Statistical estimation in the models with memory

Yuliya Mishura

Taras Shevchenko National University of Kyiv

International Workshop
"Statistics of Stochastic Processes in Discrete and Continuous Time"

October 11–12, 2022
Kyiv, Ukraine

1. Two approaches to consistent estimation of parameters of mixed fractional Brownian motion with trend
    - Estimation of θ when H is known
    - Estimation of H, κ and σ via quadratic variations
    - Estimation of θ when H is unknown
    - Alternative approach to parameter estimation based on ergodic theorem
2. Numerical approach to the drift parameter estimation in the model with two fractional Brownian motions
    - Maximum likelihood estimator for drift parameter θ, its properties and numerical construction
    - Alternative estimators
    - Simulations

Mixed fractional Brownian motion with trend

The results are common with Alexander Kukush, Kostiantyn Ralchenko and PhD students Stanislav Lohvinenko and Hanna Zheleznyak. We consider the following mixed fractional Brownian motion with trend

$$X_t = \theta t + \sigma W_t + \kappa B_t^H, \qquad t \in [0, T], \tag{1}$$

where W is a Wiener process, B^H is a fractional Brownian motion with Hurst index $H \in (0, 1)$, B^H is independent of W.
Our goal is to estimate all four parameters H, θ, κ and σ. We apply two approaches:

- "classical" approach
- "ergodic" approach

Let $H \in (0,1)$ be known. Assume that a trajectory of X is observed at the points $t_k^n = \frac{k}{2^n}$, $k = 0, 1, \dots, 2^{2n}$. Consider the estimator

$$\hat{\theta}_n := \frac{\sum_{k=1}^{2^{2n}-1} (t_k^n)^{\frac{1}{2}-H} (2^n - t_k^n)^{\frac{1}{2}-H} \left(X_{t_k^n} - X_{t_{k-1}^n} \right)}{\mathrm{B}(\frac{3}{2} - H, \frac{3}{2} - H) 2^{n(2-2H)}}, \tag{2}$$

Theorem 1

Let $H \in (0,1)$. The estimator $\hat{\theta}_n$ is a strongly consistent estimator of θ as $n \to \infty$. Moreover,

$$\hat{\theta}_n - \theta = O_\omega \left(n^\alpha 2^{-n[H \wedge (1-H)]} \right),$$

where α can be taken arbitrarily small.

Theorem 2

The estimator $\hat{\theta}_n$ is asymptotically normal: for $H < \frac{1}{2}$

$$2^{n/2}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \varphi(H)\sigma^2\right) \quad \text{as } n \to \infty,$$

and for $H > \frac{1}{2}$

$$2^{n(1-H)}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \psi(H)\kappa^2\right) \quad \text{as } n \to \infty.$$

where

$$\varphi(H) = \frac{\mathrm{B}(2-2H, 2-2H)}{\mathrm{B}^2(\frac{3}{2}-H, \frac{3}{2}-H)}, \quad H \in (0, \tfrac{1}{2}), \tag{3}$$

$$\psi(H) = \frac{H(2H-1)\mathrm{B}(H-\frac{1}{2}, \frac{3}{2}-H)}{\mathrm{B}(\frac{3}{2}-H, \frac{3}{2}-H)}, \quad H \in (\tfrac{1}{2}, 1). \tag{4}$$

Similarly to [Dozzi et al(2015)], we construct estimators using the quadratic variation of the form

$$V_n(X) := \sum_{i=0}^{n-1} (\Delta_i^n X)^2, \quad \text{with} \quad \Delta_i^n X := X_{\frac{i+1}{n}} - X_{\frac{i}{n}}.$$

Let $n \geq 1$. Assume that a trajectory of the process X is observed at points $t_k^n = \frac{k}{2^n}$, $k = 0, \ldots, 2^n$. We introduce the next statistic

$$\widehat{H}(k) = \frac{1}{2}\left(\log_{2+} \frac{V_{2^{k-2}}(X) - V_{2^{k-1}}(X)}{V_{2^{k-1}}(X) - V_{2^k}(X)} + 1\right), \tag{5}$$

with

$$\log_{2+} x := \begin{cases} \log_2 x & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Proposition 1

1. For $H \in (0, 1/2)$*, the statistic* (5) *is a strongly consistent estimator of* H*, moreover, for any* $\varepsilon > 0$,

$$\widehat{H}(k) = H + o_\omega\left(2^{(-1/2+\varepsilon)k}\right) \text{ as } k \to \infty. \tag{6}$$

2. For $H \in (1/2, 3/4)$ *the statistic* (5) *is a strongly consistent estimator of* H*, moreover, for any* $\varepsilon > 0$,

$$\widehat{H}(k) = H + o_\omega\left(2^{(2H-3/2+\varepsilon)k}\right) \text{ as } k \to \infty. \tag{7}$$

The quadratic variations also may be used for estimation of both κ and σ. The strong consistency of the following estimators can be established similarly to the corresponding results from [Dozzi et al(2015)].

Proposition 2 (Estimation of κ^2)

For $H \in (0, 1/2) \cup (1/2, 3/4)$, the statistic

$$\widetilde{\kappa}_k^2 := \frac{2^{k\left(2\widehat{H}(k)-1\right)}\left(V_{2^{k-1}}(X) - V_{2^k}(X)\right)}{2^{2\widehat{H}(k)-1} - 1}$$

is a strongly consistent estimator of κ^2.

