A fractional Ornstein-Uhlenbeck process and its time-changed version

Some results obtained with
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This talk is dedicated to the extraordinary person Yuliya is!

Outline

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The fractional Ornstein-Uhlenbeck process

It is the solution process of the following linear stochastic differential equation:

$$dV_t = -\frac{1}{\theta} V_t + \sigma dB_t^H, V_0 = \xi \tag{1}$$

in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
where $B^H = \{B_t^H, t \geq 0\}$ is a fractional Brownian motion (fBm) with Hurst parameter $H \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right)$,
and $\mathcal{F}_t^H, t \geq 0$ be its natural filtration.

Specifically, the fBm is the almost surely path-continuous centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with covariance function given by

$$\mathbb{E}[B_t^H B_s^H] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

At first, motivated by the wish to construct a new fractional neuronal model, we consider the following linear stochastic differential equation driven by our fBm:

$$dV_t = \left[-\frac{1}{\theta}(V_t - \widetilde{V}) + I_t \right] dt + \sigma dB_t^H, \;\; V_0 = \xi \qquad (2)$$

where ξ is a square-integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $I_t = I$ is a forcing term, $\theta > 0$ and $\widetilde{V}$ are constant.

The above equation is equivalent to the integral equation

$$V_t = \xi - \frac{1}{\theta}\int_0^t (V_s - \widetilde{V})ds + \int_0^t I_s ds + \sigma B_t^H. \tag{3}$$

We call the solution $V = V_t$ of equation (2) ffOU process with forcing term $I = I_t$ and $\widetilde{V}$ its resting term, that is to say a globally asymptotically stable equilibrium for the expected value as $I_t \equiv 0$.

Referring to Kaarakka T, Salminen P (2011), and in particular, following the lines of proposition A.1 of Cheridito et al. (2003) , we can explicitly determine the solution (denoting $\mathcal{F}_t^{H,\xi}$ the sigma-field generated by ξ and $B_s^H, s \leq t$).

Proposition

If I is a stochastic process such that its sample paths are (almost surely) integrable in any interval $[0, T]$ for $T > 0$, then equation (3) admits a unique solution whose paths are almost surely continuous. This solution can be expressed as

$$V_t = \widetilde{V} + e^{-\frac{t}{\theta}} \left(-\widetilde{V} + \xi + \int_0^t I_s e^{\frac{s}{\theta}} ds + \sigma \int_0^t e^{\frac{s}{\theta}} dB_s^H \right), \qquad (4)$$

where the integral is point-wise interpreted as a Riemann-Stieltjes integral ([Biagini et al, 2008]). In the case $H < 1/2$ it is defined via integration by parts, namely,

$$V_t = \widetilde{V} + e^{-\frac{t}{\theta}} \left(-\widetilde{V} + \xi + \int_0^t I_s e^{\frac{s}{\theta}} ds + \sigma \left(e^{\frac{t}{\theta}} B_t^H - \theta^{-1} \int_0^t e^{\frac{s}{\theta}} B_s^H ds \right) \right). \qquad (5)$$

Moreover, V_t is $\mathcal{F}_t^{H,\xi}$-adapted if I_t is a $\mathcal{F}_t^{H,\xi}$-adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$.

The mean value function

Recalling that

$$\mathbb{E}\left(\int_0^t e^{\frac{s}{\theta}}\, dB_s^H\right) = 0$$

for any $t > 0$, we immediately get the following result.

Proposition

Let I be a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $t > 0$, $I_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{E}[|I_t|] \in L^1([0, T], \lambda)$, where λ is the Lebesgue measure. Also, let $V = V_t$ be the solution of equation (2). Then

$$\mathbb{E}[V_t] = (1 - e^{-\frac{t}{\theta}})\widetilde{V} + e^{-\frac{t}{\theta}}\, \mathbb{E}[\xi] + e^{-\frac{t}{\theta}} \int_0^t e^{\frac{s}{\theta}}\, \mathbb{E}[I_s] ds. \qquad (6)$$

The covariance function

In order to exploit the results of [Cheridito et al., 2003], we introduce the stationary fractional Ornstein-Uhlenbeck process U^H with parameter $\frac{1}{\theta}$ (sfOU for short), as the only stationary Gaussian solution of equation (7)

$$dU_t^H = -\frac{1}{\theta} U_t^H dt + \sigma dB_t^H. \tag{7}$$

with the initial value

$$U_0^{H,\eta} = \eta$$

where η is the following square integrable random variable

$$\eta = \sigma \int_{-\infty}^{0} e^{\frac{s}{\theta}} dB_s^H$$

.

