Estimation of Chirp Signal Parameters

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1. Setting of the problem.

In this lecture we are going to speak about our joint results with my master student Victor Hladun.

An important generalization of classical trigonometric model in statistics of stochastic processes and signal processing are models of frequency-modulated sinusoidal signals observed against the background or random noises having different nature. The problems of estimating the parameters of such signals have been studied for a long time.

The most studied here is the case of a linearly frequency-modulated signals, which we will refer to as chirp signal.

For one harmonic it can be written as

$$A\cos\left(\varphi t + \psi t^2\right) + B\sin\left(\varphi t + \psi t^2\right),\ \ t \geqslant 0, \tag{1}$$

where A, B are amplitudes, φ is a starting frequency, ψ is a chirp rate. For discrete time t and random noise being a linear time series, a lot of results on consistency and asymptotic normality of LSE and some other estimates of a chirp signal which is one harmonic or the sum of harmonics (1) have been obtained in a large number of works mainly by Indian statisticans. See, for example,

D. Kundu, S. Nandi (2021). On Chirp and Some Related Signals Analysis: A Brief Review and Some New Results, Sankhya A, **83**(2), 844–890.

Here we consider time continuous multiple chirp signal observed with additive random strongly-dependent noise and prove LSE strong consistency of unknown signal parameters.

Suppose we observe a stochastic process

$$X(t) = g(t, \theta^0) + \varepsilon(t), \quad t \in \mathbb{R}_+, \tag{2}$$

where

$$g(t, \theta^0) = \sum_{j=1}^{N} \left(A_j^0 \cos\left(\varphi_j^0 t + \psi_j^0 t^2\right) + B_j^0 \sin\left(\varphi_j^0 t + \psi_j^0 t^2\right)\right), \tag{3}$$

$$\theta^0 = \left(A_1^0, B_1^0, \varphi_1^0, \psi_1^0, \dots, A_N^0, B_N^0, \varphi_N^0, \psi_N^0\right), \tag{4}$$

$\left(A_j^0\right)^2 + \left(B_j^0\right)^2 > 0$, $j \in \overline{1, N}$, $\varepsilon = \{\varepsilon(t),\ t \in \mathbb{R}\}$ is a stochastic processes defined on a probability space $(\Omega, \mathfrak{F}, P)$ and satisfying the next assumptions.

A. ε is a sample continuous stationary Gaussian process with zero mean and covariance function (c.f.) $B(t) = \mathsf{E}\,\varepsilon(t)\varepsilon(0)$, $t \in \mathbb{R}$, having one of the properties:
(**i**) $B(t) = L(|t|)/|t|^{\alpha}$, $\alpha \in (0, 1)$, with non-decreasing slowly varying at infinity function L;
(**ii**) $B(\cdot) \in L_1(\mathbb{R})$.

Assuming that true values of amplitudes A_j^0, B_j^0, $j = \overline{1, N}$, are different numbers, true values of frequencies φ_j^0, $j = \overline{1, N}$, and chirp rates ψ_j^0, $j = \overline{1, N}$, are different positive numbers, we arrange the chirp rates $\psi^0 = \left(\psi_1^0, ..., \psi_N^0\right)$ in increasing order and suppose

$$\psi^0 \in \Psi\left(\underline{\psi}, \overline{\psi}\right) = \left\{\psi = (\psi_1, ..., \psi_N) \in \mathbb{R}^N : 0 \leqslant \underline{\psi} < \psi_1 < ... < \psi_N < \overline{\psi} < +\infty\right\}.$$

In turn, we also introduce the parametric set

$$\Phi\left(\underline{\varphi}, \overline{\varphi}\right) = \left\{\varphi = (\varphi_1, ..., \varphi_N) : 0 \leqslant \underline{\varphi} < \varphi_j < \overline{\varphi} < +\infty,\ j = \overline{1, N}\right\}$$

such that $\varphi^0 = \left(\varphi_1^0, ..., \varphi_N^0\right) \in \Phi\left(\underline{\varphi}, \overline{\varphi}\right)$.