Proposition 3 (Estimation of σ^2)

1. For $H \in (1/4, 1/2)$, the statistic

$$\widetilde{\sigma}_k^2 := \frac{2^{1-2\widehat{H}(k)} V_{2^{k-1}}(X) - V_{2^k}(X)}{2^{1-2\widehat{H}(k)} - 1}$$

is a strongly consistent estimator of σ^2.
2. For $H \in (1/2, 1)$, the statistic

$$\widehat{\sigma}_k^2 := V_{2^k}(X)$$

is a strongly consistent estimator of σ^2.

If H is unknown, we start with an auxiliary result, which gives an upper bound for the difference between the estimator $\hat{\theta}_n$ and the same estimator with some number $h \in (0,1)$ in place of the true value of H.

Lemma 3

Let X be a mixed fractional Brownian motion with trend, defined by (1) *with Hurst index $H \in (0,1)$. Define*

$$\tilde{\theta}_n(h) = \frac{\sum_{k=1}^{2^{2n}-1} (t_k^n)^{\frac{1}{2}-h} (2^n - t_k^n)^{\frac{1}{2}-h} \left(X_{t_k^n} - X_{t_{k-1}^n} \right)}{\mathrm{B}(\frac{3}{2}-h, \frac{3}{2}-h) 2^{n(2-2h)}}, \quad h \in [0,1].$$

Then

$$\tilde{\theta}_n(h) - \hat{\theta}_n = O_\omega(|h-H|), \tag{8}$$

for all $h \in (0, H^*)$, where $H^* \in (H,1)$ is any number, and for all $n \geq 1$.

Now, we want to replace H in the expression for $\hat{\theta}_n$ in (2) by the estimator (5).

Theorem 4

For $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, \frac{3}{4})$, $\tilde{\theta}_n\left(\widehat{H}(n)\right)$ is a strongly consistent estimator of θ as $n \to \infty$.
If $H \in (0, 1/2)$, then

$$\tilde{\theta}_n\left(\widehat{H}(n)\right) - \theta = o_\omega\left(2^{-(H-\varepsilon)n}\right) \quad a.s. \text{ as } n \to \infty,$$

for any $\varepsilon > 0$.
If $H \in (1/2, 3/4)$, then

$$\tilde{\theta}_n\left(\widehat{H}(n)\right) - \theta = o_\omega\left(2^{-(\frac{3}{2}-2H-\varepsilon)n}\right) \quad a.s. \text{ as } n \to \infty,$$

for any $\varepsilon > 0$.

Let $h > 0$ be fixed. Assume that the process X is observed at points $t_k = kh$, $k = 0, 1, 2, \dots$ The estimation procedure will be based on the following ergodic result.

Lemma 5

The process

$$Y_k = X_{(k+1)h} - X_{kh} - \theta h, \quad k = 0, 1, \dots,$$

is ergodic.

The previous lemma allows us to apply the ergodic theorem for construction of estimators. Namely, if $g\colon \mathbb{R}^{l+1} \to \mathbb{R}$ is a Borel function such that $\mathbb{E}\left|g\left(Y_0, Y_h, \dots, Y_{lh}\right)\right| < \infty$, then

$$\frac{1}{N}\sum_{k=0}^{N-1} g\left(Y_{kh}, \dots, Y_{(k+l)h}\right) \to \mathbb{E}g\left(Y_0, \dots, Y_{lh}\right) \text{ a.s. as } N \to \infty. \tag{9}$$

The main idea is to obtain four different convergences by choosing different functions g, and then to construct the estimators by solving the corresponding system of four equations.

To this end, let us introduce the notation:

$$\xi_N =: \frac{1}{N}\sum_{k=0}^{N-1}\left(X_{(k+1)h} - X_{kh}\right)^2,$$

$$\eta_N =: \frac{1}{N}\sum_{k=0}^{N-1}\left(X_{(k+1)h} - X_{kh}\right)\left(X_{(k+2)h} - X_{(k+1)h}\right),$$

$$\zeta_N =: \frac{1}{N}\sum_{k=0}^{N-1}\left(X_{(k+2)h} - X_{kh}\right)\left(X_{(k+4)h} - X_{(k+2)h}\right).$$

Theorem 6

Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$. *The statistics*

$$\check{\theta}_N = \frac{X_{Nh}}{Nh},$$

$$\check{H}_N = \frac{1}{2} \log_{2+} \frac{\zeta_N - 4\check{\theta}_N^2 h^2}{\eta_N - \check{\theta}_N^2 h^2},$$

$$\check{\kappa}_N^2 = \frac{\eta_N - \check{\theta}_N^2 h^2}{h^{2\check{H}_N}\left(2^{2\check{H}_N - 1} - 1\right)},$$

$$\check{\sigma}_N^2 = \frac{\xi_N - \check{\theta}_N^2 h^2 - \check{\kappa}_N^2 h^{2\check{H}_N}}{h}$$

are strongly consistent estimators of parameters $\theta, H, \kappa^2, \sigma^2$ *respectively.*

Theorem 7

The estimator $\check{\theta}_N$ is normal, and for $H < \frac{1}{2}$

$$(Nh)^{1/2}\left(\check{\theta}_N - \theta\right) \stackrel{d}{=} \mathcal{N}\left(0, \sigma^2\right),$$

and for $H > \frac{1}{2}$

$$(Nh)^{1-H}\left(\check{\theta}_N - \theta\right) \stackrel{d}{=} \mathcal{N}\left(0, \kappa^2\right).$$

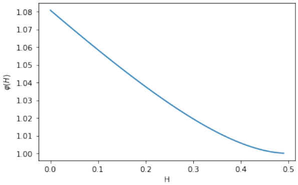

Figure: Plot of $\varphi(H)$.

