Limit theorems for additive functionals of the fractional Brownian motion

David Nualart

The University of Kansas

International Workshop “Stochastic Processes with Statistical Applications and Fractional Stochastic Calculus”
Yuliya Mishura’s anniversary
May 17, 2023

Fractional Brownian motion

The fractional Brownian motion (fBm) $B = (B_t^H, t \geq 0)$ is a zero mean Gaussian process with covariance function given by

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right).$$

$H \in (0,1)$ is called the Hurst parameter. For $H = \frac{1}{2}$, $B^{\frac{1}{2}}$ is a Brownian motion.

- *Stationary increments*: $\mathbb{E}[(B_t^H - B_s^H)^2] = |t-s|^{2H}$.
- *Regularity*: For any $\gamma < H$, with probability one, the trajectories $t \to B_t^H(\omega)$ are Hölder continuous of order γ:

$$|B_t^H(\omega) - B_s^H(\omega)| \leq G_{\gamma,T}(\omega)|t-s|^{\gamma}, \quad s,t \in [0,T].$$

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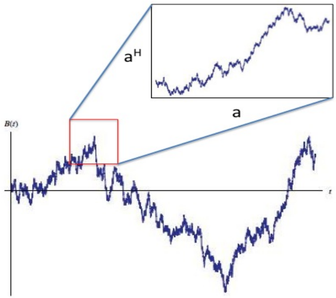
- *Self-similarity*: For all $a > 0$, the process

$$\{a^{-H} B^H_{at}, t \geq 0\}$$

is a fractional Brownian motion with Hurst parameter H.



- *Local nondeterminism:*

  For any $0 = s_0 < s_1 < \cdots < s_n < \infty$ and $u_1, \ldots, u_n \in \mathbb{R}$,

$$\mathrm{Var}\Big(\sum_{i=1}^{n} u_i \big(B^H_{s_i} - B^H_{s_{i-1}} \big) \Big) \geq k_H \sum_{i=1}^{n} u_i^2 (s_i - s_{i-1})^{2H}.$$

- For Brownian motion there is equality with $K_{\frac{1}{2}} = 1$, because the increments are independent.

Local time

Let B^H be a d-dimensional fBm with Hurst parameter H.

- The local time of the fBm B^H is formally defined as

$$L_t(x) = \int_0^t \delta(B_s^H - x) ds,$$

  for $t \geq 0$ and $x \in \mathbb{R}^d$.

- The local time is the density of the occupation measure:

$$\int_0^t f(B_s) ds = \int_{\mathbb{R}^d} f(x) L_t(x) dx.$$

- If $H < \frac{1}{d}$ there exists a version of the local time which is continuous in (t, x) (Geman-Horowitz '80). In fact,

$$\mathbb{E}[L_t(0)] = \int_0^t (2\pi s^{2H})^{-\frac{d}{2}} ds < \infty \quad \Leftrightarrow \quad H < \frac{1}{d}$$

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First order limit result

Theorem

Assume $H < \frac{1}{d}$. Let $f \in L^1(\mathbb{R}^d)$. Then, for all $t \geq 0$,

$$n^{Hd} \int_0^t f(n^H(B_s^H - \lambda))ds \to L_t(\lambda) \int_{\mathbb{R}^d} f(y)dy,$$

as $n \to \infty$, in $L^2(\Omega)$.

Proof:

We simply write

$$\begin{aligned} n^{Hd} \int_0^t f(n^H(B_s^H - \lambda))ds &= n^{Hd} \int_{\mathbb{R}^d} f(n^H(x - \lambda))L_t(x)dx \\ &= \int_{\mathbb{R}^d} f(y)L_t(n^{-H}y + \lambda)dy. \end{aligned}$$

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Second order limit result

What happens if $\int_{\mathbb{R}^d} f(y)dy = 0$?

