \mathbb{E}[\xi(0, t]]$.

Theorem

Assume

- $\mathbb{E}[\xi[0, \varepsilon]^2] < \infty$ *(or domain of attraction condition)*
- $\xi\{0\} \perp\!\!\!\perp \xi(\cdot \cap (0, \infty))$
- *For some* $\varepsilon \in (0, 1)$

$$\int_0^\varepsilon \frac{\mu_+^{-1}(t)}{t|\log t|}\,\mathrm{d}t < \infty. \qquad (*)$$

Then ξ *is explosive.*

In particular Poisson point processes with $\mu_+(t) \sim t^p$, $p > 0$

Explosion of Bellman-Harris processes

Theorem (Amini, Devroye, Griffiths, Olver 2013, Komjáthy 2016)

Let Z, W be independent such that

$$\frac{1}{t^{1-\delta}} \leq \mathbb{P}(Z > t) \leq \frac{1}{t^{\delta}}$$

for all large $t > 0$.Then $\xi = Z\delta_W$ is explosive iff for some $\varepsilon \in (0, 1)$

$$\int_0^{\varepsilon} \frac{F_W^{-1}(t)}{t|\log t|} \,\mathrm{d}t < \infty.$$

Proof: Comparison to Bellman-Harris

Lemma

Let $Y \perp\!\!\!\perp W$, $\mathbb{E}[Y] = 1$, $\mathbb{E}[Y^2] \in (1, \infty)$ *Then* $\xi' = Y\delta_0 + \delta_W$ *is explosive iff*

$$\int_0^{\varepsilon} \frac{F_W^{-1}(t)}{t|\log t|} \, \mathrm{d}t < \infty.$$

Now apply to $\xi' := \xi(0)\delta_0 + \delta_W$ with

$$W := \inf\{t > 0 : \xi(0, t] > 0\}$$

ξ' explosive $\Rightarrow \xi$ explosive

Explosion with Poisson reproduction

Suppose $\mathbb{E}[\xi\{0\}] = 1$, $\mu_+(t) := \mathbb{E}[\xi(0, t]]$

Theorem

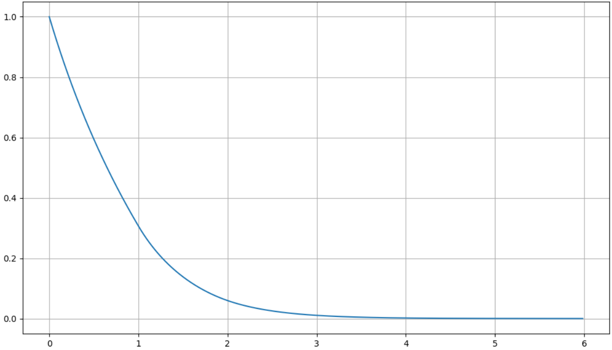
Let ξ be a Poisson point process and μ_+ be convex at zero. Then ξ is explosive iff

$$\int_0^\varepsilon \frac{\mu_+^{-1}(t)}{t|\log t|}\,\mathrm{d}t < \infty.$$

Method of proof inspired by [Grishechkin 1987]

Open Problems

- Remove convexity assumption for PPP
- $\exists$ non-explosive ξ with $\mu_+(t) \sim t^p$, $p > 1$?
- Distribution of T
- $\xi = Z\delta_W$ with Z, W dependent



Summary

- $\mathbb{E}[\xi\{0\}] > 1 \Rightarrow \xi$ explosive
- $\mathbb{E}[\xi[0,\varepsilon]] \leq 1 \Rightarrow \xi$ not explosive
- $\mathbb{E}[\xi\{0\}] = 1$, $\mathbb{E}[\xi[0,\varepsilon]^2] < \infty$ (critical offspring) $\Rightarrow$ depends on μ_+ at 0
  Sufficient for explosion:
  1. $\liminf_{t\to 0} \mu_+(t)/t\log(t)^{1+\delta} > 0$ (superlinearity)
  2. independence and

$$\int_0^{\varepsilon} \frac{\mu_+^{-1}(t)}{t|\log t|}\mathrm{d}t < \infty. \qquad (*)$$

- μ_+ convex at 0 $\Rightarrow \mathrm{PPP}(\mu)$ explosive iff $(*)$
- Thanks for your attention!