We finally refer to the process

$$U_t^{H,\eta} = e^{-\frac{t}{\theta}}\left(\eta + \sigma\int_0^t e^{\frac{s}{\theta}}\,dB_s^H\right) = \sigma\int_{-\infty}^t e^{-\frac{t-s}{\theta}}\,dB_s^H = U_t^H,\ \ t \geq 0$$

which is a stationary long-range dependent process for $H > \frac{1}{2}$, while is short-range dependent for $H < \frac{1}{2}$.

A closed form of its covariance function for $H \in (0,1)$ was obtained in [Pipiras and Taqqu, 2000], (see also [Cheridito et al., 2003]), Remark 2.4). It has a form

$$\rho(s) := \mathrm{Cov}(U_t^H, U_{t+s}^H) = \sigma^2\theta^2 C_H \int_{-\infty}^{+\infty} \frac{|x|^{1-2H}}{1+\theta^2x^2} e^{isx} dx, \tag{8}$$

where

$$C_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{9}$$

It is also shown in [Cheridito et al., 2003] that, for $s \to \infty$, the covariance function admits the following asymptotic expression for any $H \in (0,1) \setminus \{1/2\}$, $N = 1, 2, \dots$ and for fixed t:

$$\mathrm{Cov}(U_t^H, U_{t+s}^H) = \frac{1}{2}\sigma^2 \sum_{n=1}^{N} \theta^{2n} \left(\prod_{k=0}^{2n-1} (2H-k) \right) s^{2H-2n} + O(s^{2H-2N-2}).$$

In particular, for $N = 1$, we obtain

$$\mathrm{Cov}(U_t^H, U_{t+s}^H) = \sigma^2\theta^2 H(2H-1)s^{2H-2} + O(s^{2H-4}). \qquad (10)$$

For $\frac{1}{2} < H < 1$, we have $-1 < 2H - 2 < 0$, then

$$\mathrm{Cov}(U_t^H, U_{t+s}^H) \sim Ks^{2H-2}$$

as $s \to +\infty$, for $K = \sigma^2\theta^2 H(2H-1)$, saying, as usual, that $f \sim g$ if $\lim_{s\to+\infty} \frac{f(s)}{g(s)} = 1$.

Hence the long-range dependence of the process U^H for $H > \frac{1}{2}$ follows.

Conversely, for $0 < H < \frac{1}{2}$ we have $-2 < 2H - 2 < -1$ whence the short-range dependence follows.

Using these results and Theorem 2.3 in [Cheridito et al., 2003] we can get the covariance function in the following symmetric form.

Lemma

Let $U^{H,x} = U_t^{H,x}, t \geq 0$ be a fOU process solving equation (7) with initial value $x \in \mathbb{R}$. Then its covariance does not depend on $x \in \mathrm{R}$ and has a form

$$R_H(t,s) := \mathrm{Cov}(U_t^{H,x}, U_s^{H,x})$$

$$= \sigma^2\theta^2 C_H \int_{\mathrm{R}} \left(e^{isy} - e^{-\frac{s}{\theta}}\right)\left(\overline{e^{ity} - e^{-\frac{t}{\theta}}}\right)\frac{|y|^{1-2H}}{1+\theta^2 y^2}dy. \quad (11)$$

The proof is essentially based on

$$U_t^{H,x} - \mathbb{E}[U_t^{H,x}] = \sigma \int_0^t e^{-\frac{t-v}{\theta}} dB_v^H,$$

then, we have for $s \geq t$

$$\begin{aligned} R_H(t,s) &= \sigma^2 \mathbb{E}\left[\left(\int_0^t e^{-\frac{t-v}{\theta}} dB_v^H\right)\left(\int_0^s e^{-\frac{s-u}{\theta}} dB_u^H\right)\right] \\ &= \sigma^2 \left(\mathbb{E}\left[\left(\int_{-\infty}^t e^{-\frac{t-v}{\theta}} dB_v^H - \int_{-\infty}^0 e^{-\frac{t-v}{\theta}} dB_v^H\right)\right.\right. \\ &\quad \times \left.\left.\left(\int_{-\infty}^s e^{-\frac{s-u}{\theta}} dB_u^H - \int_{-\infty}^0 e^{-\frac{s-u}{\theta}} dB_u^H\right)\right]\right) \\ &= \rho(s-t) - e^{-\frac{t}{\theta}}\rho(s) - e^{-\frac{s}{\theta}}\rho(t) + e^{-\frac{t+s}{\theta}}\rho(0). \end{aligned} \tag{12}$$

