LSE Consistency of the Symmetric Textured Surface Parameters

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International Workshop "Statistics of Stochastic Processes in Discrete and Continuous Time", October 11-12, 2022

Kyiv, Ukraine

1. Setting of the problem.

This lecture presents our joint result with my PhD student O.V. Dykyi. We consider here a trigonometric regression model

$$X(t) = g(t,\theta^0)+\varepsilon(t), \;\; t = (t_1, \dots, t_M) \in \mathbb{R}_+^M = [0,\infty)^M, \; M \geqslant 3, \tag{1.1}$$

where

$$g(t,\theta^0) = \sum_{k=1}^{N} \left(A_k^0 \cos\left(\sum_{l=1}^{M} \varphi_{lk}^0 t_l\right) + B_k^0 \sin\left(\sum_{l=1}^{M} \varphi_{lk}^0 t_l\right)\right), \tag{1.2}$$

$$\begin{aligned} \theta^0 &= \left(\theta_1^0, \theta_2^0, \dots, \theta_{M+2}^0, \dots, \theta_{(M+2)(N-1)+1}^0, \dots, \theta_{(M+2)(N-1)+M+1}^0, \theta_{(M+2)N}^0\right) \\ &= \Bigg(\underbrace{A_1^0, B_1^0, \varphi_{11}^0, \dots, \varphi_{M1}^0}_{M+2 \;\; parameters}, \dots, \underbrace{A_N^0, B_N^0, \varphi_{1N}^0, \dots, \varphi_{MN}^0}_{M+2 \;\; parameters}\Bigg) \end{aligned} \tag{1.3}$$

$$\left(A_k^0\right)^2 + \left(B_k^0\right)^2 > 0, \;\; k \in \overline{1,N},$$

is the vector of true values of unknown parameters;
$\varepsilon = \left\{\varepsilon(t), \;\; t \in \mathbb{R}^M\right\}$ is a random field defined on a complete probability space $(\Omega, \mathfrak{F}, P)$.

A. ε is a sample continuous homogeneous Gaussian field with zero mean and c.f. $B(t) = \mathsf{E}\,\varepsilon(t)\varepsilon(0)$, $t \in \mathbb{R}^M$, satisfying one of the conditions:
(**i**) ε is isotropic field also and

$$B(t) = \tilde{B}(\|t\|) = L(\|t\|)\|t\|^{-\alpha}, \ \alpha \in \left(0, M - \left[\frac{M}{2}\right]\right),$$

with nondecreasing slowly varying at infinity function L;
(**ii**) $B(\cdot) \in L_1(\mathbb{R}^M)$.

In the case $M = 2$ a lot of discrete modifications of such a regression model were studied in numerous works on signal and image processing due to the applications to texture analysis. A number of useful references to publications in this area can be found in the recent paper

Ivanov A.V., Savych I.M. On the LSE asymptotic normality of the multivariate symmetric textured surface parameters, Theor. Probability and Math. Statist., **105**(2021), p. 151 – 169.

Moreover in this paper asymptotic normality of the consistent LSE θ_T of the parameter (1.3) in the model (1.1) is obtained under condition **A** with $\alpha \in (0, M)$ and some additional assumption on the field ε spectral density.

Consider the space $\mathbb{R}^{MN} = \underbrace{\mathbb{R}^N \times ... \times \mathbb{R}^N}_{M \text{ times}}$, and in each space $\mathbb{R}^N$ for some fixed numbers $0 \leqslant \underline{\varphi}_l < \overline{\varphi}_l < \infty$ we define the sets

$$(1.4) \qquad \Lambda_l = \left\{ \varphi_l = (\varphi_{l1}, ..., \varphi_{lN}) \in \mathbb{R}^N : \ 0 \leqslant \underline{\varphi}_l < \varphi_{lk} < \overline{\varphi}_l < \infty, \ k = \overline{1, N} \right\}, \ l = \overline{1, M},$$

containing all the corresponding true values of frequencies in (1.3) for fixed l.

