On Construction, Properties and Simulation of Haar-Based Multifractional Processes

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Introduction

Fractional Brownian motion (fBm) $\{B_H(t),\ t \geq 0\}$ was introduced by Benoit Mandelbrot and John Van Ness in 1968.

fBm is a centered self-similar Gaussian process with stationary dependent increments, which can be defined as follows:

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})} \left\{ \int_{-\infty}^{0} \left[(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}} \right] dB(s) \right.$$
$$\left. + \int_0^t (t-s)^{H-\frac{1}{2}} dB(s) \right\}. \tag{1}$$

It has a constant Hurst parameter $H \in (0,1)$, which regulates the roughness of the process.

Y. Mishura. (2008) *Stochastic Calculus for Fractional Brownian Motion and Related Processes.* Springer, Berlin.

One of the most frequently used stochastic processes, the standard Brownian motion $\{B(t),\ t \geq 0\}$, is a particular case of fBm with the value of the Hurst parameter equals to $1/2$.

Due to dependent increments, fBm is a more realistic model compared to the standard Brownian motion. Moreover, fBm could model more natural phenomena due to its long memory.

fBm has found numerous applications, for example, in Finance, Telecommunication, Queuing networks, Image processing...

However, fBm is unable to model changes in the roughness of trajectories over time. To address this limitations of fBm, several models of multifractional processes $\{B_{H(t)}(t),\ t \geq 0\}$, have been introduced.

It should be noted that multifractional processes are not stationary and do not have stationary increments.

There are several methods for constructing them, resulting in different processes even when their Hölder regularities are identical. Therefore, simulation approaches also vary substantially.

O.Banna, Yu.Mishura, K.Ralchenko, S.Shklyar. Fractional Brownian Motion: Approximations and Projections. Wiley, 2019.

1. "Naive approach".

2. Approach based on integral approximation.

This approach is related to the pioneering works of

A.Benassi, S.Jaffard, D.Roux(1997) Gaussian processes and pseudo differential Elliptic operators. Rev. Mat. Iberoam. 13 (1), 19–89.

R.-F. Peltier and J. Lévy Véhel. Multifractional Brownian Motion: Definition and Preliminary Results. Report 2645, INRIA, 1995.

The theoretical construction of multifractional processes by Peltier and Véhel was based on the replacement of the constant Hurst parameter in the stochastic integral in (1) with a deterministic Hurst function $H(t) \in (0, 1)$ and was named Classical Multifractional Brownian Motion.

3. Functional random series representation.
Another approach in constructing multifractional processes was based on random wavelet series. The first such construction of Multifractional Processes with Random Exponent by substituting the Hurst parameter with a stochastic process was suggested in

A. Ayache and M. S. Taqqu. Multifractional processes with random exponent. Publicacions Matematiques, 49(2):459–486, 2005.

A similar approach to the general construction of non-stationary processes was considered in

Y. Kozachenko. A. Olenko. O. Polosmak. On convergence of general wavelet decompositions of nonstationary stochastic processes. Electron. J. Probab. 18 1 - 21, 2013.

One of the main challenges is to propose a construction that can provide a feasible representation for theoretical studies and simulations for wide classes of functions $H(t)$. For example, for Hurst functions that can abruptly change over time.

This talk presents the Haar wavelet approach in constructing a new class of multifractional processes. The Haar wavelets were introduced by A.Haar in 1910 and are a system of piecewise constant square wave functions. Among the wavelet methods, which utilize an analytic expression, the Haar wavelet approach is the simplest and the fastest one. Numerous standard algorithms exist for the Haar wavelet computations which can achieve the required precision with a minimum number of grid points.

Background

For a process $\{X(t),\ t \in [0,1]\}$ with continuous sample paths, the function below quantifies the roughness of the process at a fixed time point t.

Definition 1.