Covariance function of ffOU

Now we proceed with the covariance function for ffOU process, i.e., fOU process with non-zero forcing term I. Denote

$$c(u, v) = \mathrm{Cov}(I_u, I_v)$$

From now on, we assume that the initial value $\xi = x \in \mathbb{R}$.

Lemma

With the notation above specified, let ξ be a degenerate random variable. Suppose I is a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that:

(i) *For any $t > 0$, $I_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.*

(ii) *For any $T > 0$, $t \mapsto \mathbb{E}[|I_t|]$ is integrable in $[0, T]$.*

(iii) *For any $(t, s) \in \mathbb{R}^2$ such that $t, s > 0$, $(u, v) \mapsto c(u, v)$ is integrable in $[0, t] \times [0, s]$.*

(iv) *For any $t, s > 0$ random variables I_t and B_s^H are uncorrelated.*

Then

$$\mathrm{Cov}(V_t, V_s) = e^{-\frac{t+s}{\theta}} \iint_{[0,t]\times[0,s]} e^{\frac{u+v}{\theta}} c(u, v) du dv + R_H(t, s). \tag{13}$$

Covariance as the function of time and Hurst index. Asymptotic behavior of covariance

Non-random forcing term and consider $R_H(t, t+s)$.

To start, note that in [Cheridito et al., 2003] the authors proved the following asymptotic expansion for fixed $t \geq 0$, $N = 1, 2, \ldots$ and $s \to +\infty$:

$$R_H(t, t+s) =$$
$$\frac{1}{2}\sigma^2 \sum_{n=1}^{N} \theta^{2n} \left(\prod_{k=0}^{2n-1} (2H - k) \right) \left[s^{2H-2n} - e^{-\frac{t}{\theta}} (t+s)^{2H-2n} \right]$$
$$+O(s^{2H-2N-2}),$$

which, for $N = 1$, becomes

$$R_H(t, t+s) = \sigma^2\theta^2 H(2H-1)(s^{2H-2} - e^{-\frac{t}{\theta}}(t+s)^{2H-2}) + O(s^{2H-4})$$

so that also

$$R_H(t, t+s) \sim K_t s^{2H-2}$$

with a constant

$$K_t = \sigma^2\theta^2 H(2H-1)(1-e^{-t})$$

For this reason we can conclude that $U_t^{H,x}$, as well as U_t^H, demonstrates a time non-homogeneous long-range dependence for $H > \frac{1}{2}$ and a time non-homogeneous short-range dependence for $H < \frac{1}{2}$.

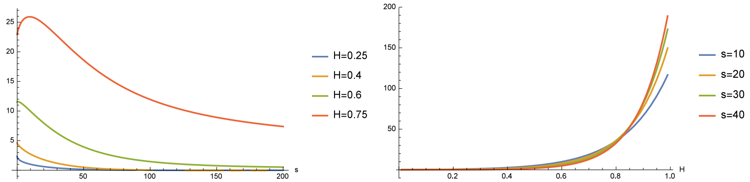

Figure: The covariance function $R_H(t, t+s)$ for $\theta = 30$, $\sigma = 1$, $t = 10$ for a non-random I: on the left as a function of s for different values of H; on the right as a function of H for different values of s.

Figure 1 on the left demonstrates that the tails of $s \mapsto R_H(t, t+s)$ as $s \to \infty$ depend on the Hurst parameter and they are slower in convergence to 0 as H grows.

Random Forcing Term

Now, let I be stochastic. Denote $C_V(t,s) = \mathrm{Cov}(V_t, V_s)$.

In general, it is not obvious that $C_V(t,s) \to 0$ as $s \to +\infty$.