Introduce the functional

$$(1.5) \qquad Q_T(\theta) = T^{-M} \int\limits_{[0,T]^M} [X(t) - g(t, \theta)]^2 \, dt, \qquad dt = \prod_{l=1}^{M} dt_l.$$

According to standard definition a random vector

$$\theta_T = (A_{1T}, B_{1T}, \varphi_{11,T}, ..., \varphi_{M1,T}, ..., A_{NT}, B_{NT}, \varphi_{1N,T}, ..., \varphi_{MN,T}) \tag{1.6}$$

is called LSE of θ^0, if it minimizes (1.5) on the parametric set $\Theta^c \subset \mathbb{R}^{(M+2)N}$. In Θ^c amplitudes A_k, B_k, $k = \overline{1, N}$, can take any values and the frequencies φ_l, $l = \overline{1, M}$, take values in the closed set

$$\Lambda^c = \mathop{\times}_{l=1}^{M} \Lambda_l^c.$$

To prove strong consistency of the θ_T it is necessary to provide convergence, as $T \to \infty$, to zero a.s. of the fractions

$$\frac{\sin\ T\left(\varphi_{lk,T} - \varphi_{lj,T}\right)}{T\left(\varphi_{lk,T} - \varphi_{lj,T}\right)}, \qquad \frac{\sin\ T\left(\varphi_{lk,T} - \varphi_{lj}^0\right)}{T\left(\varphi_{lk,T} - \varphi_{lj}^0\right)}, \qquad k \neq j; \tag{1.7}$$

$$\frac{\sin\ T\varphi_{lk,T}}{T\varphi_{lk,T}}, \qquad l = \overline{1, M}, \qquad k, j = \overline{1, N}. \tag{1.8}$$

However the use of previous definition of LSE does not allow to check the behaviour of fractions (1.7), (1.8), as $T \to \infty$.
In the paper

Walker A.M. On the estimation of harmonic component in a time series with stationary dependent residuals, Advances in Appl. Probability , **5**(1973), p. 217 – 241,

the author modified the definition of the frequencies LSE in the classical formulation of the detecting hidden periodicities problem so that the terms (1.7), (1.8) tend to zero, as $T \to \infty$.

In our setting the sense of similar modification is that LSE (1.6) is defined as an absolute minimum point of (1.5) on parametric set depending on T and distinguishing frequencies properly, as $T \to \infty$.

Following

Brillinger D.R. Regression for randomly sampled spatial series: the trigonometric case, J. Appl. Probab., **23**(1986), p. 273 – 289,

consider monotonically non-decreasing families of open sets $\Lambda_{lT} \subset \Lambda_l$, $l = \overline{1, M}$, $T > T_0 > 0$, such that $\bigcup_{T > T_0} \Lambda_{lT} = \tilde{\Lambda}_l$, $\left(\tilde{\Lambda}_l\right)^c = \Lambda_l^c$, and satisfying the following conditions.

B. For $l = \overline{1, M}$ and $k, k' = \overline{1, N}$

1) $\varphi_l^0 = \left(\varphi_{l1}^0, \dots, \varphi_{lN}^0\right) \in \Lambda_{lT}$, $T > T_0$;
2) $\lim_{T \to \infty} \inf\limits_{\varphi_l \in \Lambda_{lT}} T \left|\varphi_{lk} - \varphi_{lk'}\right| = \infty$, $k \neq k'$;
3) $\lim_{T \to \infty} \inf\limits_{\varphi_l \in \Lambda_{lT}} T \varphi_{lk} = \infty$.

The meaning of 2) and 3) is to cover cases of close true frequencies and close to zero true frequencies.

Definition 1.1. Any random vector θ_T (see (1.6) such that it is a point of $Q_T(\theta)$ (see (1.5)) absolute minimum on the parametric set $\Theta_T^c \subset \mathbb{R}^{(M+2)N}$, where amplitudes $A_k, B_k,\ k = \overline{1, N}$, can take any values and angular frequencies take values in the set

$$\Lambda_T^c = \bigtimes_{l=1}^{M} \Lambda_{lT}^c, \qquad T > T_0 > 0,$$

is called LSE in the Walker-Brillinger sense.

It goes without saying, instead of the LSE family $\{\theta_T,\ T > T_0\}$ one can consider more realistic LSE sequence $\{\theta_{T_n},\ n \geqslant 1\},\ T_n \to \infty$, as $n \to \infty$, and prove its consistency.

2. Main result.

Theorem 2.1. Let conditions **A** and **B** be satisfied. Then LSE θ_T in the Walker-Brillinger sense is a strongly consistent estimate of the parameter θ^0, namely:

$$A_{kT} \to A_k^0,\ B_{kT} \to B_k^0,\ T\left(\varphi_{lk,T} - \varphi_{lk}^0\right) \to 0,\ a.s.,\ as\ T \to \infty,\ l = \overline{1, M},\ k = \overline{1, N}. \tag{2.1}$$

In the rest of the lecture, we will cover the main details of the proof of this theorem.