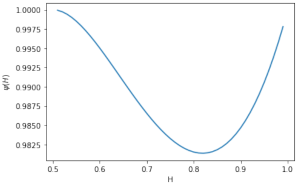

Figure: Plot of $\psi(H)$.

Remark 1

Estimators $\hat{\theta}_n$ and $\check{\theta}_N$ can be compared in terms of relative efficiency. Ratios of respective variances of their (asymptotic) distributions depend on Hurst index H and are defined in (3) and (4) as $\varphi(H), H \in \left(0, \frac{1}{2}\right)$, and $\psi(H), H \in \left(\frac{1}{2}, 1\right)$. Estimator $\check{\theta}_N$ is relatively more efficient when $H \in \left(0, \frac{1}{2}\right)$. Estimator $\hat{\theta}_n$ is relatively more efficient when $H \in \left(\frac{1}{2}, 1\right)$.

1. Two approaches to consistent estimation of parameters of mixed fractional Brownian motion with trend
   - Estimation of θ when H is known
   - Estimation of H, κ and σ via quadratic variations
   - Estimation of θ when H is unknown
   - Alternative approach to parameter estimation based on ergodic theorem

2. Numerical approach to the drift parameter estimation in the model with two fractional Brownian motions
   - Maximum likelihood estimator for drift parameter θ, its properties and numerical construction
   - Alternative estimators
   - Simulations

The section is devoted to the numerical estimation of the drift parameter $\theta \in \mathbb{R}$ in the continuous-time linear model

$$X_t = \theta t + B_t^{H_1} + B_t^{H_2}, \quad t \in [0, T],$$

where B^{H_1} and B^{H_2} are two independent fractional Brownian motions with different Hurst indices satisfying condition $1/2 \leq H_1 < H_2 < 1$.
As usual, we can apply maximum likelihood estimator for the drift parameter, but the presence of two indices leads to a significant complication of the estimator, as will be seen from the representations below. So, the theoretical basis for its construction is contained in the paper [Cai et al(2016)] for the case $1/2 = H_1 < H_2 < 1$ and in the papers [M.(2016), M.&Voronov(2015)] for $1/2 < H_1 < H_2 < 1$. More precisely, paper [M.(2016)] includes the construction of the maximum likelihood estimator, its form and properties.

It is stated that MLE estimator, up to a constant normalizer, equals the square-integrable martingale divided by its quadratic characteristics, and the numerator is the integral of the solution of some integral equation with respect to the observed process X, see (13). In this sense, numerical construction of MLE is reduced to the numerical solution of the Fredholm integral equation of the 2nd kind with singular kernel.

In our case function L is not either continuous or bounded. In this connection, we need a new theorem for approximation of the solutions of the integral equation. In the paper [M.&Voronov(2015)] alternative representation of the kernel is considered, and we apply both of the representations.

Taking into account the considerable technical difficulties arising during the implementation of the maximum likelihood estimator, we introduce two alternative estimators, one is constructed according to the last value of the observable process, another one is of the integral form. Their advantages include the relative simplicity of construction, and the disadvantage is the larger asymptotic variance, in comparison with the maximum likelihood estimator.

MLE for drift parameter θ

Introduce the following continuous-time linear model

$$X_t = \theta t + B_t^{H_1} + B_t^{H_2}, \quad t \in [0, T],$$

where B^{H_1} and B^{H_2} are two independent fractional Brownian motions with different Hurst indices $1/2 \le H_1 < H_2 < 1$. We consider the problem of estimation of unknown drift parameter θ, assuming that the parameters H_1 and H_2 are known.
Let us introduce the following notation

$$\begin{aligned} \beta_H &= \left(\frac{H(2H-1)}{B(H-1/2, 2-2H)} \right)^{\frac{1}{2}}, \\ c_H &= \left(\frac{\Gamma(3-2H)}{2H\Gamma^3(3/2-H)\Gamma(1/2+H)} \right)^{\frac{1}{2}}, \\ \gamma_H &= B(3/2-H, 3/2-H)c_H, \end{aligned} \tag{10}$$

where $\Gamma(\cdot)$ is the Gamma function and $B(\cdot,\cdot)$ is the Beta function.

The maximum likelihood estimator of θ was constructed in the paper [M.(2016)]. The representation of the estimator on $[0, T]$ involves a solution of the integral Fredholm equation of the 2nd kind

$$h_T(u) + \frac{1}{2-2H_1}\int_0^T h_T(s)K(s,u)ds = 1, \quad u \in [0, T]. \tag{11}$$

with weakly singular kernel

$$K(s,u) = u^{1-2H_1}\int_0^{s\wedge u} \varphi(s,v)\varphi(u,v)dv,$$

where

$$\varphi(s,v) = \frac{\partial}{\partial s} K_{H_1,H_2}(s,v)$$

and

$$K_{H_1,H_2}(s,v) = \beta_{H_2} c_{H_1} v^{\frac{1}{2}-H_2}\int_s^v (s-z)^{\frac{1}{2}-H_1} z^{H_2-H_1}(z-v)^{H_2-\frac{3}{2}}dz.$$