The following lemma establishes some hypotheses under which $C_V(t,s) \to 0$ as $s \to +\infty$.

Lemma

Suppose the forcing term I verifies the hypotheses of previous Lemma. Then $\lim_{s\to+\infty} C_V(t,s) = 0$ under one of the following additional hypothesis:

- $\mathcal{L}1$ *The function $v \mapsto \int_0^t e^{\frac{u+v}{\theta}} c_I(u,v)du$ is in $L^1(\mathrm{R}^+)$;*
- $\mathcal{L}2$ *The following properties are verified:*
  - (*i*)
  $$\lim_{s\to+\infty} \int_0^s \int_0^t e^{\frac{u+v}{\theta}} c_I(u,v)dudv = \infty;$$
  - (*ii*)
  $$\lim_{s\to+\infty} \int_0^t e^{\frac{u}{\theta}} c_I(u,s)du = 0.$$

Moreover, property (ii) of hypothesis $\mathcal{L}2$ is assured if the following properties hold:

- (*iii*) *There exists $k \in L^1([0,t])$ such that $|c_I(u,s)| \leq k(u)$ for almost all $u \in [0,t]$ and $s \geq 0$.*
- (*iv*) $\lim_{s\to+\infty} c_I(u,s) = 0$.

Theorem

Suppose the forcing term I verifies the hypotheses of previous Lemma. Then, $C_V(t,s) \sim Ks^{2H-2}$ for some constant K under one of the following additional hypothesis: hypothesis $\mathcal{L}1$ from Lemma 3 or

$\mathcal{L}2'$ *The following properties are verified: property (i) from hypothesis $\mathcal{L}2$ from Lemma 3 and*

$(ii)'$

$$\lim_{s\to+\infty} s^{2-2H}\int_0^t e^{\frac{u}{\theta}} c_I(u,s)du = 0.$$

Moreover, property $(ii)'$ of hypothesis $\mathcal{L}2'$ is assured by the following properties:

$(iii)'$ *there exists $k \in L^1([0,t])$ such that $s^{2-2H}|c_I(u,s)| \le k(u)$ for almost all $u \in [0,t]$ and $s \ge 0$;*

$(iv)'$ $\lim_{s\to+\infty} s^{2-2H} c_I(u,s) = 0.$

By this theorem, we know that under hypothesis $\mathcal{L}1$ or $\mathcal{L}2'$,

$$C(t,s) \sim K_t s^{2H-2}$$

Hence, in such case V_t exhibits a time non-homogeneous long-range dependence for $H > \frac{1}{2}$ and a time non-homogeneous short-range dependence for $H < \frac{1}{2}$.

In particular, this behavior is induced only by the noise.

For $H > \frac{1}{2}$, if we replace hypothesis $(ii)'$ with

$(ii)''$

$$\lim_{s \to +\infty} s^{2-2H} \int_0^t e^{\frac{u}{\theta}} c(u,s) du = +\infty$$

we have again a time non-homogeneous long-range dependence, but this time

this behavior is induced by the covariance of the process I.

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this behavior is induced by the covariance of the process I.

For $H < \frac{1}{2}$ we cannot conclude the same assertion.

The fact that V preserves its correlation for long times when $H > \frac{1}{2}$, may be useful in the field of neuronal modeling when one wants to include memory effects.

Now, we study the time-changed fractional OU (TCfOU) process and some convergence results concerning its one-dimensional distribution.
In particular, here, we show results concerning that,

- despite the time change, the process admits a Gaussian limit random variable.
- We prove that the process converges towards the time-changed OU (TCOU) as the Hurst index $H \to 1/2^+$, with locally uniform convergence of one-dimensional distributions.
- We also achieve convergence in the Skorohod J_1-topology of the TCfOU process as $H \to 1/2^+$ in the space of cádlág functions.

The fractional Ornstein-Uhlenbeck process (again)

In order to put in evidence the dependence on H, we adopt a slightly different notation and we consider again the fractional OU (fOU) process as the solution of the fractional Brownian motion (fBm)-driven equation

$$dU_H(t) = -\frac{1}{\theta} U_H(t)dt + \sigma dB_H(t),$$

where $\theta, \sigma \in \mathbb{R}^+$ and $B_H = \{B_H(t), t \geq 0\}$ is a fBm with Hurst parameter $H \in (0, 1)$.
Considering $H > 1/2$, we focus on the case of its long-range dependence.