The Hölder exponent of the stochastic process $X(\cdot)$ at point $t \in [0,1]$ is defined by

$$\alpha_X(t) = \sup\left\{\alpha \in \mathbb{R}_+ : \limsup_{h\to 0} \frac{|X(t+h) - X(t)|}{|h|^{\alpha}} = 0\right\}.$$

Let $\{H_j : j \in \mathbb{Z}_+\}$ be a sequence of Lipchitz functions defined on $[0,1]$ with values in a fixed compact interval $[\underline{H}, \overline{H}] \subset (0,1)$.

Definition 2.

For each $j \in \mathbb{Z}_+$, we define the norm of H_j by

$$\nu_j := \|H_j\|_{Lipz} := \sup_{x\in[0,1]} |H_j(x)| + \sup_{0\le x'<x''} \frac{|H_j(x'') - H_j(x')|}{x'' - x'}.$$

Assumption 1.

For the sequence of Lipchitz functions $\{H_j : j \in \mathbb{Z}_+\}$ there exists a constant $C > 0$ such that, for all $j \in \mathbb{Z}_+$, it holds

$$\nu_j \leq C(1 + j).$$

Definition 3.

The Haar mother wavelet h is defined via the indicator function $\mathbf{1}_A(\cdot)$ of a set A, as

$$h(s) := \mathbf{1}_{[0,1/2)}(s) - \mathbf{1}_{[1/2,1)}(s), \quad s \in \mathbb{R}.$$

For all $j \in \mathbb{Z}_+$, $k \in \{0, \ldots, 2^j - 1\}$ and $s \in \mathbb{R}$ their dyadic translations and dilations are given by

$$\begin{aligned} h_{j,k}(s) &:= 2^{j/2} h(2^j s - k) \\ &= 2^{j/2} \left(\mathbf{1}_{[k/2^j,(k+1/2)/2^j)}(s) - \mathbf{1}_{[(k+1/2)/2^j,(k+1)/2^j)}(s) \right). \end{aligned}$$

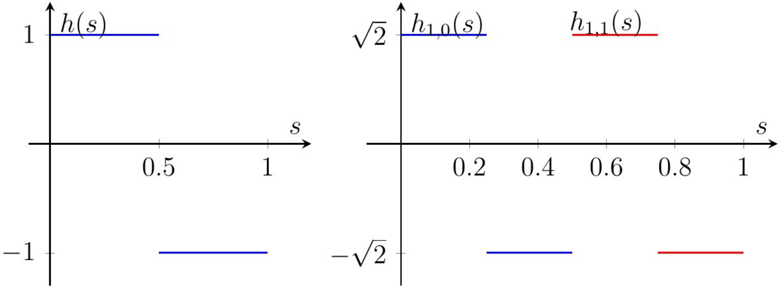


Figure: Haar wavelets

Let $\{\varepsilon_{j,k} : j \in \mathbb{Z}_+ \text{ and } k \in \{0, \ldots, 2^j - 1\}\}$ be a sequence of independent $\mathcal{N}(0,1)$ Gaussian random variables.

Definition 4.

Let the functions $\{H_j(t)\}$ satisfy Assumption 2. The process defined on the interval $[0, 1]$ as

$$X(t) := \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \left(\int_0^1 (t-s)_+^{H_j(k/2^j)-1/2} h_{j,k}(s) ds \right) \varepsilon_{j,k}, \tag{2}$$

will be called the Gaussian Haar-based multifractional process. The abbreviation GHBMP will be used to denote processes from this class.

Note that the GHBMP processes are centered Gaussian, but do not have stationary increments.

L_2 convergence and Hölder continuity of GHBMP

The first result shows that the series in the definition of GHBMP is L_2-convergent.

Theorem 1.

For all fixed $t \in [0,1]$ it holds

$$\mathbb{E}X^2(t) = \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \left| \int_0^1 (t-s)_+^{H_j(k/2^j)-1/2} h_{j,k}(s)ds \right|^2 < +\infty.$$

Therefore, the GHBMP processes are properly defined in L_2-sense.

Let $h^{[\lambda]}(\cdot)$ be defined on $\mathbb{R}$ as

$$h^{[\lambda]}(x) := \int_{\mathbb{R}} (x-s)_+^{\lambda-\frac{1}{2}} h(s)ds$$

$$= \left(\lambda + \frac{1}{2}\right)^{-1} \left((x)_+^{\lambda+\frac{1}{2}} - 2\left(x - \frac{1}{2}\right)_+^{\lambda+\frac{1}{2}} + (x-1)_+^{\lambda+\frac{1}{2}} \right).$$

Proposition 1.