It was proved in the paper [M.&Voronov(2015)] that the integral Fredholm operator
$K\colon C([0,T] \to C([0,T]$ with the kernel $K(s,u)$ is a compact linear operator therefore it admits Fredholm alternative according to which for any sequence $T_m \to \infty$ that does not include the respective eigenvalues, there exists a unique solution $h_{T_m} \in C[0,T_m]$ of the integral Fredholm equation (11). More precisely, we can take the sequence $T_m \to \infty$ in such a way that

$$\lambda_m = -(2-2H_1)T_m^{2H_1-2H_2} \tag{12}$$

will be not an eigenvalue. Consider only such T_m, but for the technical simplicity denote T the upper bound of integration.
Let us introduce the following stochastic process:

$$Y_t = c_{H_1}\int_0^t s^{\frac{1}{2}-H_1}(t-s)^{\frac{1}{2}-H_1}\,dX_s.$$

It was stated in [M.(2016)] that the maximum likelihood estimator has the form

$$\hat{\theta}_T = \frac{\int_0^T h_T(t)dY_t}{\gamma_{H_1}(2-2H_1)\int_0^T h_T(t)t^{1-2H_1}dt}. \tag{13}$$

Moreover, in [M.(2016)] the strong consistency of $\hat{\theta}_T$ was established, assuming additionally that $H_2 - H_1 > \frac{1}{4}$. This result was generalized in [M.&Voronov(2015)] to the case of arbitrary $\frac{1}{2} \le H_1 < H_2 < 1$.
Let us summarize the properties of the estimator $\hat{\theta}_T$.

Theorem 8

The estimator $\hat{\theta}_T$ is strongly consistent, unbiased, and the corresponding estimation error is normal

$$\hat{\theta}_T - \theta \sim \mathcal{N}\left(0, \frac{1}{\gamma_{H_1}^2 (2 - 2H_1) \int_0^T h_T(s) s^{1-2H_1} ds}\right).$$

Moreover,

$$T^{1-H_2}\left(\hat{\theta}_T - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, V_{H_1,H_2}\right),$$

where the asymptotic variance equals

$$V_{H_1,H_2} = \frac{H_1\left(H_1 - \frac{1}{2}\right)\Gamma\left(\frac{3}{2} - H_1\right)\left(H_2 - \frac{1}{2}\right)\Gamma\left(\frac{3}{2} - H_2\right)}{(1 - H_1) B\left(\frac{3}{2} - H_1, \frac{3}{2} - H_1\right) B\left(2H_1 - H_2 + \frac{1}{2}, \frac{3}{2} - H_1\right)}. \quad (14)$$

Kernels and their representations

Let us investigate the kernel $K(s,u)$, $s,u \in [0,T]$, in more detail because its properties are principal for further numerical calculations. Namely, let us introduce several different representations for $K(s,u)$.
We start with the representation through Gauss hypergeometric functions, which allows us to calculate the kernel numerically. According to [M.&Voronov(2015), Lemma 1],

$$K(s,u) = \begin{cases} L(s,u)|s-u|^{-\alpha}, & s \neq u, \\ 0, & s = u, \end{cases}$$

where $\alpha = 1 - 2(H_2 - H_1) > 0$,

$$L(s,u) = \left(\beta_{H_2} c_{H_1} (H_2 - H_1)\right)^2 (s \wedge u)^{1-2H_1} u^{2H_1-1} \varkappa_0(s,u),$$

and the function $\varkappa_0(s,u)$ could be represented for $u < s$

$$\varkappa_0(s,u) = \int_0^1 (1-y)^{1-2H_2} \left(1 - \frac{u}{s}y\right)^{2H_1-1} y^{H_2-H_1-1} \times G\left(\frac{u(1-y)}{s-uy}\right) G\left(\frac{s(1-y)}{s-uy}\right) dy,$$

and for $u > s$

$$\varkappa_0(s,u) = \int_0^1 (1-y)^{1-2H_2} \left(1 - \frac{s}{u}y\right)^{2H_1-1} y^{H_2-H_1-1} \times G\left(\frac{s(1-y)}{u-sy}\right) G\left(\frac{u(1-y)}{u-sy}\right) dy.$$

It is bounded and belongs to $C([0,T]^2 \setminus \{(0,0)\})$. Here

$$G(s) = B\left(\tfrac{3}{2} - H_1, H_2 - \tfrac{1}{2}\right) F\left(H_1 - H_2, \tfrac{3}{2} - H_1, 1 - H_1 + H_2; 1 - s\right)$$
$$+ (1-s) B\left(\tfrac{3}{2} - H_1, H_2 + \tfrac{3}{2}\right) F\left(H_1 - H_2 + 1, \tfrac{3}{2} - H_1, 2 - H_1 + H_2; 1 - s\right),$$

$F(a,b,c;x)$ is the Gauss hypergeometric function.