Consider monotonically non-decreasing family of open sets $\Psi_T \subset \Psi\left(\underline{\psi}, \overline{\psi}\right)$, $T > T_0 > 0$, containing vector ψ^0, such that $\left(\bigcup\limits_{T > T_0} \Psi_T\right)^c = \Psi^c\left(\underline{\psi}, \overline{\psi}\right)$, with the following properties.

B. 1) $\lim\limits_{T\to\infty} \inf\limits_{\substack{1 \leqslant j \leqslant N-1, \\ \psi \in \Psi_T}} T^2 (\psi_{j+1} - \psi_j) = +\infty;$

2) $\lim\limits_{T\to\infty} \inf\limits_{\psi\in\Psi_T} T^2 \psi_1 = +\infty.$

Definition 1. Any random vector

$$\theta_T = (A_{1T}, B_{1T}, \varphi_{1T}, \psi_{1T}, ..., A_{NT}, B_{NT}, \varphi_{NT}, \psi_{NT}) \tag{5}$$

such that it is a point of the functional

$$Q_T(\theta) = T^{-1} \int\limits_0^T [X(t) - g(t,\theta)]^2 \, dt \tag{6}$$

absolute minimum on the parametric set $\Theta_T^c \subset \mathbb{R}^{4N}$, where amplitudes A_j, B_j, $j = \overline{1,N}$, can take any values, and parameters (φ, ψ) take values in the set $\Phi^c(\underline{\psi}, \overline{\psi}) \times \Psi_T^c$, $T > T_0 > 0$, is called LSE of the parameter θ^0.

So, we study just such an estimate, namely: θ_T is defined on the parametric set Θ_T^c depending on T.

The main result of the presentation is the folowing.

Theorem 1. Let the conditions **A** and **B** be satisfied. Then LSE θ_T in strongly consistent estimate of the parameter θ^0 in the sense that
$A_{jT} \to A_j^0,\ B_{jT} \to B_j^0,\ T\left(\varphi_{jT} - \varphi_j^0\right) \to 0,\ T^2\left(\psi_{jT} - \psi_j^0\right) \to 0$, a.s., as $T \to \infty,\ j = \overline{1, N}$.

One auxiliary fact, which is also of independent mathematical interest, plays important role in the proof of **Theorem 1**.

2. Uniform law of large numbers.

Theorem 2. Under condition **A**

$$\xi_T = \sup_{\varphi,\psi\in\mathbb{R}} \left| T^{-1} \int_0^T e^{-i(\varphi t+\psi t^2)} \varepsilon(t)dt \right| \to 0 \ a.s., \ as \ T \to \infty. \tag{7}$$

Below we sketch the proof of the convergence (7).

Step 1. Making suitable changes of variables in the integral

$$T^{-2}\int_0^T\int_0^T exp\left\{-i\left(\varphi(t-s)+\psi(t^2-s^2)\right)\right\}\varepsilon(t)\varepsilon(s)dtds$$

in order to obtain a majorant independent of the parameters φ and ψ, we arrive at the nontrivial inequality

$$\mathsf{E}\,\xi_T^2 \leqslant 2^{3/2}T^{-2}\int_0^T\left(\int_0^{T-u}\left(\int_0^{T-u-v}\int_0^{T-u-v}\mathsf{E}\,\varepsilon(u+v+s)\varepsilon(u+s)\varepsilon(v+s)\varepsilon(s)\ \times\right.\right.$$

$$\left.\left.\times\,\varepsilon(u+v+t)\varepsilon(u+t)\varepsilon(v+t)\varepsilon(t)dtds\right)^{1/2}dv\right)^{1/2}du. \qquad (8)$$

