Parameter estimation in mixed fractional models

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International Workshop
"Stochastic processes with statistical applications and fractional stochastic calculus"

May 17, 2023
Kyiv, Ukraine

Mixed fractional Brownian motion with trend

We consider the following mixed fractional Brownian motion with trend

$$X_t = \theta t + \sigma W_t + \kappa B_t^H, \qquad t \in [0, T], \tag{1}$$

where W is a Wiener process, B^H is a fractional Brownian motion with Hurst index $H \in (0, 1)$, B^H is independent of W.
Our goal is to estimate all four parameters H, θ, κ and σ. We apply two approaches:

- "classical" approach
- "ergodic" approach

Exact MLE: limitations

Let the observation vector be

$$\mathrm{X} = \left(X_h, X_{2h}, \dots X_{Nh}\right)^\top, \qquad h > 0.$$

Obviously, X has N-dimensional Gaussian distribution with pdf

$$f(\mathrm{X}, \theta, H, \sigma^2, \kappa^2) = (2\pi)^{-N/2} \left(\det \Gamma\right)^{-1/2} \exp\left\{-\frac{1}{2}(\mathrm{X} - \theta \mathrm{t})^\top \Gamma^{-1} (\mathrm{X} - \theta \mathrm{t})\right\},$$

where $\mathrm{t} = (h, 2h, \dots, Nh)^\top$, Γ is the covariance matrix of X,

$$\Gamma_{ij} = \operatorname{cov}\left(X_{ih}, X_{jh}\right) = \sigma^2 h \min(i, j) + \frac{\kappa^2 h^{2H}}{2}\left(i^{2H} + j^{2H} - |i - j|^{2H}\right).$$

Then, MLE of $(\theta, H, \sigma^2, \kappa^2)$ can be obtained by maximization of $f(\mathrm{X}, \theta, H, \sigma^2, \kappa^2)$ w.r.t. $(\theta, H, \sigma^2, \kappa^2)$.

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- **'Hybrid approach' to maximization of likelihood:**

Dufitinema J, Pynnönen S, Sottinen T (2022). Maximum likelihood estimators from discrete data modeled by mixed fractional Brownian motion with application to the Nordic stock markets. *Comm. Statist. Simulation Comput.*, **51**(9), 5264–5287.

Estimation of θ when H is known

Let $H \in (0,1)$ be known. Assume that a trajectory of X is observed at the points $t_k^n = \frac{k}{2^n}$, $k = 0, 1, \dots, 2^{2n}$. Consider the estimator

$$\hat{\theta}_n := \frac{\sum_{k=1}^{2^{2n}-1} (t_k^n)^{\frac{1}{2}-H} (2^n - t_k^n)^{\frac{1}{2}-H} \left(X_{t_k^n} - X_{t_{k-1}^n} \right)}{\mathrm{B}(\frac{3}{2} - H, \frac{3}{2} - H) 2^{n(2-2H)}}, \tag{2}$$

Theorem 1

Let $H \in (0,1)$. The estimator $\hat{\theta}_n$ is a strongly consistent estimator of θ as $n \to \infty$. Moreover,

$$\hat{\theta}_n - \theta = O_\omega \left(n^\alpha 2^{-n[H \wedge (1-H)]} \right),$$

where α can be taken arbitrarily small.

Theorem 2

The estimator $\hat{\theta}_n$ is asymptotically normal: for $H < \frac{1}{2}$

$$2^{n/2}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \varphi(H)\sigma^2\right) \quad \text{as } n \to \infty,$$

and for $H > \frac{1}{2}$

$$2^{n(1-H)}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \psi(H)\kappa^2\right) \quad \text{as } n \to \infty.$$

where

$$\varphi(H) = \frac{\mathrm{B}(2-2H, 2-2H)}{\mathrm{B}^2(\frac{3}{2}-H, \frac{3}{2}-H)}, \quad H \in (0, \tfrac{1}{2}), \tag{3}$$

$$\psi(H) = \frac{H(2H-1)\mathrm{B}(H-\frac{1}{2}, \frac{3}{2}-H)}{\mathrm{B}(\frac{3}{2}-H, \frac{3}{2}-H)}, \quad H \in (\tfrac{1}{2}, 1). \tag{4}$$

Estimation of H, κ and σ via quadratic variations

Similarly to [Dozzi et al(2015)][1], we construct estimators using the quadratic variation of the form

$$V_n(X) := \sum_{i=0}^{n-1} (\Delta_i^n X)^2 , \quad \text{with} \quad \Delta_i^n X := X_{\frac{i+1}{n}} - X_{\frac{i}{n}} .$$