The covariance function of the GHBMP process defined by (2) *equals*

$$\mathrm{Cov}(X(t), X(t')) = \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} 2^{-2jH_{j,k}} h^{[H_{j,k}]}(2^j t - k) h^{[H_{j,k}]}(2^j t' - k).$$

Theorem 2.

For all $t', t'' \in [0,1]$ *it holds*

$$\mathbb{E}(|X(t'') - X(t')|^2) \le c|t'' - t'|^{2\underline{H}}.$$

Therefore, the paths of X *are Hölder continuous functions of any order strictly lower than* $\underline{H}$.

Bounds for the Hölder exponent

For the following results we need the assumption.

Assumption 2.

For all $t \in (0,1)$ it holds that

$$H(t) := \liminf_{j \to +\infty} H_j(t) < \underline{H} + 1/2.$$

First we obtain a lower bound for the Hölder exponent $\alpha_X(t)$.

Proposition 2.

There is a universal event of probability 1, denoted by Ω_1^*, such that on Ω_1^*, it holds

$$\alpha_X(t) \geq H(t) \quad \textit{for all} \quad t \in (0,1).$$

For all integers $J \in \mathbb{Z}_+$ and $K \in \{0, \ldots, 2^J - 1\}$, we set $d_{J,K} := K/2^J$ and use the notation $\Delta_{J,K} := X(d_{J,K+1}) - 2X(d_{J+1,2K+1}) + X(d_{J,K})$.

Lemma 1.

There exists an event Ω_1^* of probability 1 and a random variable C_1^* with finite moments of any order, such that the inequality

$$|\Delta_{J,K}| \leq C_1^* \sigma(\Delta_{J,K}) \sqrt{\log(3 + J + K)}$$

holds on Ω_1^* for all $J \in \mathbb{Z}_+$ and $K \in \{0, \ldots, 2^J - 1\}$.

Lemma 2.

For each fixed $t_0 \in (0, 1)$ and $\varepsilon > 0$, there exists a constant C, which only depends on ε and t_0, such, that for all $J \in \mathbb{Z}_+$ and $K \in \{0, \ldots, 2^J - 1\}$ satisfying

$$\left| \frac{K}{2^J} - t_0 \right| \leq (1 + J)^{-4}.$$

it holds

$$\sigma(\Delta_{J,K}) \leq C 2^{-J(H(t_0) - \varepsilon)}.$$

Proposition 3.

There is a universal event of probability 1, denoted by Ω_2^*, such that on Ω_2^*, it holds

$$\alpha_X(t) \leq H(t) \quad \textit{for all} \quad t \in (0,1).$$

For all $J \in \mathbb{N}$ and $K \in \{0, \ldots, 2^J - 1\}$, we set

$$\Delta^1_{J,K} := X(d_{J,K+1}) - X(d_{J,K}) = \sum_{j=0}^{+\infty} \sum_{\{j,k:\, d_{j,k} \leq d_{J,K+1}\}} a_{j,k}(J,K)\varepsilon_{j,k},$$

where

$$\begin{aligned} a_{j,k}(J,K) &:= \int_0^{d_{J,K+1}} \left((d_{J,K+1} - s)^{H_{j,k}-\frac{1}{2}} - (d_{J,K} - s)_+^{H_{j,k}-\frac{1}{2}} \right) h_{jk}(s) ds \\ &= 2^{-jH_{j,k}} \left(h^{[H_{j,k}]} \left(2^j d_{J,K+1} - k \right) - h^{[H_{j,k}]} \left(2^j d_{J,K} - k \right) \right). \end{aligned}$$