Step 2. For further estimation of the right hand side of (8) we use condition **A** and apply Isserlis' theorem to the product of 8 values of the stationary Gaussian process ε. The number of terms in the Isserlis' sum is $7!! = 105$, and each summand is the product of some 4 values of the process ε c.f., that is

$$\mathsf{E}\,\varepsilon(u+v+s)\varepsilon(u+s)\varepsilon(v+s)\varepsilon(s)\varepsilon(u+v+t)\varepsilon(u+t)\varepsilon(v+t)\varepsilon(t) = \sum_{r=1}^{105} b_r(u,v,t-s). \quad (9)$$

The terms $b_r(u,v,t-s)$ can be relatively easy calculated and in fact, not all of them depend on all the variables u, v, $t-s$. From (8) and (9) under condition **A**(**i**) we obtain

$$\mathsf{E}\,\xi_T^2 \leqslant 2^{3/2}\sum_{r=1}^{105} T^{-2}\int_0^T\left(\int_0^{T-u}\left(\int_0^{T-u-v}\int_0^{T-u-v} b_r(u,v,t-s)dtds\right)^{1/2}dv\right)^{1/2}du \leqslant$$

$$2^{3/2}\sum_{r=1}^{105}\int_0^1\left(\int_0^1\left(\int_0^1\int_0^1 b_r\left(Tu,Tv,T(t-s)\right)dtds\right)^{1/2}dv\right)^{1/2}du. \quad (10)$$

Step 3. Any $b_r(Tu, Tv, T(t-s))$ is a product of 4 values of c.f., and we bound 3 of them by $B(0)$ to estimate then a correspondent integral of the 4-th value using condition **A**(**i**). This approach coarsens the inequalities, but makes it easier to write down the answer.

In final analysis for any $\delta > 0$ and $T > T_0(\delta)$ we arrive at the inequality

$$\mathsf{E}\,\xi_T^2 \leqslant 2\sqrt{2}\left(75\sqrt[4]{2}(1-\alpha)^{-\frac{1}{4}}+25\left(1-\frac{\alpha}{2}\right)^{-\frac{1}{2}}+5\left(1-\frac{\alpha}{4}\right)^{-1}\right)(1+\delta)^{\frac{1}{4}}B^{\frac{3}{4}}(0)B^{\frac{1}{4}}(T). \tag{11}$$

Similarly, under condition **A**(**ii**), using notation $\|B\|_1 = \int\limits_{-\infty}^{\infty}|B(t)|dt$, we obtain

$$\mathsf{E}\,\xi_T^2 \leqslant 2\sqrt{2}\sum_{r=1}^{105}\int\limits_0^1\left(\int\limits_0^1\left(\int\limits_0^1\int\limits_0^1 b_r\left(Tu, Tv, T(t-s)\right)dtds\right)^{1/2}dv\right)^{1/2}du \leqslant$$

$$\leqslant\left(\frac{1536+64\sqrt{2}}{7\sqrt{3}}\right)B^{\frac{3}{4}}(0)\|B\|_1 T^{-\frac{1}{4}}+2\|B\|_1 T^{-1},$$

that is

$$\mathsf{E}\,\xi_T^2 = O\left(T^{-1/4}\right),\ \ as\ T\to\infty. \tag{12}$$

Step 4. Using the relations (11) and (12) we arrive in the standard way at convergence (7) in the formulation of **Theorem 1**.