The time-change

In what follows we will consider subordinators, i.e.increasing and starting from zero (hence positive) Lévy processes (see [Bertoin, 1996]). Each subordinator admits a Laplace exponent $\Phi(\lambda)$, i.e. a function $\Phi : [0, +\infty) \to \mathbb{R}$ such that

$$\mathbb{E}[e^{-\lambda\sigma(t)}] = e^{-t\Phi(\lambda)}, \;\; t \geq 0, \;\; \lambda \geq 0.$$

In particular, such Laplace exponents Φ are Bernstein functions (see [Schilling, Song and Vondracek, 2012]) and therefore we can define the characteristic triplet of Φ, given by $(a_\Phi, b_\Phi, \nu_\Phi)$, where $a_\Phi, b_\Phi \geq 0$ are constants and ν_Φ is a Lévy measure on $(0, +\infty)$ such that

$$\int_0^{+\infty} (t \wedge 1)\nu_\Phi(dt) < +\infty.$$

Indeed, let us recall that any Bernstein function Φ can be represented in a unique way by means of the characteristic triple as

$$\Phi(\lambda) = a_\Phi + b_\Phi \lambda + \int_0^{+\infty} (1 - e^{-t\lambda}) \nu_\Phi(dt).$$

Vice versa, each Bernstein function Φ determines a unique (non-killed when $a_\Phi = 0$) subordinator $\sigma_\Phi(t)$ whose Laplace exponent is Φ.

Here we will consider Φ to be such that $a_\Phi = 0$ and if $b_\Phi = 0$ then we assume that $\nu_\Phi(0, +\infty) = +\infty$. Now let us define the inverse subordinator associated to Φ as

$$E_\Phi(t) := \inf\{y > 0 : \ \sigma_\Phi(y) > t\}.$$

It was established in [Meerschaert and Scheffler,, 2008] that our hypotheses on Φ are enough to guarantee that $E_\Phi(t)$ admits a probability density function $f_\Phi(s; t)$ for each $t > 0$.

A particular case is given by the choice $\Phi(\lambda) = \lambda^\alpha$ for $\alpha \in (0, 1)$. Indeed, in this case, we have the α-stable subordinator $\sigma_\alpha(t)$. According to [Meerschaert and Straka, 2013], $\sigma_\alpha(t)$ and $E_\alpha(t)$ are absolutely continuous random variables for any $t > 0$.

Let us also fix some number $\theta > 0$ and focus on the fractional Ornstein-Uhlenbeck process as

$$U_H(t) = e^{-\frac{t}{\theta}} \int_0^t e^{\frac{s}{\theta}} dB^H(s), \qquad t \geq 0, \tag{14}$$

which is a Gaussian process with one-dimensional density

$$p_H(t,x) = \frac{1}{\sqrt{2\pi V_{2,H}(t)}} e^{-\frac{x^2}{2V_{2,H}(t)}}, \ x \in \mathbb{R}, \ t \geq 0, \tag{15}$$

where

$$V_{2,H}(t) = e^{-2\frac{t}{\theta}} \int_0^t \int_0^t e^{\frac{v+u}{\theta}} |u-v|^{2H-2} du dv, \ t \geq 0$$

is the variance of $U_H(t)$. Let us also denote by $V_{n,H}(t) = \mathbb{E}[|U_H(t)|^n]$ the nth absolute moment of $U_H(t)$.

We construct the time-changed fractional Ornstein-Uhlenbeck process by considering a fractional Ornstein-Uhlenbeck process $U_H(t)$, together with an independent inverse subordinator $E_\Phi(t)$, and defining

$$U_{H,\Phi}(t) := U_H(E_\Phi(t)).$$

Let us state some properties of $V_{n,H,\Phi}(t) := \mathbb{E}[|U_{H,\Phi}(t)|^n]$.

Proposition

(i) $V_{2n,H,\Phi}(t)$ *is finite for any* $t > 0$ *and* $n \in \mathbb{N}$.

