Drift parameters estimation in the Cox–Ingersoll–Ross model

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Outline

Introduction

The Cox–Ingersoll–Ross (CIR) process is a very famous object and is a unique solution of the following stochastic differential equation

$$dr_t = (a - br_t)dt + \sigma\sqrt{r_t}dW_t, \quad r_t\Big|_{t=0} = r_0 > 0 \qquad (1)$$

where $W = \{W_t, t \geq 0\}$ is a Wiener process and a, b, σ are positive constants.

We investigate two estimators of the parameter (a, b) by continuous observations of a sample path of CIR process $r = \{r_t, t \in [0, T]\}$ and prove their strong consistency as $T \to \infty$. The first one is the standard maximum likelihood estimator, which was constructed and studied in [Ben Alaya and Kebaier(2012), Ben Alaya and Kebaier(2013)]. Compared to the known results, we establish the strong consistency instead of weak one. Note that the maximum likelihood estimator is well-defined only if $2a \geq \sigma^2$, because, in particular, it contains the integral $\int_0^T \frac{1}{r_t} dt$, which exists with probability one if and only if $2a \geq \sigma^2$, see [Ben Alaya and Kebaier(2012), Prop. 4].

For this reason, we decided to create some statistics that converges regardless of whether $2a \geq \sigma^2$ or not. On this way we created a different estimator of the vector parameter (a, b), which is strongly consistent for all positive a, σ and b. Another advantage of the new alternative estimator is that it has simpler form, therefore, it is computationally faster. It includes only two statistics of the process r, namely the Lebesgue integrals $\int_0^T r_t\,dt$ and $\int_0^T r_t^2\,dt$, see Theorem 13 below. At the same time the maximum likelihood estimator depends on two Lebesgue integrals, $\int_0^T r_t\,dt$ and $\int_0^T \frac{dt}{r_t}$, on the stochastic integral $\int_0^T \frac{dr_t}{r_t}$ and on the process itself.

Proposition 1

Assume that $2a \geq \sigma^2$. Then

1. *The unique solution $r = \{r_t, t \geq 0\}$ of equation (1) is positive with probability 1:*
$$\inf\{t \geq 0 : r_t = 0\} = +\infty \quad a.s. \tag{2}$$
(with convention $\inf \emptyset = +\infty$). Moreover,
$$\mathbf{P}\{\limsup_{t\to\infty} r_t = +\infty\} = \mathbf{P}\{\liminf_{t\to\infty} r_t = 0\} = 1. \tag{3}$$

2. *The process r is ergodic and it has continuous stationary density that corresponds to gamma distribution and has the following form:*
$$p_\infty(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \mathbb{1}_{x>0}, \quad \alpha = \frac{2a}{\sigma^2},\ \beta = \frac{2b}{\sigma^2}. \tag{4}$$

3. *For any function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $\int_{\mathbb{R}^+} |f(x)| p_\infty(x) dx < \infty$ we have that*
$$\frac{1}{T}\int_0^T f(r_t)dt \to \int_{\mathbb{R}} f(x) p_\infty(x) dx, \quad a.s., \text{ as } T \to \infty. \tag{5}$$

Corollary 2

In the case $2a > \sigma^2$ we have the following asymptotic relations:

$$\frac{1}{T}\int_0^T r_t\,dt \to \int_{\mathbb{R}} x p_\infty(x)\,dx = \frac{\alpha}{\beta} = \frac{a}{b}, \quad a.\,s.,\ as\ T \to \infty, \tag{6}$$

$$\frac{1}{T}\int_0^T \frac{1}{r_t}\,dt \to \int_{\mathbb{R}} \frac{1}{x} p_\infty(x)\,dx = \frac{\beta}{\alpha - 1} = \frac{b}{a - \sigma^2/2}, \quad a.\,s.,\ as\ T \to \infty, \tag{7}$$

$$\frac{1}{T}\int_0^T r_t^2\,dt \to \int_{\mathbb{R}} x^2 p_\infty(x) dx = \left(\frac{\alpha}{\beta}\right)^2 + \frac{\alpha}{\beta^2} = \frac{a^2}{b^2} + \frac{a\sigma^2}{2b^2},\ a.s.,\ T \to \infty. \tag{8}$$

Remark 1

It was proved in [Deelstra and Delbaen(1995), Thm. 1] that the convergence (6) holds also in the case $0 < 2a \le \sigma^2$. The convergence (8) is valid for all positive a and σ, this will be justified in Theorem 12 below.