3. Sketch of the proof of Theorem 1.

Part 1. Consider a system of linear equations for A_{jT}, B_{jT}, $j = \overline{1, N}$, which is a subsystem of the system of normal equations for LSE θ_T:

$$\frac{\partial Q_T(\theta_T)}{\partial A_p} = \frac{\partial Q_T(\theta_T)}{\partial B_p} = 0, \quad p = \overline{1, N},$$

and rewrite it in more detail in the form

$$\begin{cases} \sum_{j=1}^{N} a_{jp}^{(1)} A_{jT} + \sum_{j=1}^{N} b_{jp}^{(1)} B_{jT} = c_p^{(1)}, \quad p = \overline{1, N}; \\ \sum_{j=1}^{N} a_{jp}^{(2)} A_{jT} + \sum_{j=1}^{N} b_{jp}^{(2)} B_{jT} = c_p^{(2)}, \quad p = \overline{1, N}. \end{cases} \tag{13}$$

In the system (13) the following notation are used. Let $\cos(\varphi_{jT} t + \psi_{jT} t^2) = \cos_j(t)$, $\sin(\varphi_{jT} t + \psi_{jT} t^2) = \sin_j(t)$.

Then the coefficients of system (13) can be written as

$$a_{jp}^{(1)} = T^{-1}\int_0^T \cos_j(t)\cos_p(t)dt, \quad a_{jp}^{(2)} = T^{-1}\int_0^T \cos_j(t)\sin_p(t)dt,$$

$$b_{jp}^{(1)} = T^{-1}\int_0^T \sin_j(t)\cos_p(t)dt, \quad b_{jp}^{(2)} = T^{-1}\int_0^T \sin_j(t)\sin_p(t)dt,$$

$$c_{p}^{(1)} = T^{-1}\int_0^T X(t)\cos_p(t)dt, \quad c_{p}^{(2)} = T^{-1}\int_0^T X(t)\sin_p(t)dt,$$

Let also denote by $o_T(1)$, $T > 0$, possibly, different stochastic processes such that $o_T(1) \to 0$ a.s., as $T \to \infty$.

Using condintion **B**, it can be shown thet for $j, p = \overline{1, N}$

$$\begin{gathered} a_{jp}^{(1)} = o_T(1),\ j \neq p,\ a_{pp}^{(1)} = \frac{1}{2} + o_T(1);\ a_{jp}^{(2)} = o_T(1), \\ b_{jp}^{(1)} = o_T(1);\ b_{jp}^{(2)} = o_T(1), j \neq p,\ b_{pp}^{(2)} = \frac{1}{2} + o_T(1),\ j, p = \overline{1, N}. \end{gathered} \tag{14}$$

The following simple statement helps to do this.

Lemma 1. Let α_T, β_T, $T > 0$, be some functions and $\beta_T \to +\infty$, as $T \to \infty$. Then

$$\int_0^1 \cos(\alpha_T t + \beta_T t^2) dt, \ \int_0^1 \sin(\alpha_T t + \beta_T t^2) dt \to 0, \ as \ T \to \infty. \quad (15)$$

The proof of Lemma 1 uses the properties of Fresnel integrals

$$C(x) = \int_0^x \cos(t^2) dt, \ \int_0^x \sin(t^2) dt, \ x > 0. \quad (16)$$

Using notation

$$x_{pT} = T^{-1} \int_0^T \cos((\varphi_{pT} - \varphi_p^0) t + (\psi_{pT} - \psi_p^0) t^2) dt, \quad (17)$$

$$y_{pT} = T^{-1} \int_0^T \sin((\varphi_{pT} - \varphi_p^0) t + (\psi_{pT} - \psi_p^0) t^2) dt, \quad (18)$$

condition **B** and **Theorem 2** we get

$$c_p^{(1)} = \frac{1}{2}\left[A_p^0 x_{pT} - B_p^0 y_{pT}\right] + o_T(1),\ p = \overline{1, N}, \tag{19}$$

$$c_p^{(2)} = \frac{1}{2}\left[A_p^0 y_{pT} + B_p^0 x_{pT}\right] + o_T(1),\ p = \overline{1, N}, \tag{20}$$

Appliying relations (14), (19), (20) to the system (13) we obtain the following expressions for LSE A_{jT}, B_{jT}:

$$A_{jT} = A_j^0 x_{jT} - B_j^0 y_{jT} + o_T(1),\ B_{jT} = A_j^0 y_{jT} + B_j^0 x_{jT} + o_T(1),\ j = \overline{1, N}. \tag{21}$$