Now rewrite the equation (11) in the equivalent form

$$u^{\frac{1}{2}-H_1} = h_T(u)u^{\frac{1}{2}-H_1} + (2-2H_1)^{-1}\int_0^T h_T(s)s^{\frac{1}{2}-H_1}s^{H_1-\frac{1}{2}}u^{\frac{1}{2}-H_1}K(s,u)ds,$$

or

$$u^{\frac{1}{2}-H_1} = \tilde{h}_T(u) + \int_0^T \tilde{h}_T(s)K_1(s,u)ds, \tag{15}$$

where

$$\begin{aligned} K_1(s,u) &= (2-2H_1)^{-1}s^{H_1-\frac{1}{2}}u^{\frac{1}{2}-H_1}K(s,u) \\ &= (2-2H_1)^{-1}s^{H_1-\frac{1}{2}}u^{H_1-\frac{1}{2}}\int_0^{s\wedge u}\varphi(s,v)\varphi(u,v)dv. \end{aligned}$$

is a symmetric in s and u kernel, $\tilde{h}_T(u) = h_T(u)u^{\frac{1}{2}-H_1} \in L_2[0,T]$.

Note that the integral operator $Af(x) = \int_0^T K_1(s,u)f(s)ds$ is positive definite on $L_2[0,T]$.

Approximate solution of the integral equation

The main problem with the kernel is that in its representation

$$K_1(s,u) = \begin{cases} L_1(s,u)|s-u|^{-\alpha}, & s \neq u, \\ 0, & s = u, \end{cases}$$

with $\alpha = 1 - 2(H_2 - H_1)$ the numerator

$$L_1(s,u) = \frac{(\beta_{H_2} c_{H_1}(H_2 - H_1))^2}{2 - 2H_1} (s \wedge u)^{1-2H_1} u^{H_1 - 1/2} s^{H_1 - 1/2} \varkappa_0(s,u), \quad (16)$$

is neither continuous nor bounded, but consists of the product of a continuous (except zero) and bounded function $\varkappa_0(s,u)$ and a factor $(s \wedge u)^{1-2H_1} u^{H_1-1/2} s^{H_1-1/2}$ tending to infinity on the axes. In this connection, taking into account well-known approximate methods for the calculation of the solutions of integral equations with singular kernels, let us construct a sequence of approximations $K_1^{(n)}(s,u)$ of the kernel $K_1(s,u)$ so that

$$K_1^{(n)}(s,u) = \begin{cases} L_1^{(n)}(s,u)|s-u|^{-\alpha}, & s \neq u, \\ 0, & s = u, \end{cases}$$

where $L_1^{(n)}(s,u)$ be a continuous bounded function. Namely, put $L_1^{(n)}(s,u) = L_1(s \vee \frac{1}{n}, u \vee \frac{1}{n})$. Then, obviously, $L_1^{(n)}(s,u) = L_1(s,u)$ for $s, u \geq 1/n$. Moreover, for any fixed $n \geq 1/T$, $L_1^{(n)}(s,u)$ is a bounded continuous function. This property ensures, by virtue of the Fredholm alternative, the existence and uniqueness of a solution to the integral equation of the second kind

$$u^{\frac{1}{2}-H_1} = \tilde{h}_T^{(n)}(u) + \int_0^T \tilde{h}_T^{(n)}(s) K_1^{(n)}(s,u) ds, \tag{17}$$

at any interval $[0, T]$ except the interval that corresponds to the eigenvalue in the same way as in (12).

Note that the kernel $K_1^{(n)}(s,u)$ has a standard weak singularity on a diagonal and no more singularities, therefore the integral operator A_n corresponding to the kernel $K_1^{(n)}(s,u)$ is compact from $L_2[0,T]$ into $L_2[0,T]$. Since any compact operator has no more than countable number of eigenvalues, we have a sequence $T_m \to \infty$ such that all equations, (15) and (17) for any n have a unique solution, h_{T_m} and $h_{T_m}^{(n)}$, respectively. In what follows, index m will be omitted.

Theorem 9

Let T be any point such that all equations (15) and (17) have unique solutions, h_T and $h_T^{(n)}$, respectively, on $[0,T]$. Also, let $H_2 - H_1 > \frac{1}{4}$. Then

$$\|h_T^{(n)} - h_T\|_{L_2[0,T]} \to 0, \text{ as } n \to \infty.$$

Description of the numerical method

To find an approximate solution of the Fredholm equation (11) we propose to use a modified product-integration method for weakly singular kernels, see, e. g., [Kythe&Puri(2011)]. The algorithm is described in detail in [Makogin et al(2021)], where the case of the kernel with bounded numerator $L(s,u)$ is studied. It is based on the modification proposed by Neta in [Neta(1984)]. Here we describe the main steps of this numerical method, adapted for our integral equation.
More precisely, let us consider the following generalization of the equation (18):

$$h_T(u) + \frac{1}{2-2H_1}\int_0^T h_T(s)K(s,u)ds = g(u), \quad u \in [0,T], \tag{18}$$

where the right-hand side $g \in L^2[0,T]$ is an arbitrary given function. We can transform this equation as described above.

Then we get the following analogue of (15):

$$\tilde{g}(u) = \tilde{h}_T(u) + \int_0^T \tilde{h}_T(s)K_1(s,u)ds$$

with $\tilde{g}(u) = g(u)u^{\frac{1}{2}-H_1}$. The corresponding approximate equation has the form

$$\tilde{g}(u) = \tilde{h}_T^{(n)}(u) + \int_0^T \tilde{h}_T^{(n)}(s)K_1^{(n)}(s,u)ds, \tag{19}$$

We start with a given number $N \geq 1$ of equally spaced points $0 = t_1 < t_2 < \cdots < t_N = T$ and $\delta = t_{j+1} - t_j = \frac{T}{N-1}$ (i. e. $t_j = (j-1)\delta$, $j = 1, \ldots, N$). By the product-integration rule

$$\int_{t_j}^{t_{j+1}} L_1^{(n)}(s,u)|u-s|^{-\alpha}\tilde{h}_T(s)ds$$
$$\approx \int_{t_j}^{t_{j+1}} \frac{L_j(u)(t_{j+1}-s)h_j + (s-t_j)L_{j+1}(u)h_{j+1}}{\delta}|u-s|^{-\alpha}ds,$$

where $L_j(u) = L_1^{(n)}(t_j,u)$ and $h_j = \tilde{h}_T^{(n)}(t_j)$.