Maximum likelihood estimation

The maximum likelihood estimator for the couple (a, b) is constructed by maximizing of

$$\mathcal{L} = \exp\left\{\int_0^T \frac{a - br_t}{\sigma^2 r_t} dr_t - \frac{1}{2}\int_0^T \frac{(a - br_t)^2}{\sigma^2 r_t} dt\right\}.$$

with respect to (a, b). It has the following form:

$$\hat{a}_T = \frac{\int_0^T r_t dt \int_0^T \frac{dr_t}{r_t} - T \cdot (r_T - r_0)}{\int_0^T r_t dt \cdot \int_0^T \frac{dt}{r_t} - T^2}; \tag{9}$$

$$\hat{b}_T = \frac{(r_0 - r_T)\int_0^T \frac{dt}{r_t} + T\int_0^T \frac{dr_t}{r_t}}{\int_0^T r_t dt \cdot \int_0^T \frac{dt}{r_t} - T^2}. \tag{10}$$

Theorem 3

Assume that $2a > \sigma^2$. Then the estimator $(\hat{a}_T, \hat{b}_T)$ is strongly consistent.

This estimator is also asymptotically normal [Ben Alaya and Kebaier(2013), Thm. 5]. We study its rate of convergence in probability. Namely, we prove the following

Theorem 4

Let $2a > \sigma^2$. For any $\frac{1}{2} \le q < 2a/\sigma^2$ and any $\varepsilon > 0$

$$\mathbf{P}\left(|\hat{a}_T - a| > \varepsilon\right) \le C\varepsilon^{-2q} T^{-q}, \tag{11}$$

$$\mathbf{P}\left(\left|\hat{b}_T - b\right| > \varepsilon\right) \le C\varepsilon^{-2q} T^{-q}, \tag{12}$$

as $T \to \infty$.

The following result shows that convergences (6) and (7) hold also in $L_p(\Omega)$, $1 \le p < 4a/\sigma^2$, in particular, they hold in $L_2(\Omega)$ (recall that $2a > \sigma^2$). Moreover, we can estimate the uniform rate of convergence in L_p-norm.

Lemma 5

Assume that $2a > \sigma^2$. Then

1. *For any $p \ge 1$ there exists a constant $C_p > 0$ such that*

$$\mathbf{E}\left|\frac{1}{T}\int_0^T r_t\,dt - \frac{a}{b}\right|^p \le C_p T^{-p/2}, \tag{13}$$

2. *For any $1 \le p < 4a/\sigma^2$ there exists a constant $C_p > 0$ such that*

$$\mathbf{E}\left|\frac{1}{T}\int_0^T \frac{1}{r_t}\,dt - \frac{b}{a - \sigma^2/2}\right|^p \le C_p T^{-p/2}, \tag{14}$$

Discretization of MLE

In order to discretize the maximum likelihood estimator $(\hat{a}_T, \hat{b}_T)$, let us fix some positive integer n, consider the interval $[0, n]$ and divide it into equal parts of size $\delta_n = \frac{1}{n^\beta}$, $\beta > 0$. Let $t_k = \frac{k}{n^\beta}$, $m_n = n^{\beta+1}$. Instead of $(\hat{a}_T, \hat{b}_T)$ we consider

$$\hat{a}_n = \frac{\sum_{k=0}^{m_n-1} r_{t_k}\delta_n \sum_{k=0}^{m_n-1} \frac{r_{t_{k+1}}-r_{t_k}}{r_{t_k}} - n\cdot(r_n - r_0)}{\sum_{k=0}^{m_n-1} r_{t_k}\delta_n \cdot \sum_{k=0}^{m_n-1} \frac{\delta_n}{r_{t_k}} - n^2}; \tag{15}$$

$$\hat{b}_n = \frac{(r_0 - r_n)\sum_{k=0}^{m_n-1} \frac{\delta_n}{r_{t_k}} + n\sum_{k=0}^{m_n-1} \frac{r_{t_{k+1}}-r_{t_k}}{r_{t_k}}}{\sum_{k=0}^{m_n-1} r_{t_k}\delta_n \cdot \sum_{k=0}^{m_n-1} \frac{\delta_n}{r_{t_k}} - n^2}. \tag{16}$$