Obviously, $|x_{jT}|$, $|y_{jT}| \leqslant 1$, then as it follows from (21)

$$|A_{jT}| \leqslant |A_j^0| + |B_j^0| + o_T(1),\ |B_{jT}| \leqslant |A_j^0| + |B^0| + o_T(1),\ j = \overline{1, N}. \tag{22}$$

Part 2. Let

$$G_T(\theta_1,\theta_2) = T^{-1}\int_0^T (g(t,\theta_1) - g(t,\theta_2)^2\, dt,\ \ \theta_1,\theta_2 \in \Theta_T^c.$$

From LSE θ_T definition we obtain

$$0 \geqslant Q_T(\theta_T) - Q_T(\theta^0) = G(\theta_T,\theta^0) + 2T^{-1}\int_0^T \varepsilon(t)\left(g(t,\theta^0) - g(t,\theta_T)\right) dt. \tag{23}$$

By **Theorem 2** and (22)

$$T^{-1}\int_0^T \varepsilon(t)\left(g(t,\theta^0) - g(t,\theta_T)\right) dt \to 0\ a.s.,\ \ as\ T \to \infty,$$

and therefore due to (23)

$$G_T(\theta_T,\theta^0) \to 0\ a.s.,\ \ as\ T \to \infty. \tag{24}$$

Using notaiton

$$g_{jT}(t) = A_{jT}\cos_j(t) + B_{jT}\sin_j(t) - A_j^0\cos_j^0(t) - B_j^0\sin_j^0(t),$$

we get

$$G_T(\widehat{\theta}_T, \theta^0) = \sum_{j=1}^{N} T^{-1}\int_0^T g_{jT}^2 dt + 2\sum_{j<p} T^{-1}\int_0^T g_{jT}(t)g_{pT}(t)dt. \tag{25}$$

From **Lemma1**, condintion **B**, inequalities (22) we find that the 2nd sum in (25) is $o_T(1)$.

On the other hand

$$T^{-1}\int_0^T g_{jT}^2 dt = \frac{1}{2}\left[A_{jT}^2 + B_{jT}^2 + (A_j^0)^2 + (B_j^0)^2\right] - \left(A_{jT}A_j^0 + B_{jT}B_j^0\right)x_{jT} + \\ + \left(A_{jT}B_j^0 - A_j^0 B_{jT}\right)y_{jT} + o_T(1),\ j = \overline{1,N}. \tag{26}$$

Substituting equalities (21) into expressions (26) we derive from (24)

$$\sum_{j=1}^{N}\left((A_j^0)^2+(B_j^0)^2\right)(1-x_{jT}^2-y_{jT}^2)\to 0 \ a.s., \ as\ T\to\infty, \tag{27}$$

or

$$x_{jT}^2+y_{jT}^2\to 1 \ a.s., \ as\ T\to\infty, \ j=\overline{1,N}. \tag{28}$$

Part 3. Introducing the notation $\lambda_{jT}=T(\varphi_{jT}-\varphi_j^0)$, $\mu_{jT}=T^2(\psi_{jT}-\psi_j^0)$, we can write

$$x_{jT}=\int_0^1\cos(\lambda_{jT}t+\mu_{jT}t^2)dt, \ y_{jT}=\int_0^1\sin(\lambda_{jT}t+\mu_{jT}t^2)dt, \ j=\overline{1,N}. \tag{29}$$

Let $\Omega_0\subset\Omega$, $\mathsf{P}(\Omega_0)=1$, be a random event for which (28) is valid. If for any elementary event $\omega\in\Omega_0$

$$\lambda_{jT},\mu_{jT}\to 0 \ a.s., \ as\ T\to\infty, \ j=\overline{1,N}, \tag{30}$$

Then (28) is true by Lebesgue majorized convergence theorem.