For the event Ω_2^* of probability 1 and for all $J \geq 3$ and $L \in \mathcal{L}_J$ it holds

$$\Delta^1_{J,Le_J} = \tilde{\Delta}^1_{J,Le_J} + \check{\Delta}^1_{J,Le_J},$$

and

$$\limsup_{J\to+\infty} \left\{ 2^{J(H(t_0)+\delta\gamma/4)} \max_{L\in\mathcal{L}_J(t_0)} \left|\Delta^1_{J,Le_J}\right| \right\} \geq C > 0,$$

where

$$\mathcal{L}_J(t_0) = \left\{ L \in \mathcal{L}_J : |t_0 - d_{J,Le_J}| \leq 2Je_J 2^{-J} \right\}.$$

It contradicts the statement that there exists $\bar{\omega} \in \Omega_2^*$ and $\bar{t}_0(\bar{\omega}) \in (0,1)$, such that

$$\limsup_{h\to+\infty} \left\{ |h|^{-H(\bar{t}_0(\bar{\omega}))-2\delta} \left| X(\bar{t}_0(\bar{\omega})+h) - X(\bar{t}_0(\bar{\omega})) \right| \right\} = 0.$$

Therefore, on Ω_2^*, for each $t_0 \in (0,1)$,

$$\alpha_X(t_0) \leq H(t_0) + 2\delta.$$

Now, by combining Propositions 2 and 3, we obtain the following result, which demonstrates that the GHBMP has the required Hölder exponent.

Theorem 3.

There is a universal event of probability 1, *denoted by* Ω^*, *such that on* Ω^*, *it holds*

$$\alpha_X(t) = H(t) \quad \textit{for all} \quad t \in (0,1).$$

Simulation and statistical analysis

To simulate the GHBMP the truncated version of the formula (2) was used, where the first summation was considered up to the level J :

$$X_J(t) := \sum_{j=0}^{J} \sum_{k=0}^{2^j-1} \left(\int_0^1 (t-s)_+^{H_j(k/2^j)-1/2} h_{j,k}(s) ds \right) \varepsilon_{j,k}.$$

The following estimation method of the Hurst function was applied

$$\hat{H}_N^Q(I_{N,n}) := \min \left\{ \max \left\{ \log_{Q^2} \left(\frac{V_N(I_{N,n})}{V_{QN}(I_{N,n})} \right), 0 \right\}, 1 \right\},$$

where $\{I_{N,n}\}$ is a finite sequence of compact subintervals of $[0,1]$ and $Q \geq 2$ is a fixed integer.

$V_N(I_{N,n})$ denotes the generalized quadratic variation of the multifractional process X on $I_{N,n}$ that is defined as

$$V_N(I_{N,n}) := |\nu_N(I_{N,n})|^{-1} \sum_{k \in \nu_N(I_{N,n})} |d_{N,k}|^2,$$

where $|\nu_N(I)|$ denotes the cardinality of the set $\nu_N(I) := \{k \in \{0, \dots, N - L\} : \ k/N \in I\}$ and

$$d_{N,k} = \sum_{l=0}^{L} a_l X\left(\frac{k+l}{N}\right)$$

are the generalized increments of the multifractional process X.

Here, $0 \leq k \leq N - L$, $N \geq L$, where $L \geq 2$ is an arbitrary fixed integer and $\{a_l\}$ are coefficients defined for $l \in \{0, \dots, L\}$ by $a_l := (-1)^{L-l}\binom{L}{l}$.

In each of the following four cases, the multifractional processes were simulated using $J = 20$ and the equally spaced grid of $(2^{18} + 1) = 262145$ time points in the time interval $[0, 1]$. To estimate the Hurst function, $Q = 2$ and $L = 2$ were chosen and the partition $\{I_{N,n}\}$ of the time interval $[0, 1]$ into $N = 100$ subintervals was used.