(ii) *It holds that*

$$V_{2n,H,\Phi}(t) = \int_0^{+\infty} V_{2n,H}(s) f_\Phi(s,t) ds.$$

(iii) $V_{2n,H,\Phi}(t)$ *is increasing in* t *for any* $n \in \mathbb{N}$ *and*

$$\lim_{t \to +\infty} V_{2n,H,\Phi}(t) = V_{2n,H}(\infty) = \frac{\left(2\theta^{2H} H\Gamma(2H)\right)^n \Gamma\left(\frac{2n+1}{2}\right)}{\sqrt{\pi}}.$$

Lemma

Function $V_{2,H}$ satisfies the relations

$$\lim_{t\to 0^+} \frac{V_{2,H}(t)}{t^{2H}} = 1 \text{ and } \lim_{t\to+\infty} V_{2,H}(t) = \theta^{2H} H\Gamma(2H).$$

Moreover, $V_{2,H} \in C^1(0,+\infty)$ and its derivative satisfies the relations

$$\lim_{t\to 0^+} \frac{V'_{2,H}(t)}{t^{2H-1}} = 2H \text{ and } \lim_{t\to+\infty} e^{\frac{t}{\theta}} t^{2-2H} V'_{2,H}(t) = 2H(2H-1)\theta.$$

We first focus on the one-dimensional convergence of $U_H(t)$ to the (classical) OU process $U(t)$, i.e.

$$U_H \to U \text{ (in distribution) as } H \to 1/2^+.$$

Recalling that $U_H(0) = 0$ and $\mathbb{E}[U_H(t)] = 0$, about the convergence of the variance and that of all the moments we have:

Proposition

For any $n \geq 1$ it holds that $\lim_{H \to \frac{1}{2}^+} V_{n,H}(t) = V_{n,\frac{1}{2}}(t)$ and the convergence is uniform in $[0, +\infty)$.

Now let us observe that, since $\mathbb{E}[U_H(t)] = 0$, $V_{2,H}(t) \to V_{2,\frac{1}{2}}(t)$, and $U_H(t)$ is a Gaussian random variable, we have that, for fixed $t > 0$,

$$U_H(t) \xrightarrow{d} U_{\frac{1}{2}}(t).$$

Thus we already have the weak one-dimensional convergence of the fractional Ornstein-Uhlenbeck process to the classical one as $H \to 1/2^+$. However, we can improve this kind of convergence, showing that the density $p_H(x,t)$ converges uniformly to $p_{1/2}(x,t)$.

Theorem

It holds that

$$\lim_{H \to \frac{1}{2}^+} p_H(t,x) = p_{\frac{1}{2}}(t,x)$$

for any $t \in (0,+\infty)$ *and for any* $x \in \mathbb{R}$.

The limit distribution of the TCfOU process

Now, let us explore the limit distribution of the TCfOU process as $t \to +\infty$.

It is well known that (for negative drift parameters) the fractional Ornstein-Uhlenbeck process is ergodic and admits Gaussian limit distribution with density function

$$p_H(\infty, x) = \frac{1}{\sqrt{2\pi V_H}} e^{-\frac{x^2}{2V_H}}$$

where $V_H := V_{2,H}(\infty)$.

By using this result we can exploit the limit distribution of the TCfOU process, i.e.

$$TCfOU \to fOU$$

for the time-limit distribution.

Proposition

Let $p_{H,\Phi}(t,x)$ be the probability density function of $U_{H,\Phi}(t)$. Then it holds that

$$\lim_{t \to +\infty} p_{H,\Phi}(t,x) = p_H(\infty, x). \tag{16}$$

Moreover, $U_{H,\Phi}(t) \xrightarrow{d} Z$ as $t \to +\infty$, where $Z \sim \mathcal{N}(0, V_H)$.

Let us observe that if we are able to establish equality (16), then the weak convergence result directly follows.

The proof relies on the following facts:

- $$p_{H,\Phi}(t,x) = \int_0^{+\infty} p_H(s,x) f_\Phi(s;t) ds = \mathbb{E}[p_H(E_\Phi(t),x)], \tag{17}$$
  where p_H is defined by equality (15),
- $$p_H(E_\Phi(t),x) \leq \frac{1}{\sqrt{2\pi E_\Phi(1)}}. \tag{18}$$
  valid for $t > 1$ being $E_\Phi(t)$ a.s. non-decreasing,
- $\mathbb{E}\left[\frac{1}{\sqrt{V_{2,H}(E_\Phi(1))}}\right] < +\infty$
- the use dominated convergence theorem to conclude that limit relation (16) holds, as $E_\Phi(t) \to +\infty$ as $t \to +\infty$ almost surely (being $a_\Phi = 0$ by assumption).