Let $n \geq 1$. Assume that a trajectory of the process X is observed at points $t_k^n = \frac{k}{2^n}$, $k = 0, \dots, 2^n$. We introduce the next statistic

$$\widehat{H}(k) = \frac{1}{2} \left(\log_{2+} \frac{V_{2^{k-2}}(X) - V_{2^{k-1}}(X)}{V_{2^{k-1}}(X) - V_{2^k}(X)} + 1 \right), \tag{5}$$

with

$$\log_{2+} x := \begin{cases} \log_2 x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

[1] Dozzi, M., Mishura, Yu., Shevchenko, G. (2015). Asymptotic behavior of mixed power variations and statistical estimation in mixed models. *Stat. Inference Stoch. Process*, *18*(2), 151—175.

Proposition 1

1. For $H \in (0, 1/2)$, *the statistic* (5) *is a strongly consistent estimator of* H, *moreover, for any* $\varepsilon > 0$,

$$\widehat{H}(k) = H + o_\omega\left(2^{(-1/2+\varepsilon)k}\right) \text{ as } k \to \infty. \tag{6}$$

2. For $H \in (1/2, 3/4)$ *the statistic* (5) *is a strongly consistent estimator of* H, *moreover, for any* $\varepsilon > 0$,

$$\widehat{H}(k) = H + o_\omega\left(2^{(2H-3/2+\varepsilon)k}\right) \text{ as } k \to \infty. \tag{7}$$

The quadratic variations also may be used for estimation of both κ and σ. The strong consistency of the following estimators can be established similarly to the corresponding results from [Dozzi et al(2015)].

Proposition 2 (Estimation of κ^2)

For $H \in (0, 1/2) \cup (1/2, 3/4)$, the statistic

$$\widetilde{\kappa}_k^2 := \frac{2^{k\left(2\widehat{H}(k)-1\right)}\left(V_{2^{k-1}}(X) - V_{2^k}(X)\right)}{2^{2\widehat{H}(k)-1} - 1}$$

is a strongly consistent estimator of κ^2.

Proposition 3 (Estimation of σ^2)

1. For $H \in (1/4, 1/2)$, the statistic

$$\widetilde{\sigma}_k^2 := \frac{2^{1-2\widehat{H}(k)} V_{2^{k-1}}(X) - V_{2^k}(X)}{2^{1-2\widehat{H}(k)} - 1}$$

is a strongly consistent estimator of σ^2.
2. For $H \in (1/2, 1)$, the statistic

$$\widehat{\sigma}_k^2 := V_{2^k}(X)$$

is a strongly consistent estimator of σ^2.

Estimation of θ when H is unknown

If H is unknown, we start with an auxiliary result, which gives an upper bound for the difference between the estimator $\hat{\theta}_n$ and the same estimator with some number $h \in (0,1)$ in place of the true value of H.

Lemma 3

Let X be a mixed fractional Brownian motion with trend, defined by (1) *with Hurst index $H \in (0,1)$. Define*

$$\tilde{\theta}_n(h) = \frac{\sum_{k=1}^{2^{2n}-1} (t_k^n)^{\frac{1}{2}-h} (2^n - t_k^n)^{\frac{1}{2}-h} \left(X_{t_k^n} - X_{t_{k-1}^n} \right)}{\mathrm{B}(\frac{3}{2}-h, \frac{3}{2}-h) 2^{n(2-2h)}}, \quad h \in [0,1].$$

Then

$$\tilde{\theta}_n(h) - \hat{\theta}_n = O_\omega(|h-H|), \tag{8}$$

for all $h \in (0, H^*)$, where $H^* \in (H,1)$ is any number, and for all $n \geq 1$.

Now, we want to replace H in the expression for $\hat{\theta}_n$ in (2) by the estimator (5).

Theorem 4

For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$, $\tilde{\theta}_n\left(\widehat{H}(n)\right)$ is a strongly consistent estimator of θ as $n \to \infty$.
If $H \in (0, 1/2)$, then

$$\tilde{\theta}_n\left(\widehat{H}(n)\right) - \theta = o_\omega\left(2^{-(H-\varepsilon)n}\right) \quad \text{a.s. as } n \to \infty,$$

for any $\varepsilon > 0$.
If $H \in (1/2, 3/4)$, then

$$\tilde{\theta}_n\left(\widehat{H}(n)\right) - \theta = o_\omega\left(2^{-(\frac{3}{2}-2H-\varepsilon)n}\right) \quad \text{a.s. as } n \to \infty,$$

for any $\varepsilon > 0$.