3. Some comments on the proof.

The next uniform law of large numbers plays a decisive role in the proof of Theorem 2.1.

Let $\varphi = (\varphi_1, ..., \varphi_M)$, $\langle \cdot, \cdot \rangle$ is inner product in $\mathbb{R}^M$.

Theorem 3.1. Under condition **A**

$$\xi_T = \sup_{\varphi \in \mathbb{R}^M} \left| T^{-M} \int\limits_{[0,T]^M} exp\left\{-i\langle \varphi, t\rangle\right\} \varepsilon(t)dt \right| \to 0 \ a.s., \ \ as \ T \to \infty. \tag{3.1}$$

Denote by $\eta_T(\varphi)$ the expression under the supremum sign in (3.1). Then

$$\eta_T^2(\varphi) = T^{-2M} \int\limits_{[0,T]^{2M}} exp\left\{-i\langle \varphi, t - s\rangle\right\} \varepsilon(t)\varepsilon(s)dtds. \tag{3.2}$$

Let $b_j,\ j=\overline{1,2^M}$, be the ordered collection of binary vectors of the length M "symmetric" about the middle of their list in the sense that $b_{2^M-j}=\overline{b}_j,\ j=\overline{1,2^{M-1}}$, where $\overline{b}=(\overline{\beta}_1,...,\overline{\beta}_M)$, if $b=(\beta_1,...,\beta_M)$. Here $\beta_i=0\ or\ 1,\ i=\overline{1,M}$.

Simple but cumbersome change of variables in (3.2) leads to the inequalities

$$\mathsf{E}\,\xi_T^2=\mathsf{E}\sup_{\varphi\in\mathbb{R}^M}\eta_T^2(\varphi)\leqslant$$

$$\leqslant 2\sum_{j=1}^{2^M-1}T^{-2M}\int\limits_{[0,T]^M}\mathsf{E}\left|\int\limits_{\Pi_T(u)}\varepsilon\left(v+\langle b_j,u\rangle\right)\varepsilon\left(v+\langle\overline{b}_j,u\rangle\right)dv\right|du\leqslant \tag{3.3}$$

$$\leqslant 2\sum_{j=1}^{2^M-1}T^{-2M}\int\limits_{[0,T]^M}\Psi_j^{1/2}(u)du,$$

where $\Pi_T(u)=[0,T-u_1]\times...\times[0,T-u_M]$, (3.4) $$\Psi_j(u) = \int\limits_{\Pi_T^2(u)} \mathsf{E}\,\varepsilon\left(v + \langle b_j, u\rangle\right)\varepsilon\left(v + \langle \overline{b}_j, u\rangle\right)\varepsilon\left(w + \langle b_j, u\rangle\right)\varepsilon\left(w + \langle \overline{b}_j, u\rangle\right) dv dw,$$

$j = \overline{1, 2^{M-1}}$. To deal with integrals (3.4) and (3.3) we apply standard Isserlis' theorem and tedious evaluation of the resulting expressions using an explicit formula for the covariance function $\tilde{B}$ of the field ε from conditon **A**(**i**). Finally, we get the bound

(3.5) $$\mathsf{E}\,\xi_T^2 = O\left(\tilde{B}^{1/2}(T)\right) = O\left(B^{1/2}(T)T^{-\alpha/2}\right), \quad as \quad T \to \infty.$$

In turn, from simpler reasoning, we conclude that under the condition **A**(**ii**)

(3.6) $$\mathsf{E}\,\xi_T^2 = O\left(T^{-\frac{M}{2}}\right), \quad as \quad T \to \infty.$$

Standard argument shows that (3.1) follows from (3.5), as well as from (3.6).