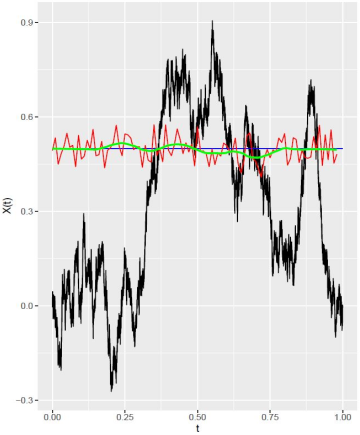
The GHBMP was simulated for

- a constant Hurst function $H(t) \equiv 0.5$;
- $H(t) = 0.2 + 0.45t$;
- $H(t) = 0.5 - 0.4\sin(6\pi t)$;
- using the sequence of functions

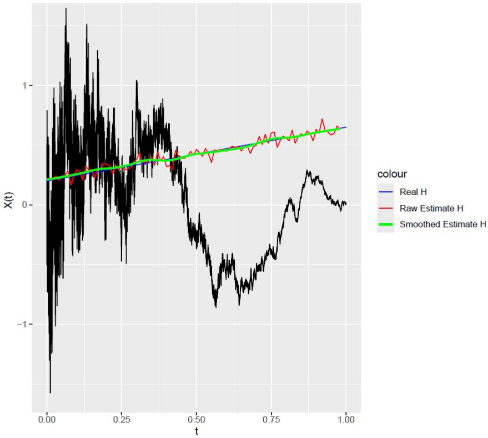
$$H_j(t) = \begin{cases} 1/4, & \text{if } t \in [0, 1/2 - 1/(2j)]\,; \\ jt/2 + (1/2 - j/4)\,, & \text{if } t \in [1/2 - 1/(2j), 1/2 + 1/(2j)]\,; \\ 3/4, & \text{if } t \in [1/2 + 1/(2j), 1]\,, \end{cases}$$

that converges to the Hurst function with a discontinuity

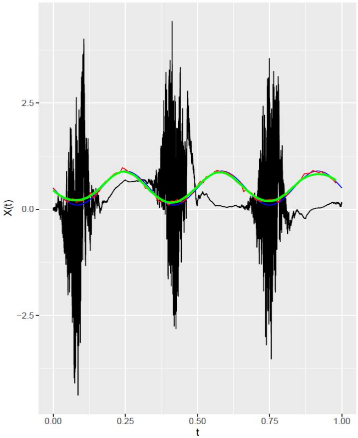
$$\lim_{j \to \infty} H_j(t) = H(t) = \begin{cases} 1/4, & \text{if } t \in [0, 1/2]\,; \\ 3/4, & \text{if } t \in [1/2, 1]\,. \end{cases}$$



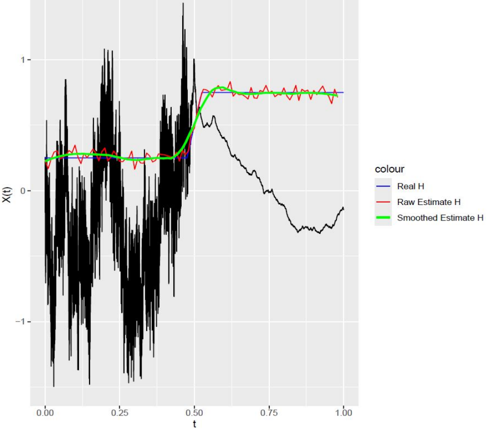
(a) Case of $H \equiv 0.5$



(b) Case of $H(t) = 0.2 + 0.45t$



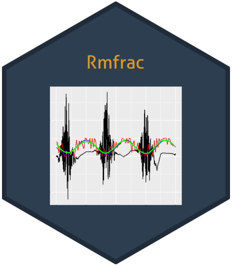
(c) $H(t) = 0.5 - 0.4\sin(6\pi t)$



(d) $H(t)$ with discontinuity

Figure: Realisations of GHBMP and corresponding Hurst functions

R package Rmfrac



The functions available in **Rmfrac** package can be categorized into four main groups:

- simulation of multifractional processes,
- estimation of the Hurst function,
- analyses of multifractional processes,
- plotting functions for each class of outputs from the main groups.