Theorem (Hu-N.-Xu '14)

Suppose $\frac{1}{d+2} < H < \frac{1}{d}$ *and* $f : \mathbb{R}^d \to \mathbb{R}$ *satisfies* $\int_{\mathbb{R}^d} |f(y)|(1 + |y|^{\frac{1}{H}-d})dy < \infty$ *and* $\int_{\mathbb{R}^d} f(y)dy = 0$. *Then,*

$$n^{\frac{Hd+1}{2}} \int_0^t f(n^H B_s^H)ds \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(0)}$$

in $C([0,\infty))$, *as* $n \to \infty$, *where* $\widetilde{W}$ *is a Brownian motion independent of* B^H *and*

$$\|f\|^2_{\frac{1}{H}-d} = -\int_{\mathbb{R}^{2d}} f(x)f(y)|x-y|^{\frac{1}{H}-d}dxdy.$$

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$$C_{H,d} = \frac{2^{1-1/(2H)}}{(1-Hd)\pi^{d/2}} \Gamma\left(\frac{Hd+2H-1}{2H}\right).$$

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- The proof is based on the method of moments and uses the local nondeterministic property.
- One can show tightness and the convergence holds in $C([0,\infty))$.
- By self-similarity, we can replace $n^H B^H_s$ by B^H_{sn} and making the change of variable $ns \to u$, we obtain

$$n^{\frac{Hd-1}{2}} \int_0^{nt} f(B^H_u)du \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}}\|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(0)}.$$

- When $d=1$ we need $H > \frac{1}{3}$. In the particular case $d=1$ and $H=\frac{1}{2}$, we obtain

$$n^{-\frac{1}{4}} \int_0^{nt} f(B^{\frac{1}{2}}_u)du \xrightarrow{\mathcal{L}} \sqrt{2}\|f\|_1 \widetilde{W}_{L_t(0)},$$

which was proved by Papanicolau-Stroock-Varadhan '77 using martingale methods.

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Suppose $d = 1$

Questions:

- What happens if $H \leq \frac{1}{3}$?
- Can we obtain a second order limit result when $\int_{\mathbb{R}} f(y)dy \neq 0$?

Second order limit result for $\int_{\mathbb{R}} f(y)dy \neq 0$:

Theorem (Jaramillo-Nourdin-N.-Peccati '23)

Suppose $H > \frac{1}{3}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies $\int_{\mathbb{R}} |f(y)|(1+|y|)dy < \infty$. Then

$$n^{\frac{H+1}{2}} \left(\int_0^t f(n^H(B_s^H - \lambda))ds - n^{-H} L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) \xrightarrow{f.d.d.} \sqrt{C_{H,f}} \widetilde{W}_{L_t(\lambda)},$$

as $n \to \infty$, where f.d.d means convergence in law of the finite-dimensional distributions, $\widetilde{W}$ is a Brownian motion independent of B^H, and $C_{H,f}$ is a constant depending on H and f.

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- This approach was used to establish a CLT for the modulus of continuity of the Brownian local time (Hu-N. '09):

$$h^{-\frac{1}{2}}\left(\int_{\mathbb{R}}(L_t(x+h)-L_t(x))^2dx-4th\right)\xrightarrow{\mathcal{L}}\frac{8}{\sqrt{3}}\left(\int_{\mathbb{R}}(L_t(x))^2dx\right)^{\frac{1}{2}}\eta,$$

as $h\to 0$, where η is a $N(0,1)$ random variable independent of the Brownian motion.

Integral representation of fBm

We can assume that

$$B_t^H = \int_0^t K_H(t,s)dW_s,$$

where W is a standard Brownian motion and

$$K_H(t,s) = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

if $H > \frac{1}{2}$ and

$$K_H(t,s) = \left[\frac{2H}{(1-2H)\beta(1-2H, H+\frac{1}{2})}\right]^{\frac{1}{2}} \times \left[\left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{1}{2}} u^{H-\frac{3}{2}} du\right],$$

if $H < \frac{1}{2}$.