Consider further the system of linear equations for $A_{kT},\ B_{kT}, k=\overline{1,N}$, which is a subsystem of the system of normal equations for finding θ_T:

$$\left.\frac{\partial Q_T(\theta)}{\partial A_p}\right|_{\theta=\theta_T} = \left.\frac{\partial Q_T(\theta)}{\partial B_p}\right|_{\theta=\theta_T} = 0, \quad p=\overline{1,N},$$

and write it in the form

$$\left\{\begin{array}{l} \sum\limits_{k=1}^{N} a_{kp}^{(1)} A_{kT} + \sum\limits_{k=1}^{N} b_{kp}^{(1)} B_{kT} = c_p^{(1)}, \quad p=\overline{1,N}; \\ \sum\limits_{k=1}^{N} a_{kp}^{(2)} A_{kT} + \sum\limits_{k=1}^{N} b_{kp}^{(2)} B_{kT} = c_p^{(2)}, \quad p=\overline{1,N}. \end{array}\right. \tag{3.7}$$

Introduce the notation

$$\begin{array}{c}\sin\\ \cos\end{array}\left(\sum_{l=1}^{M}\varphi_{lk,T}t_l\right) = \begin{array}{c}\sin_k\\ \cos_k\end{array}(t), \qquad \begin{array}{c}\sin\\ \cos\end{array}\left(\sum_{l=1}^{M}\varphi_{lk,T}^{0}t_l\right) = \begin{array}{c}\sin_k^0\\ \cos_k^0\end{array}(t).$$

Denote also by $o_T(1)$, $T > 0$, generally speaking, different stochastic processes converging to zero a.s., as $T \to \infty$.

Using condition **B** we find for the coefficients of the system (3.7)

$$a_{kp}^{(1)} = T^{-M} \int\limits_{[0,T]^M} \cos_k(t)\cos_p(t)dt = o_T(1),\ k \neq p;\ a_{pp}^{(1)} = \frac{1}{2} + o_T(1);$$

$$a_{kp}^{(2)} = T^{-M} \int\limits_{[0,T]^M} \cos_k(t)\sin_p(t)dt = b_{kp}^{(1)} = o_T(1); \tag{3.8}$$

$$b_{kp}^{(2)} = T^{-M} \int\limits_{[0,T]^M} \sin_k(t)\sin_p(t)dt = o_T(1),\ k \neq p;\ b_{pp}^{(2)} = \frac{1}{2} + o_T(1).$$

Set for $l = \overline{1, M}$, $p = \overline{1, N}$

$$x_{lp} = \frac{\sin\ T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)}{T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)}, \qquad y_{lp} = \frac{1 - \cos T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)}{T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)}. \tag{3.9}$$

Then the integrals

$$T^{-M} \int\limits_{[0,T]^M} \begin{matrix} \sin \\ \cos \end{matrix} \left(\sum_{l=1}^{M} \left(\varphi_{lp,T} - \varphi_{lp}^{0} \right) t \right) dt = \begin{matrix} S_M \\ C_M \end{matrix} \; (x_{lp}, y_{lp}) = \begin{matrix} S_{Mp} \\ C_{Mp} \end{matrix}$$

are some homogeneous polynomials of x_{lp}, y_{lp}, $l = \overline{1, M}$, and due to Theorem 3.1 it can be proved that in (3.7) the constant terms are

$$c_p^{(1)} = T^{-M} \int_{[0,T]^M} X(t) \cos_p(t) dt = \frac{1}{2} A_p^0 C_{Mp} - \frac{1}{2} B_p^0 S_{Mp} + o_T(1),$$

(3.10)

$$c_p^{(2)} = T^{-M} \int_{[0,T]^M} X(t) \sin_p(t) dt = \frac{1}{2} A_p^0 S_{Mp} + \frac{1}{2} B_p^0 C_{Mp} + o_T(1), \quad p = \overline{1, N}.$$

From (3.7)–(3.10) we get

$$A_{pT} = A_p^0 C_{Mp} - B_p^0 S_{Mp} + o_T(1),$$

(3.11)

$$B_{pT} = A_p^0 S_{Mp} + B_p^0 C_{Mp} + o_T(1),$$

and

(3.12) $$|A_{pT}|, \; |B_{pT}| \leqslant |A_p^0| + |B_p^0| + o_T(1), \; p = \overline{1, N}.$$

Using the function

$$\Phi_T(\theta_1, \theta_2) = T^{-M} \int\limits_{[0,T]^M} (g(t, \theta_1) - g(t, \theta_2))^2 \, dt,$$

from Definition 1.1 of LSE θ_T we obtain

$$0 \geqslant Q_T(\theta_T) - Q_T(\theta^0) = \Phi_T(\theta_T, \theta^0) + 2T^{-M} \int\limits_{[0,T]^M} \varepsilon(t) \left(g(t, \theta^0) - g(t, \theta_T) \right) dt \ a.s. \tag{3.13}$$