Then we have

$$\tilde{h}_T^{(n)}(u) = \sum_{j=1}^{N-1} \left(L_j(u) h_j \psi_j^1(u) + L_{j+1}(u) h_{j+1} \psi_{j+1}^2(u) \right) + \tilde{g}(t), \qquad (20)$$

where weights are assigned as follows

$$\psi_j^1(u) = \frac{1}{\delta} \int_{t_j}^{t_{j+1}} (t_{j+1} - s)|u - s|^{-\alpha} ds,$$
$$\psi_j^2(u) = \frac{1}{\delta} \int_{t_{j-1}}^{t_j} (s - t_{j-1})|u - s|^{-\alpha} ds.$$

Substituting $u = t_i$ in (20) and rearranging the sums we obtain the following system of linear equations for approximation of integral equation (19)

$$h_i = \sum_{j=1}^{N} L_{i,j} \left(\psi_{j,i}^1 + \psi_{j,i}^2 \right) h_j + g_i, \quad i = 1, \ldots, N, \qquad (21)$$

where $L_{i,j} = L_1^{(n)}(t_j, t_i)$, $\psi_{j,i}^l = \psi_j^l(t_i)$ for $l = 1, 2$, and $g_i = \tilde{g}(t_i)$.

In (21) we assume that $\psi^1_{N,i} = \psi^2_{0,i} = 0$, for all i. From the practical point of view, it is convenient to choose $n = \frac{1}{\delta}$. Then

$$L_{i,j} = \begin{cases} L_1(t_j, t_i), & i > 1,\ j > 1, \\ L_1(1/n, t_i), & j = 1, \\ L_1(t_j, 1/n), & i = 1. \end{cases}$$

The system (21) can be written in matrix form

$$\mathrm{H}_N = \mathrm{K}_N \mathbf{H}_N + \mathbf{G}_N, \tag{22}$$

where H_N and G_N are vectors whose components are h_i and g_i, $i = 1, \ldots, N$, respectively, and K_N is $N \times N$-matrix with components

$$K_{i,j} = L_{i,j}\left(\psi^1_{j,i} + \psi^2_{j,i}\right), \quad i, j = 1 \ldots, N.$$

The approximate solution is obtained by solving a linear system of algebraic equations (22).

Let us consider a numerical example, demonstrating for the quality of the approximate method.

Example 10

We compare the approximate solution of equation (18) with the exact solution given by $h_T(u) \equiv 1,\ t \in [0, T]$. The right-hand side of (18) is computed by

$$g(u) = 1 + \frac{1}{2 - 2H_1} \int_0^T K(s, u) ds, \quad u \in [0, T], .$$

The maximum absolute errors between the approximate and exact solution for $N = 500$ and various values of T, H_1 and H_2 are presented in Table 1. We see that the method works even when the condition $H_2 - H_1 > \frac{1}{4}$ is violated.

Table: Maximum norm of the error

T	$H_1 = 0.6$, $H_2 = 0.7$	$H_1 = 0.6$, $H_2 = 0.8$	$H_1 = 0.6$, $H_2 = 0.9$	$H_1 = 0.7$, $H_2 = 0.9$
5	$2.97 \cdot 10^{-5}$	$6.42 \cdot 10^{-7}$	$3.77 \cdot 10^{-9}$	$1.97 \cdot 10^{-8}$
10	$1.07 \cdot 10^{-5}$	$1.58 \cdot 10^{-6}$	$3.75 \cdot 10^{-8}$	$5.60 \cdot 10^{-9}$
25	$1.26 \cdot 10^{-6}$	$5.27 \cdot 10^{-6}$	$5.21 \cdot 10^{-7}$	$1.43 \cdot 10^{-7}$
50	$8.85 \cdot 10^{-5}$	$6.27 \cdot 10^{-6}$	$2.59 \cdot 10^{-6}$	$3.31 \cdot 10^{-7}$
100	$2.90 \cdot 10^{-4}$	$1.23 \cdot 10^{-5}$	$1.37 \cdot 10^{-5}$	$5.66 \cdot 10^{-7}$
200	$2.53 \cdot 10^{-3}$	$3.67 \cdot 10^{-5}$	$4.91 \cdot 10^{-5}$	$1.73 \cdot 10^{-5}$

Alternative estimators

Consider now two alternative methods of estimation of the drift parameter. Recall that we consider the model of the form

$$X_t = \theta t + B_t^{H_1} + B_t^{H_2}, \tag{23}$$

where $B_t^{H_1}$, and $B_t^{H_2}$ are two independent fractional Brownian motions. We assume now that $0 < H_1 < H_2 < 1$. As it was proved in [Kozachenko et al(2015)], for any $p > 1$ and any $H \in (0,1)$ there exists a nonnegative random variable $\xi(p,H)$ such that for all $t \geq 0$,

$$\sup_{0 \leq s \leq t} |B_s^H| \leq \left(\left(t^H |\log t|^p\right) \vee 1\right) \xi(p,H),$$

and there exists such a number $c_\xi(p,H) > 0$ that for any $0 < y < c_\xi(p,H)$, $\mathbb{E}\exp\{y\xi^2(p,H)\} < \infty$. This means that

$$\frac{B_T^H}{T} \to 0, \quad \text{a. s. as } T \to \infty. \tag{24}$$

As a result, we can formulate the following theorem.