Malliavin calculus

- $\mathcal{S}$ is the space of random variables of the form

$$F = f(W(h_1), \dots, W(h_n)),$$

where $h_i \in \mathfrak{H} = L^2(\mathbb{R}_+)$, $W(h_i) = \int_0^\infty h_i(t)dW_t$ and $f \in C_b^\infty(\mathbb{R}^n)$.

- If $F \in \mathcal{S}$ we define its *derivative* by

$$D_s F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), \dots, W(h_n))h_i(s).$$

DF is a random variable with values in $\mathfrak{H}$.

- Sobolev spaces: For $p \geq 1$, $\mathbb{D}^{k,p} \subset L^p(\Omega)$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|DF\|_{k,p} = \sum_{j=0}^{k} \left(\mathbb{E}(\|D^j F\|^p_{\mathfrak{H}^{\otimes j}})\right)^{1/p}.$$

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Clark-Ocone formula

For any random variable $F \in \mathbb{D}^{1,2}$ we have

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dW_t,$$

where $\{\mathcal{F}_t, t \geq 0\}$ is the filtration generated by the Brownian motion W.

Stochastic integral representation of the local time:

Applying the Clarck-Ocone formula we can show that

$$L_t(\lambda) = \mathbb{E}[L_t(\lambda)] + \int_0^t \mathbb{E}[D_r L_t(\lambda) | \mathcal{F}_r] dW_r$$

$$= \int_0^t p_{s^{2H}}(\lambda) ds + \int_0^t \left(\int_r^t p'_{\mu_{r,s}}(B_{r,s} - \lambda) K_H(s,r) ds \right) dW_r,$$

where

$$B_{r,s} = \int_0^r K_H(s,\theta) dW_\theta, \qquad \mu_{r,s} = \int_r^s K_H^2(s,\theta) d\theta,$$

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Proof:

(i) One one hand we have

$$\mathbb{E}[L_t(\lambda)] = E\left[\int_0^t \delta(B_s^H - \lambda)ds\right] = \int_0^t p_{s^{2H}}(\lambda)ds.$$

(ii) On the other hand, for $r \leq t$,

$$\mathbb{E}[D_r L_t(\lambda)|\mathcal{F}_r] = \lim_{\epsilon \to 0} \mathbb{E}\left[D_r \int_0^t p_\epsilon(B_s^H - \lambda)ds \middle| \mathcal{F}_r \right]$$
$$= \lim_{\epsilon \to 0} \mathbb{E}\left[\int_r^t K_H(s,r)\mathbb{E}[p'_\epsilon(B_s^H - \lambda)|\mathcal{F}_r]ds \right],$$

because

$$D_r B_s^H = D_r \int_0^s K_H(s,\theta)dW_\theta = K_H(s,r)\mathbf{1}_{[0,s]}(r).$$

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because

$$D_r B_s^H = D_r \int_0^s K_H(s,\theta)dW_\theta = K_H(s,r)\mathbf{1}_{[0,s]}(r).$$

(iii) Making the decomposition, for $r \leq s$,

$$B_s^H = \int_0^r K_H(s,\theta)dW_\theta + \int_r^s K_H(s,\theta)dW_\theta,$$

and taking into account that $B_{r,s} = \int_0^r K_H(s,\theta)dW_\theta$ is $\mathcal{F}_r$-measurable and $\int_r^s K_H(s,\theta)dW_\theta$ is independent of $\mathcal{F}_r$, we obtain

$$\begin{aligned}\mathbb{E}[p'_\epsilon(B_s^H - \lambda)|\mathcal{F}_r] &= \mathbb{E}\left[p'_\epsilon \left(B_{r,s} + \int_r^s K_H(s,\theta)dW_\theta - \lambda \right) \middle| \mathcal{F}_r \right] \\ &= p'_{\epsilon+\mu_{r,s}}\left(B_{r,s} - \lambda\right),\end{aligned}$$

where $\mu_{r,s} = \int_r^s K_H^2(s,\theta)d\theta$.
Therefore, letting $\epsilon \to 0$,

$$\mathbb{E}[D_r L_t(\lambda)|\mathcal{F}_r] = \int_r^t K_H(s,r)p'_{\mu_{r,s}}\left(B_{r,s} - \lambda\right) ds.$$