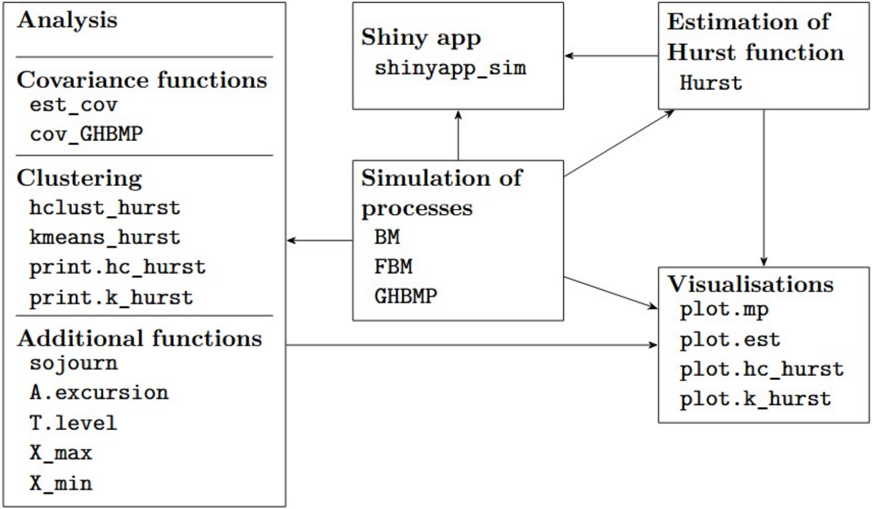


Figure: Structure of the **Rmfrac** package

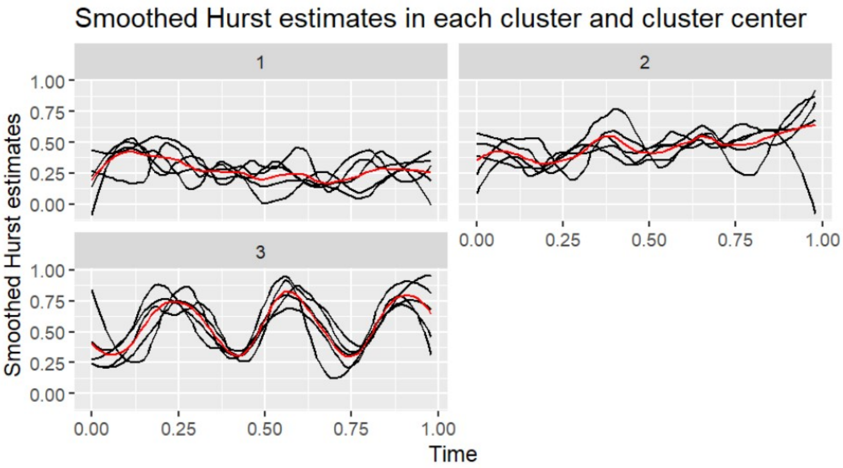


Figure: Clustering trajectories

```
R> sojourn(Process_1,0.8,level='lower',subI=c(0.5,0.8),plot=TRUE)

[1] 0.1833525

R> A.excursion(Process_1,0.8,level='lower',subI=c(0.5,0.8),plot=TRUE)

[1] 0.05567839
```

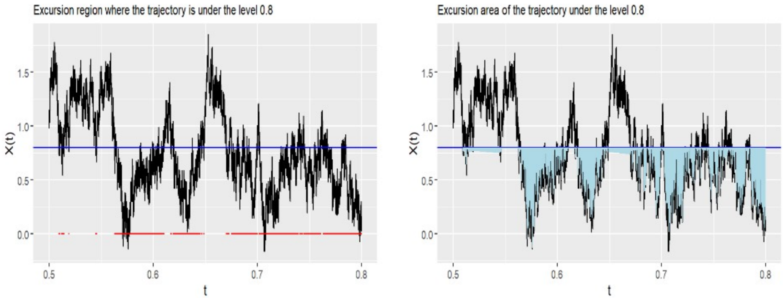


Figure: Estimation of geometric functionals

Problems for future research

To advance the proposed approach, future studies can be focused on:

- developing similar models for processes with random Hurst functions;
- modifying the approach for the multidimensional case of random fields;
- studying convergence rates in different metrics;
- developing and justifying statistical inference methods for the corresponding classes of multifractional processes;
- extending the R package to other models making them accessible to applied researchers.

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