Alternative approach to parameter estimation based on ergodic theorem

Let $h > 0$ be fixed. Assume that the process X is observed at points $t_k = kh$, $k = 0, 1, 2, \dots$ The estimation procedure will be based on the following ergodic result.

Lemma 5

The process

$$Y_k = X_{(k+1)h} - X_{kh} - \theta h, \quad k = 0, 1, \dots,$$

is ergodic.

The previous lemma allows us to apply the ergodic theorem for construction of estimators. Namely, if $g\colon \mathbb{R}^{l+1} \to \mathbb{R}$ is a Borel function such that $\mathbb{E}\left|g\left(Y_0, Y_h, \dots, Y_{lh}\right)\right| < \infty$, then

$$\frac{1}{N}\sum_{k=0}^{N-1} g\left(Y_{kh}, \dots, Y_{(k+l)h}\right) \to \mathbb{E}g\left(Y_0, \dots, Y_{lh}\right) \text{ a.s. as } N \to \infty. \tag{9}$$

The main idea is to obtain four different convergences by choosing different functions g, and then to construct the estimators by solving the corresponding system of four equations.

To this end, let us introduce the notation:

$$\xi_N =: \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+1)h} - X_{kh}\right)^2,$$

$$\eta_N =: \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+1)h} - X_{kh}\right) \left(X_{(k+2)h} - X_{(k+1)h}\right),$$

$$\zeta_N =: \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+2)h} - X_{kh}\right) \left(X_{(k+4)h} - X_{(k+2)h}\right).$$

Strong consistency

Theorem 6

Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. *The statistics*

$$\check{\theta}_N = \frac{X_{Nh}}{Nh},$$

$$\check{H}_N = \frac{1}{2} \log_{2+} \frac{\zeta_N - 4\check{\theta}_N^2 h^2}{\eta_N - \check{\theta}_N^2 h^2},$$

$$\check{\kappa}_N^2 = \frac{\eta_N - \check{\theta}_N^2 h^2}{h^{2\check{H}_N} \left(2^{2\check{H}_N - 1} - 1\right)},$$

$$\check{\sigma}_N^2 = \frac{\xi_N - \check{\theta}_N^2 h^2 - \check{\kappa}_N^2 h^{2\check{H}_N}}{h}$$

are strongly consistent estimators of parameters $\theta, H, \kappa^2, \sigma^2$ *respectively.*

Normality of $\check{\theta}_N$

Theorem 7

The estimator $\check{\theta}_N$ is normal, and for $H < \frac{1}{2}$

$$(Nh)^{1/2}\left(\check{\theta}_N - \theta\right) \stackrel{d}{=} \mathcal{N}\left(0, \sigma^2\right),$$

and for $H > \frac{1}{2}$

$$(Nh)^{1-H}\left(\check{\theta}_N - \theta\right) \stackrel{d}{=} \mathcal{N}\left(0, \kappa^2\right).$$

Comparison of $\hat{\theta}_n$ and $\check{\theta}_N$

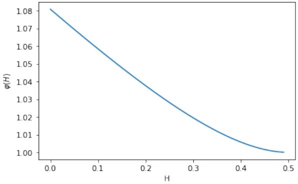

Figure: Plot of $\varphi(H)$.

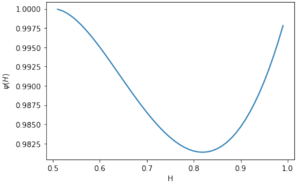

Figure: Plot of $\psi(H)$.

Remark 1

Estimators $\hat{\theta}_n$ and $\check{\theta}_N$ can be compared in terms of relative efficiency. Ratios of respective variances of their (asymptotic) distributions depend on Hurst index H and are defined in (3) and (4) as $\varphi(H), H \in \left(0, \frac{1}{2}\right)$, and $\psi(H), H \in \left(\frac{1}{2}, 1\right)$. Estimator $\check{\theta}_N$ is relatively more efficient when $H \in \left(0, \frac{1}{2}\right)$. Estimator $\hat{\theta}_n$ is relatively more efficient when $H \in \left(\frac{1}{2}, 1\right)$.