Sketch of the proof of the theorem:

(i) We want to show that

$$Z_t^{(n)}(f) \xrightarrow{f.d.d.} \sqrt{C_{H,d}}\|f\|_{\frac{1}{H}-d}\widetilde{W}_{L_t(\lambda)},$$

where

$$Z_t^{(n)}(f) := n^{\frac{H+1}{2}}\left(\int_0^t f(n^H(B_s^H-\lambda))ds - n^{-H}L_t(\lambda)\int_{\mathbb{R}} f(x)dx\right).$$

(ii) Using the occupation formula, we can write

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(ii)i From the representation of the local time, we obtain

$$Z_t^{(n)}(f) = n^{\frac{1-H}{2}} \int_0^t G_{r,t}^{f,n} dW_r + n^{\frac{1-H}{2}} R_t^{(f,n)},$$

where

$$G_{r,t}^{f,n} = \int_{\mathbb{R}} \int_r^t f(x) \left(p'_{\mu_{r,s}}(B_{r,s} - \frac{x}{n^H} - \lambda) - p'_{\mu_{r,s}}(B_{r,s} - \lambda) \right) \times K_H(s,r) ds dx$$

and

$$R_t^{f,n} = \int_{\mathbb{R}} \int_0^t f(x) \left(p_{s^{2H}}(\frac{x}{n^H} + \lambda) - p_{s^{2H}}(\lambda) \right) ds dx.$$

(iv) We have

$$\lim_{n\to\infty} \sup_{0\le t\le T} n^{\frac{1-H}{2}} |R_t^{(f,n)}| = 0.$$

(v) To show the convergence

$$n^{\frac{1-H}{2}} \int_0^t G_{r,t}^{f,n} dW_r \xrightarrow{f.d.d.} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_t(\lambda)},$$

where $\widetilde{W}$ is a Brownian motion independent of B^H, we use martingale methods.

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(vi) If we fix $t > 0$, the process

$$M_u^{(n)} = n^{\frac{1-H}{2}} \int_0^u G_{r,t}^{f,n} dW_r, \qquad u \geq 0,$$

with the convention $G_{r,t}^{f,n} = 0$ if $r > t$, is a martingale, that satisfies

$$\langle M^{(n)} \rangle_u \xrightarrow{P} C_{H,d} \|f\|^2_{\frac{1}{H}-d} L_{t \wedge u}(\lambda) \tag{1}$$

and

$$\langle M^{(n)}, W \rangle_u \xrightarrow{P} 0, \tag{2}$$

uniformly in $u \in [0, T]$, for each fixed $T > 0$.

(vii) (1) and (2) imply:

$$M_u^{(n)} \xrightarrow{\mathcal{L}} \sqrt{C_{H,d}} \|f\|_{\frac{1}{H}-d} \widetilde{W}_{L_{t \wedge u}(\lambda)}, \tag{3}$$

where $\widetilde{W}$ is a Brownian motion independent of W.

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Proof that (1) *and* (2) *imply* (3):

(viii) Let $W^{(n)}$ be the Brownian motion such that $M_u^{(n)} = W^{(n)}_{\langle M^{(n)} \rangle_u}$, $u \in [0, T]$.