Theorem 6

Let $a > \sigma^2$. Then for any $\beta > 0$, $\hat{a}_n \to a$, $\hat{b}_n \to b$ in probability as $n \to \infty$, moreover,

$$\mathbf{P}\left(|\hat{a}_n - a| > \varepsilon\right) \le C\varepsilon^{-2q} n^{-q(1\wedge\beta)}, \tag{17}$$

$$\mathbf{P}\left(\left|\hat{b}_n - b\right| > \varepsilon\right) \le C\varepsilon^{-2q} n^{-q(1\wedge\beta)} \tag{18}$$

for any $\frac{1}{2} \le q \le \frac{a}{2\sigma^2}$.
If $\beta \ge 2$, then

$$\hat{a}_n \to a, \quad \hat{b}_n \to b \quad \text{a.s. as } n \to \infty. \tag{19}$$

If $\beta > 1$, then

$$\sqrt{n}\begin{pmatrix} \hat{a}_n - a \\ \hat{b}_n - b \end{pmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}, \Sigma), \quad \text{as } n \to \infty,$$

where

$$\Sigma = \begin{pmatrix} \frac{a(2a-\sigma^2)}{b} & 2a - \sigma^2 \\ 2a - \sigma^2 & 2b \end{pmatrix}.$$

Discrete LLNs

In this subsection we prove discrete analogs of the convergences (6) and (7). Moreover, we establish the rates of convergence in terms of L_p-norm. Recall that we consider the equidistant partition of the interval $[0, n]$ with the step $\delta_n = n^{-\beta}$, $\beta > 0$, and $m_n = n^{1+\beta}$ is the number of subintervals.

Lemma 7

Let $2a > \sigma^2$, $\beta > 0$. Then

$$\frac{1}{n} \sum_{k=0}^{m_n - 1} r_{t_k} \delta_n \to \frac{a}{b}, \quad \textit{a.s., as } n \to \infty. \tag{20}$$

Moreover, for any $p \geq 1$,

$$\mathbf{E} \left| \sum_{k=0}^{m_n - 1} r_{t_k} \delta_n - \frac{a}{b} \right|^p \leq C n^{-(\beta \wedge 1)p/2}. \tag{21}$$

Lemma 8

Let $a > \sigma^2$ *and* $\beta > 0$. *Then for any* $1 \le p < \frac{2a}{\sigma^2}$,

$$\mathbf{E}\left|\frac{1}{n}\sum_{k=0}^{m_n-1}\frac{\delta_n}{r_{t_k}}\right|^p \le C. \tag{22}$$

If $1 \le p < \frac{a}{\sigma^2}$, *then*

$$\mathbf{E}\left|\frac{1}{n}\sum_{k=0}^{m_n-1}\frac{1}{r_{t_k}}\,\delta_n - \frac{b}{a-\sigma^2/2}\right|^p \le Cn^{-(\beta\wedge 1)p/2}. \tag{23}$$

If $1 \le p < \frac{a}{\sigma^2}$ *and additionally* $\beta \ge 2$, *then*

$$\frac{1}{n}\sum_{k=0}^{m_n-1}\frac{1}{r_{t_k}}\,\delta_n \to \frac{b}{a-\sigma^2/2}, \quad a.s., \text{ as } n\to\infty. \tag{24}$$

An alternative approach to drift parameters estimation

The disadvantage of the maximum likelihood estimators is that they work only if $a > \sigma^2/2$, however, a priori we do not know if this relation holds for the observed process. To avoid this circumstance, in this section we will introduce a new estimator for the parameter (a, b) based on the statistics $\int_0^T r_t\,dt$ and $\int_0^T r_t^2\,dt$. First, we will prove that the convergence (8) remains valid in the case $0 < a \le \sigma^2/2$. To this end, we start with several auxiliary results. The first result gives the asymptotics of two normalized Lebesgue integrals