By Theorem 3.1 and (3.12) the 2nd term in the right-hand side of (3.13) is $o_T(1)$, and we arrive at the convergence

$$\Phi_T(\theta_T, \theta^0) \to 0 \ a.s., \ as \ T \to \infty. \tag{3.14}$$

Substitute now the expressions (3.11) into (3.14). Then after reduction of similar terms we get

$$\Phi_T(\theta_T, \theta^0) = \frac{1}{2} \sum_{p=1}^{N} \left((A_p^0)^2 + (B_P^0)^2) \right) \left(1 - C_{Mp}^2 - S_{Mp}^2 \right) + o_T(1). \tag{3.15}$$

The last sum can be written in a more convenient form.

Lemma 3.2. For any $M \geqslant 3$ and $p = \overline{1, N}$

$$C^2_{Mp}(x_{lp}, y_{lp}) + S^2_{Mp}(x_{lp}, y_{lp}) = \prod_{l=1}^{M} \left(x^2_{lp} + y^2_{lp} \right). \tag{3.16}$$

▲ For $M = 3$

$$C_{3p} = x_{1p}x_{2p}x_{3p} - y_{1p}y_{2p}x_{3p} - y_{1p}x_{2p}y_{3p} - x_{1p}y_{2p}y_{3p},$$
$$S_{3p} = y_{1p}x_{2p}x_{3p} + x_{1p}y_{2p}x_{3p} + x_{1p}x_{2p}y_{3p} - y_{1p}y_{2p}y_{3p},$$

$$C^2_{3p} + S^2_{3p} = (x^2_{1p} + y^2_{1p})(x^2_{2p} + y^2_{2p})(x^2_{3p} + y^2_{3p}).$$

Assume (3.16) is true and show that the similar identity is correct for $M+1$ as well. Using obvious iterative formulas

$$C_{M+1,p} = C_{Mp}x_{M+1,p} - S_{Mp}y_{M+1,p}, \; S_{M+1,p} = S_{Mp}x_{M+1,p} + C_{Mp}y_{M+1,p}, \tag{3.17}$$

we find,

$$C_{M+1,p}^2 + S_{M+1,p}^2 = (x_{M+1,p} - S_{Mp}y_{M+1,p})^2 + (S_{Mp}x_{M+1,p} + C_{Mp}y_{M+1,p})^2 =$$

$$= \left(C_{Mp}^2 + S_{Mp}^2\right)\left(x_{M+1,p}^2 + y_{M+1,p}^2\right) = \prod_{l=1}^{M+1}(x_{lp}^2 + y_{lp}^2). \quad \blacksquare$$

Note that according to the formula (3.9)

$$x_{lp}^2 + y_{lp}^2 = \left(\frac{\sin\left(T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)/2\right)}{T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)/2}\right)^2, \quad l = \overline{1,M}, \ \ p = \overline{1,N}. \tag{3.18}$$

It follows from (3.15), Lemma 3.2 (formula (3.16), and (3.18)

$$\Phi_T(\theta_T, \theta^0) = \frac{1}{2}\sum_{p=1}^{N}\left((A_p^0)^2 + (B_p^0)^2\right)\left(1 - \prod_{l=1}^{M}\left(\frac{\sin\left(T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)/2\right)}{T\left(\varphi_{lp,T} - \varphi_{lp}^0\right)/2}\right)^2\right) + o_T(1). \tag{3.19}$$

Together with (3.14) this means that

$$T\left(\varphi_{lp,T} - \varphi_{lp}^0\right) \to 0 \ a.s., \ \ as\ T \to \infty, \ \ l = \overline{1,M}, \ \ p = \overline{1,N}.$$

From (3.9) it follows

$$x_{lp} \to 1,\ y_{lp} \to 0,\ a.s.,\ as\ T \to \infty,\ l = \overline{1, M},\ p = \overline{1, N}. \tag{3.20}$$

In turn, from (3.20) and recurrent formulas (3.17) we derive $C_{Mp} \to 1,\ S_{Mp} \to 0\ a.s.$, as $T \to \infty,\ p = \overline{1, N}$.

Finally from (3.11) we obtain

$$A_{pT} \to A_p^0,\ B_{pT} \to B_p^0\ a.s.,\ as\ T \to \infty,\ p = \overline{1, N}. \tag{3.21}$$

Thank you for your attention!