Then, an asymptotic version of Knight's theorem together with (1) and (2) imply

$$(W, W^{(n)}, \langle M^{(n)} \rangle) \xrightarrow{\mathcal{L}} (W, \widetilde{W}, \langle M^{(\infty)} \rangle),$$

where $\widetilde{W}$ is a Brownian motion independent of W and

$$\langle M^{(\infty)} \rangle_u = C_{H,d} \|f\|^2_{\frac{1}{H}-d} L_{t\wedge u}(\lambda).$$

This implies the following convergence in law for each $u \in [0, T]$:

$$M_u^{(n)} = W^{(n)}_{\langle M^{(n)} \rangle_u} \xrightarrow{\mathcal{L}} \widetilde{W}_{\langle M^{(\infty)} \rangle_u}.$$

Proof that (1) *and* (2) *imply* (3)*:*

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Case $H = \frac{1}{3}$

Theorem (Jaramillo-Nourdin-N.-Peccati '23)

Suppose $H = \frac{1}{3}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *satisfies* $\int_{\mathbb{R}} |f(y)|(1+|y|^2)dy < \infty$. *Then for any* $t > 0$ *and* $\lambda \in \mathbb{R}$ *we have*

$$(\log n)^{-\frac{1}{2}} n^{\frac{1+H}{2}} \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) \xrightarrow{f.d.d.} \sqrt{C_f}\widetilde{W}_{L_t(\lambda)},$$

as $n \to \infty$.

Case $H < \frac{1}{3}$

Theorem (Jaramillo-Nourdin-N.-Peccati '22)

Suppose $H < \frac{1}{3}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfies $\int_{\mathbb{R}} |f(y)|(1 + |y|^{\nu})dy < \infty$ for some $\nu > 1$. Then for any $t > 0$ and $\lambda \in \mathbb{R}$ we have

$$n^H \left(n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t(\lambda) \int_{\mathbb{R}} f(x)dx \right) \xrightarrow{L^2(\Omega)} L_t'(\lambda) \int_{\mathbb{R}} yf(y)dy,$$

as $n \to \infty$.

Sketch of the proof:

Set

$$\mathcal{D}_n := n^H \int_0^t f(n^H(B_s^H - \lambda))ds - L_t(\lambda) \int_{\mathbb{R}} f(x)dx - n^{-H} L_t'(\lambda) \int_{\mathbb{R}} yf(y)dy$$

We have

$$\lim_{n\to\infty} n^{2H} \mathbb{E}[|\mathcal{D}_n|^2] = 0,$$

which follows from

$$n^H \mathcal{D}_n = \int_{\mathbb{R}} f(y) n^H \left(L_t(\frac{y}{n^H} + \lambda) - L_t(\lambda) - \frac{y}{n^H} L_t'(\lambda) \right) dy.$$

Open problems:

- Tightness in the case $H \geq \frac{1}{3}$.
- d-dimensional fBm:
  (i) For $\frac{1}{d+2} \leq H < \frac{1}{d}$ we expect convergence in law.
  (ii) For $H < \frac{1}{d+2}$ we expect convergence in $L^2(\Omega)$ to some derivatives of the local time.

References

[1] A. Jaramillo, I. Nourdin, D. Nualart and G. Peccati: Limit theorems for additive functionals of the fractional Brownian motion. *Ann. Probab.* To appear.

[2] A. Jaramillo, I. Nourdin and G. Peccati: Approximation of local times: zero energy and weak derivatives. *Ann. Appl. Probab.* **31**, 2143-2191, 2021.

[3] Y. Hu, D. Nualart and F. Xu: Central limit theorem for an additive functional of the fractional Brownian motion. *Ann. Probab.* **42**, 168-203, 2014.

[4] D. Nualart and F. Xu: Central limit theorem for an additive functional of the fractional Brownian motion II. *Electron. Commun. Probab.* **18**, no. 74, 2013.

[5] G. C. Papanicolau, D. Stroock and S. R. S. Varadhan: *Martingale approach to some limit theorems.* Duke Univ. Math. Ser., Vol. III, 1977.

Thanks for your attention!