This result was expected. Indeed, the action of the time-change consists in change of the time scale that generally does not affect the limit distributions.

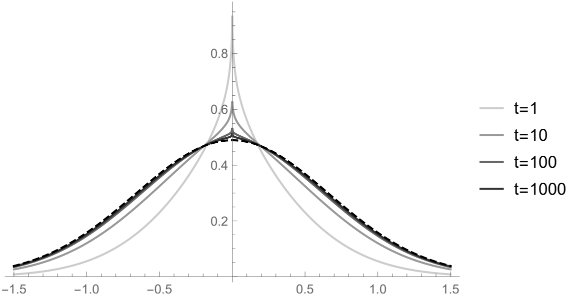

Figure: The density function $p_{H,\Phi}(t,x)$ for $\Phi(\lambda) = \sqrt{\lambda}$, $H = 3/4$, $\theta = 1$ and different values of t. The dashed line represents the limit distribution.

Weak one-dimensional convergence of the TCfOU process, $TCfOU \to TCOU$

Now we want to discuss the one-dimensional convergence of the TCfOU process to the time-changed Ornstein-Uhlenbeck process, i.e.

$$U_{H,\Phi} \to U_{\Phi}.$$

To do this, let us actually demonstrate some more strong convergence of the probability density function.

Theorem

Let $p_{H,\Phi}(t,x)$ *be the probability density function of* $U_{H,\Phi}(t)$ *and* $p_{\frac{1}{2},\Phi}(t,x)$ *the probability density function of* $U_{\Phi}(t)$*. Then it holds*

$$\lim_{H\to\frac{1}{2}^+} p_{H,\Phi}(t,x) = p_{\frac{1}{2},\Phi}(t,x) \tag{19}$$

for any $t>0$ *and* $x\in\mathbb{R}$*. Moreover, for any compact set* $K\subset\mathbb{R}\setminus\{0\}$ *it holds* $p_{H,\Phi}\to p_{\frac{1}{2},\Phi}$ *uniformly in* $[0,+\infty)\times K$ *as* $H\to\frac{1}{2}^+$*.*

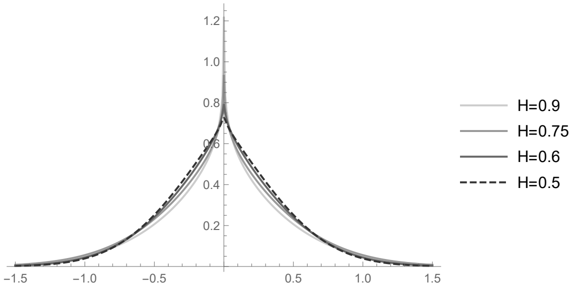

Figure: The density function $p_{H,\Phi}(t,x)$ for $\Phi(\lambda)=\sqrt{\lambda}$, $t=1$, $\theta=1$ and different values of H. The dashed line represents the case $H=1/2$.

As for the fractional Ornstein-Uhlenbeck process, we can establish also the uniform convergence of the absolute moments.

Proposition

It holds that $V_{n,H,\Phi} \to V_{n,\frac{1}{2},\Phi}$ *as* $H \to \frac{1}{2}^{+}$ *uniformly in* $[0,+\infty)$.

A functional limit theorem for the TCfOU process

Let us consider the function space $C = C(0, +\infty)$ equipped with the uniform norm. The processes
$U_{H,\Phi} = \{U_{H,\Phi}(t),\ t \geq 0\}$ (TCfOU),
$U_{\Phi} = \{U_{\Phi}(t),\ t \geq 0\}$, (TCOU)
$U_H = \{U_H(t),\ t \geq 0\}$ (fOU),
$U = \{U(t),\ t \geq 0\}$(OU),
can be seen as C-valued random variables and then we can study the weak convergence in C of such random variables.