Theorem 11

The process $\hat{\theta}_T^{(1)} = \frac{X_T}{T}$ *is a strongly consistent estimator for* θ*, and for* $0 < H_1 < H_2 < 1$,

$$\lim_{T\to\infty} T^{2-2H_2}\mathbb{E}(\hat{\theta}_T^{(1)} - \theta)^2 = 1.$$

A disadvantage of the estimator $\hat{\theta}_T$ is that it is constructed according to only the final observation X_T. In order to reduce the dependence on only the fixed observation, we can consider "smoothed" estimator of the form

$$\hat{\theta}_T^{(2)} = \frac{\int_0^T X_t dt}{T}.$$

Note that

$$\int_0^T X_t dt = \theta\frac{T^2}{2} + \int_0^T B_t^{H_1} dt + \int_0^T B_t^{H_2} dt.$$

Taking (24) into account and applying L'Hospital's rule, we immediately get that

$$\frac{\int_0^T B_t^{H_1} dt}{T^2} \to 0 \qquad \text{and} \qquad \frac{\int_0^T B_t^{H_2} dt}{T^2} \to 0 \quad \text{a.s.}$$

as $T \to \infty$. As a result, we can state the following theorem.

Theorem 12

The process $\hat{\theta}_T^{(2)} = \frac{2}{T^2} \int_0^T X_t dt$ is a strongly consistent estimator of θ, and for $0 < H_1 < H_2 < 1$,

$$\lim_{T \to \infty} T^{2-2H_2} \mathbb{E}(\hat{\theta}_T^{(2)} - \theta)^2 = \frac{2}{1 + H_2}. \tag{25}$$

Remark 2

Let us compare the asymptotic variances of estimators $\hat{\theta}_T^{(1)}$ and $\hat{\theta}_T^{(2)}$. For $H_2 \in (0,1)$, $\frac{2}{1+H_2} > 1$, therefore the asymptotic variance of $\hat{\theta}_T^{(2)}$ is bigger. Asymptotic variances of θ_T and $\hat{\theta}_T^{(i)}$, $i = 1,2$ will be compared numerically.

Remark 3

The proposed estimators and methods may be applied (with obvious modifications) to more general model

$$X_t = \theta t + \sigma_1 B_t^{H_1} + \sigma_2 B_t^{H_2}, \quad t \in [0, T],$$

where σ_1 and σ_2 are known positive constants.

Simulations

In this section we would like to compare the behavior of three drift parameter estimators by numerical simulations. For each set of parameters (H_1, H_2) we start with solving the integral Fredholm equation (11) by the numerical method described above. Then we generate 1000 trajectories of the process X choosing the true value of drift parameter $\theta = 1$. The empirical means and empirical variances of the estimates for various values of the time horizon T are reported in Tables 2 and 3, respectively.

Table: Sample means of the estimates $\hat{\theta}_T$, $\hat{\theta}_T^{(1)}$ and $\hat{\theta}_T^{(2)}$

T		$H_1 = 0.6$, $H_2 = 0.7$	$H_1 = 0.6$, $H_2 = 0.8$	$H_1 = 0.6$, $H_2 = 0.9$	$H_1 = 0.7$, $H_2 = 0.9$
5	$\hat{\theta}_T$	0.99843	1.02050	0.97179	0.97511
	$\hat{\theta}_T^{(1)}$	0.98028	0.98521	0.97315	1.03375
	$\hat{\theta}_T^{(2)}$	0.97844	0.98677	0.97360	1.03327
10	$\hat{\theta}_T$	0.98254	1.01568	0.99384	0.96281
	$\hat{\theta}_T^{(1)}$	0.99878	1.00275	0.98399	1.02414
	$\hat{\theta}_T^{(2)}$	0.98802	0.99007	0.98274	1.02640
25	$\hat{\theta}_T$	0.97020	1.02370	0.99230	0.98650
	$\hat{\theta}_T^{(1)}$	1.00689	1.00846	0.98988	1.00861
	$\hat{\theta}_T^{(2)}$	0.99646	1.00128	0.98437	1.01577
50	$\hat{\theta}_T$	0.98163	1.02387	0.97414	0.99587
	$\hat{\theta}_T^{(1)}$	1.00919	1.01427	1.00170	1.02509
	$\hat{\theta}_T^{(2)}$	1.00714	1.00861	0.99198	1.02039
100	$\hat{\theta}_T$	0.97936	1.02123	0.98676	1.00517
	$\hat{\theta}_T^{(1)}$	1.01171	1.00776	1.00770	1.04042
	$\hat{\theta}_T^{(2)}$	1.00845	1.01059	1.00311	1.03198
200	$\hat{\theta}_T$	0.98830	0.99949	1.00481	0.99481
	$\hat{\theta}_T^{(1)}$	1.01638	0.99784	1.00371	1.04151
	$\hat{\theta}_T^{(2)}$	1.01299	1.00583	1.00202	1.03841