Asymptotic normality of vector $(\phi_N, \xi_N, \eta_N, \zeta_N)$

Let us denote $\phi_N = \frac{X_{Nh}}{N}$,

$$\tau_N = (\phi_N, \xi_N, \eta_N, \zeta_N), \qquad \tau_0 = (\mathbb{E}\phi_N, \mathbb{E}\xi_N, \mathbb{E}\eta_N, \mathbb{E}\zeta_N). \tag{10}$$

$$\tilde{\rho}(i) = \operatorname{cov}(\Delta X_0, \Delta X_i) = \sigma^2 h \mathbb{1}_{\{i=0\}} + \kappa^2 h^{2H} \rho(i), \quad i \in \mathbb{Z}, \tag{11}$$

where $\rho(i) := \frac{1}{2}\left(|i+1|^{2H} - 2|i|^{2H} + |i-1|^{2H}\right)$ denotes the autocovariance function of the stationary sequence $\{B^H_{k+1} - B^H_k, k \geq 0\}$, which is known as a fractional Gaussian noise.

Theorem 8

Let $H \in (0, \frac{1}{2})$. The vector τ_N defined by (10) is asymptotically normal, namely

$$\sqrt{N}(\tau_N - \tau_0) = \sqrt{N} \begin{pmatrix} \phi_N - \mathbb{E}\phi_N \\ \xi_N - \mathbb{E}\xi_N \\ \eta_N - \mathbb{E}\eta_N \\ \zeta_N - \mathbb{E}\zeta_N \end{pmatrix} \xrightarrow{d} \mathcal{N}(\vec{0}, \widetilde{\Sigma})$$

The elements of the asymptotic covariance matrix $\widetilde{\Sigma}$ can be presented explicitly:

$$\widetilde{\Sigma}_{11} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i), \quad \widetilde{\Sigma}_{22} = 2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)^2 + 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{33} = \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + \tilde{\rho}(i+2)\Big) + 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{44} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(6\tilde{\rho}(i) + 8\tilde{\rho}(i+1) + 3\tilde{\rho}(i+2) + 4\tilde{\rho}(i+3) + 6\tilde{\rho}(i+4) + 4\tilde{\rho}(i+5) + \tilde{\rho}(i+6)\Big) + 64\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{23} = 2\sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\tilde{\rho}(i+1) + 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{24} = 2\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + \tilde{\rho}(i+3)\Big) + 16\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{34} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + 2\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + 2\tilde{\rho}(i+3) + \tilde{\rho}(i+4)\Big)$$
$$+ 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{12} = \widetilde{\Sigma}_{13} = 2\theta h \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i), \quad \widetilde{\Sigma}_{14} = 8\theta h \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i).$$

Asymptotic normality of vector $(\check{\theta}_N, \check{H}_N, \check{\kappa}_N^2, \check{\sigma}_N^2)$

Let us introduce the notation $\vartheta = (\theta, H, \kappa^2, \sigma^2)$, $\check{\vartheta}_N = (\check{\theta}_N, \check{H}_N, \check{\kappa}_N^2, \check{\sigma}_N^2)$.

Theorem 9

Let $H \in (0, \frac{1}{2})$. The estimator $\check{\vartheta}_N$ is asymptotically normal, namely

$$\sqrt{N}\left(\check{\vartheta}_N - \vartheta\right) \xrightarrow{d} \mathcal{N}\left(\vec{0}, \Sigma^0\right)$$

with $\Sigma^0 = g'(\tau_0)\widetilde{\Sigma}\left(g'(\tau_0)\right)^\top$, where $\widetilde{\Sigma}$ is defined in Theorem 8 and

$$g'(\tau_0) = \begin{pmatrix} \frac{1}{h} & 0 & 0 & 0 \\ \frac{2\theta(l-2)}{k^2 h^{2H-1} l(l+2)} & 0 & \frac{-1}{\kappa^2 h^{2H} l \log 2} & \frac{1}{\kappa^2 h^{2H} l(l+2)\log 2} \\ \frac{-4\theta}{h^{2H-1}} \frac{2(l+2)(l-1)+cl(l-2)}{l^2(l+2)} & 0 & \frac{2}{h^{2H}} \frac{(2+c)l+2}{l^2} & \frac{-2}{h^{2H}} \frac{l(c+1)+2}{l^2(l+2)} \\ \frac{4\theta(l^2+4l-4)}{l^2} & \frac{1}{h} & -\frac{4(l+1)}{hl^2} & \frac{2}{hl^2} \end{pmatrix}.$$

where $c = \log_2 h$ and $l = 2^{2H} - 2$.

A. Kukush, Y. Mishura, S. Lohvinenko, and K. Ralchenko, *Two approaches to consistent estimation of parameters of mixed fractional Brownian motion with trend*, Stat. Inference Stoch. Process. **25** (2022), no. 1, 159–187.

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