Let us denote by C^* the dual space of C (i.e. the space of continuous linear functionals on C).
For a sequence of C-valued random variables $(X_n)_{n\in\mathbb{N}}$ and a C-valued random variable $X \in C$, it holds

$$X_n \Rightarrow X$$

if and only if for any $\mathcal{F} \in C^*$ it holds $\mathbb{E}[\mathcal{F}(X_n)] \to \mathbb{E}[\mathcal{F}(X)]$ as $n \to +\infty$.

In particular, for any $\gamma \in (0,1]$, let us denote by $\mathrm{Lip}_\gamma([0,T])$ the space of Hölder-continuous functions on $[0,T]$ with exponent γ, which is a Banach space when equipped with the norm

$$\|f\|_{\mathrm{Lip}_\gamma([0,T])} = \sup_{\substack{(t,s)\in[0,T]^2 \\ t\neq s}} \frac{|f(t)-f(s)|}{|t-s|^\gamma} + \|f\|_{L^\infty(0,T)} \cdot$$

At first, let us focus on U_H and U.

Theorem

It holds $U_H \Rightarrow U$ in C as $H \to \frac{1}{2}^+$. Moreover, for any $T > 0$ and any linear functional $\mathcal{F} \in \mathrm{Lip}_\gamma{}^*([0, T])$ for $\gamma < 1/2$ it holds $\mathbb{E}[\mathcal{F}(U_H)] \to \mathbb{E}[\mathcal{F}(U)]$.

Remark

The bound $\gamma < 1/2$ is sharp since U belongs to $\mathrm{Lip}_\gamma([0,T])$ for $\gamma < 1/2$ but not for $\gamma = 1/2$.

Now we need to provide a similar result for

$$U_{H,\Phi} \text{ and } U_\Phi.$$

With this purpose, let us consider the set D of cadlag functions on $[0,+\infty)$ and Λ the set of strictly increasing functions $g:[0,+\infty)\to[0,+\infty)$. Let $\iota \in \Lambda$ be the identity function on $[0,+\infty)$.
The Skorohod J_1 metric on D is, for any couple of functions $f_1, f_2 \in D$,

$$d_{J_1}(f_1,f_2) = \inf_{g\in\Lambda} \max\{\|f_1\circ g - f_2\|_{L^\infty(0,+\infty)}, \|g-\iota\|_{L^\infty(0,+\infty)}\}.$$

With this metric, the set (D, d_{J_1}) is a metric space (that we will denote only as D), thus we can consider the notion of weak convergence of D-valued random variables, which is actually weaker than the weak convergence in C.

With this in mind, let us show the following result that relies on a continuous mapping technique which is common in such kind of functional limit theorems (see, for instance, [Meerschaert and Scheffler, 2008, Scalas and Viles, 2014]).

Theorem

It holds $U_{H,\Phi} \Rightarrow U_\Phi$ in D as $H \to \frac{1}{2}^+$.

Proof.

First, note that C is a closed subspace of D, hence D^* is contained in C^*.

From $U_H \Rightarrow U$ in C we have the same convergence in D.

Consider the coupled processes (U_H, E_Φ) and (U, E_Φ).

Since E_Φ is independent of both U_H and U, we get $(U_H, E_\Phi) \Rightarrow (U, E_\Phi)$ in $D \times D$.

We remark that $U \in C$ almost surely and $E_\Phi \in D_\uparrow$ almost surely, where $D_\uparrow = \{f \in D : \ f \text{ is increasing}\}$.

Now let us denote by g the composition map on $D \times D$, i.e. $g(x, y) = x \circ y \in D$, and with $\mathrm{Disc}(g)$ the set of discontinuity points of g.

Proof (Cont.)

It follows from [Whitt,2002] that

$$\mathrm{Disc}(g) \subseteq (D \times D) \setminus ((C \times D_{\uparrow}) \cup (D \times C_{\uparrow\uparrow})),$$

where $D_{\uparrow\uparrow} = \{f \in D : \ f \text{ is strictly increasing}\}$ and $C_{\uparrow\uparrow} = C \cap D_{\uparrow\uparrow}$. Taking into account the inclusion $(U, E_\Phi) \in C \times D_{\uparrow}$ almost surely, we get that

$$\mathbb{P}((U, E_\Phi) \in \mathrm{Disc}(g)) = 0.$$

Thus we can use the continuous mapping theorem to obtain the desired convergence. □

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