Table: Sample variances of the estimates $\hat{\theta}_T$, $\hat{\theta}_T^{(1)}$ and $\hat{\theta}_T^{(2)}$

T		$H_1 = 0.6$, $H_2 = 0.7$	$H_1 = 0.6$, $H_2 = 0.8$	$H_1 = 0.6$, $H_2 = 0.9$	$H_1 = 0.7$, $H_2 = 0.9$
5	$\hat{\theta}_T$	0.64699	0.80487	0.93363	1.06066
	$\hat{\theta}_T^{(1)}$	0.65501	0.84638	1.02237	1.11990
	$\hat{\theta}_T^{(2)}$	0.82555	0.97128	1.10788	1.22260
10	$\hat{\theta}_T$	0.42978	0.56085	0.74443	0.87134
	$\hat{\theta}_T^{(1)}$	0.39102	0.55397	0.79020	0.90523
	$\hat{\theta}_T^{(2)}$	0.48277	0.66097	0.87299	0.97534
25	$\hat{\theta}_T$	0.22352	0.33755	0.56440	0.65710
	$\hat{\theta}_T^{(1)}$	0.22508	0.33907	0.60744	0.69254
	$\hat{\theta}_T^{(2)}$	0.26144	0.39350	0.65080	0.74921
50	$\hat{\theta}_T$	0.14386	0.24854	0.48419	0.55723
	$\hat{\theta}_T^{(1)}$	0.14305	0.23685	0.51244	0.56264
	$\hat{\theta}_T^{(2)}$	0.17039	0.27073	0.54097	0.61000
100	$\hat{\theta}_T$	0.08788	0.18225	0.40255	0.45579
	$\hat{\theta}_T^{(1)}$	0.09188	0.17700	0.41662	0.46105
	$\hat{\theta}_T^{(2)}$	0.11068	0.19756	0.45442	0.49821
200	$\hat{\theta}_T$	0.05658	0.12624	0.33736	0.37987
	$\hat{\theta}_T^{(1)}$	0.05493	0.13165	0.36074	0.38137
	$\hat{\theta}_T^{(2)}$	0.06928	0.14634	0.38130	0.41179

Analyzing Table 2 we see that all the estimators are clearly unbiased. Moreover, all empirical variances in Table 3 tend to zero, this confirms the consistency of the estimators. The rate of convergence is much higher for small values of H_2. This is not surprising due to the presence of the normalizing factor T^{2-2H_2} in the asymptotic relations for the variances. We observe that the maximum likelihood estimator $\hat{\theta}_T$ outperforms two other estimators in most cases, including the situation when the condition $H_2 - H_1 > 1/4$ of Theorem 9 is violated, which is quite interesting. As a rule, the estimator $\hat{\theta}_T^{(1)}$ converges a bit slower than $\hat{\theta}_T$. However, the strength of $\hat{\theta}_T^{(1)}$ is its simple form, so it is faster to compute. The estimator $\hat{\theta}_T^{(2)}$ performs notably worse than $\hat{\theta}_T$ and $\hat{\theta}_T^{(1)}$, consequently we do not recommend to use it.

- A. Kukush, Y. Mishura, S. Lohvinenko, and K. Ralchenko, *Two approaches to consistent estimation of parameters of mixed fractional Brownian motion with trend*, Stat. Inference Stoch. Process. **25** (2022), no. 1, 159–187.
- Yu. Mishura, K. Ralchenko and H. Zhelezniak, *Numerical approach to the drift parameter estimation in the model with two fractional Brownian motions*, Communications in Statistics – Simulation and Computation (2022), DOI 10.1080/03610918.2022.2099556
- C. Cai, P. Chigansky and M. Kleptsyna, *Mixed Gaussian processes: a filtering approach*, Ann. Probab. **44** (2016), no. 4, 3032–3075.
- Dozzi, M., Mishura, Yu., Shevchenko, G. (2015). Asymptotic behavior of mixed power variations and statistical estimation in mixed models. *Stat. Inference Stoch. Process, 18*(2), 151—175.
- Kubilius, K., Mishura, Yu., Ralchenko, K. (2017). *Parameter estimation in fractional diffusion models*. Cham: Springer.

Le Breton, A. (1998). Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. *Statist. Probab. Lett., 38*(3), 263—274.

Y. Kozachenko, A. Melnikov and Y. Mishura, *On drift parameter estimation in models with fractional Brownian motion*, Statistics **49** (2015), no. 1, 35–62.

P. K. Kythe, P. Puri, *Computational methods for linear integral equations*, Springer Science & Business Media, 2011.

V. Makogin, Y. Mishura, H. Zhelezniak, *Approximate solution of the integral equations involving kernel with additional singularity*, Stochastic Models, (2021), DOI: 10.1080/15326349.2021.1945933.

Y. Mishura, *Maximum likelihood drift estimation for the mixing of two fractional Brownian motions*, in Stochastic and Infinite Dimensional Analysis, 2016, pp. 263–280.

Y. Mishura and I. Voronov, *Construction of maximum likelihood estimator in the mixed fractional–fractional Brownian motion model with double long-range dependence*, Mod. Stoch. Theory Appl. **2** (2015), no. 2, 147–164.

B. Neta, *Adaptive method for the numerical solution of Fredholm integral equations of the second kind. Part II: Singular kernels*, in Numerical Solution of Singular Integral Equations, 1984, pp. 249–263.