Assume (28) is not true for some $\omega_0 \in \Omega_0$. Then there exists a sequence $T_m,\ m \geqslant 1$, such that for some $j \in \{1, ..., N\}$ and $\lambda_{jT_m} := \lambda_{jm}$, $\mu_{jT_m} := \mu_{jm}$ we can indicate the possible options of λ_{jm}, μ_{jm} convergence, as $m \to \infty$.

$(i)\ \ \lambda_{jm} \to +\infty \ or\ -\infty,\ \ \mu_{jm} \to \mu_j \neq 0;$
$(ii)\ \ \lambda_{jm} \to +\infty \ or\ -\infty,\ \ \mu_{jm} \to 0;$
$(iii)\ \ \lambda_{jm} \to \lambda_j,\ \ \mu_{jm} \to \mu_j, \lambda_j^2 + \mu_j^2 > 0;$
$(iv)\ \ \lambda_{jm} \to +\infty \ or\ -\infty,\ \ \mu_{jm} \to +\infty \ or\ -\infty;$
$(v)\ \ \lambda_{jm} \to \lambda_j,\ \ \mu_{jm} \to +\infty \ or\ -\infty;$

Set also $x_{jT_m} := x_{jm}$, $y_{jT_m} := y_{jm}$ and show that for options $(i) - (v)$

$$x_{jm}^2 + y_{jm}^2 \nrightarrow 1,\ \ as\ m \to \infty. \tag{31}$$

Let $\mu_{jm} \nrightarrow 0$, as $m \to \infty$, and $\mu_{jm} > 0$ for sufficiently large m. Then we can express $x_{jm}^2 + y_{jm}^2$ in terms of Fresnel integrals.

Designate $\gamma_{jm} = \lambda_{jm}/2\sqrt{\mu_{jm}}$, then

$$x_{jm}^2 + y_{jm}^2 = \frac{1}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \cos(t^2) dt \right)^2 + \frac{1}{\mu_{jm}} \left(\int\limits_{\gamma_{jm}}^{\sqrt{\mu_{jm}}+\gamma_{jm}} \sin(t^2) dt \right)^2 . \tag{32}$$

In the case $\mu_{jm} < 0$ for sufficiently large m and $\gamma'_{jm} = -\lambda_{jm}/2\sqrt{|\mu_{jm}|}$, we obtain a similar to (32) equality with $|\mu_{jm}|$ instead of μ_{jm} and γ'_{jm} instead of γ_{jm}.
Then for options (i), (iv), and (v)

$$x_{jm}^2 + y_{jm}^2 \to 0, \text{ as } m \to \infty. \tag{33}$$

Going over to the option (ii) using identity

$$x_{jm}^2 + y_{jm}^2 = \int\limits_0^1 \int\limits_0^1 \cos(\lambda_{jm}(t-s) + \mu_{jm}(t^2 - s^2)) dt ds$$

and Cauchy-Schwartz inequality, we arrive at the relation

$$\limsup_{m\to\infty} (x_{jm}^2 + y_{jm}^2) \leqslant \frac{1}{\sqrt{2}}. \tag{34}$$

In the option (*iii*) by the Lebesgue theorem and Cauchy-Schwartz inequality we receive

$$\lim_{m\to\infty}(x_{jm}^2 + y_{jm}^2) < 1. \tag{35}$$

Together with (27), this means that for $j = \overline{1, N}$

$$\lambda_{jT} = T(\varphi_{jT} - \varphi_j^0),\ \mu_{jT} = T^2(\psi_{jT} - \psi_j^0) \to 0\ a.s.,\ as\ T \to \infty. \tag{36}$$

From (21), (29), and (36) it follows also

$$A_{jT} \to A_j^0,\ B_{jT} \to B_j^0,\ a.s.,\ as\ T \to \infty,\ j = \overline{1, N}. \tag{37}$$

Thank you for your attention!