Lemma 9

Let $a > 0$, $b > 0$, $\sigma > 0$. Then the following normalized integrals asymptotically vanish as $T \to \infty$:

$$\frac{1}{T}\int_0^T e^{-bt} r_t\,dt \to 0 \quad a.\,s., \tag{25}$$

$$\frac{1}{T}e^{-2bT}\int_0^T e^{2bt} r_t\,dt \to 0 \quad a.\,s. \tag{26}$$

Lemma 10

Denote

$$Z_t = \int_0^t e^{bu} \sqrt{r_u}\, dW_u, \quad t \ge 0. \tag{27}$$

There exists a constant $C > 0$ *such that for all* $t \ge 0$,

$$\mathbf{E}\left[Z_t^2\right] \le Ce^{2bt}, \qquad \mathbf{E}\left[Z_t^3\right] \le Ce^{3bt}, \qquad \mathbf{E}\left[Z_t^2 r_t\right] \le Ce^{2bt}. \tag{28}$$

Lemma 11

Let the process Z *be defined by* (27)*. Then the following normalized stochastic integrals vanish as* $T \to \infty$*:*

$$\frac{1}{T} \int_0^T e^{-bt} Z_t \sqrt{r_t}\, dW_t \to 0 \quad a.s., \tag{29}$$

$$\frac{1}{T} e^{-2bT} \int_0^T e^{bt} Z_t \sqrt{r_t}\, dW_t \to 0 \quad a.s. \tag{30}$$

Theorem 12

The following convergence holds:

$$\frac{1}{T}\int_0^T r_t^2\,dt \to \frac{a^2}{b^2} + \frac{a\sigma^2}{2b^2} \quad a.s., \text{ as } T \to \infty.$$

Theorem 13

Define

$$\tilde{a}_T = \frac{\sigma^2}{2} \cdot \frac{\left(\int_0^T r_t\,dt\right)^2}{T\int_0^T r_t^2\,dt - \left(\int_0^T r_t\,dt\right)^2},$$

$$\tilde{b}_T = \frac{\sigma^2}{2} \cdot \frac{T\int_0^T r_t\,dt}{T\int_0^T r_t^2\,dt - \left(\int_0^T r_t\,dt\right)^2}.$$

Then vector $(\tilde{a}_T, \tilde{b}_T)$ *is a strongly consistent estimator of* (a, b).

Introduction

We consider a stochastic differential equation of the form

$$dr_t = (a - br_t)dt + \sigma r_t^\beta dW_t, \quad r_t\big|_{t=0} = r_0 > 0, \tag{31}$$

where $W = \{W_t, t \geq 0\}$ is a Wiener process. This equation is known as Chan–Karolyi–Longstaff–Sanders (CKLS) model, which was introduced for interest rate modeling. The particular case $\beta = \frac{1}{2}$ corresponds to the famous Cox–Ingersoll–Ross (CIR) process.

We concentrate on the case $\beta \in (\frac{1}{2}, 1)$. We generalize both approaches to drift parameters estimation developed in the previous sections for the CIR process.

Preliminaries

In what follows we assume that the following condition is satisfied.

Assumption

$a > 0,\ b > 0,\ \sigma > 0,\ \frac{1}{2} < \beta < 1,\ r_0 > 0.$

Our main goal is to estimate parameters a and b of the model (31) by continuous observations of a trajectory of r on the interval $[0, T]$. We assume that the parameters β and σ are known. This assumption is natural, because β and σ can be obtained explicitly from the observations.

The next proposition summarizes the properties of the solution of a stochastic differential equation (31).

Proposition 14

1. *The equation* (31) *has a unique strong solution* $r = \{r_t, t \geq 0\}$.
2. *The process* r *is a. s. strictly positive.*
3. *The process* r *is an ergodic diffusion with the following stationary density:*

$$p_\infty(x) = G \cdot x^{-2\beta} \exp\left\{\frac{2}{\sigma^2}\left(\frac{a \cdot x^{1-2\beta}}{1-2\beta} - \frac{b \cdot x^{2-2\beta}}{2-2\beta}\right)\right\}, \quad x > 0.$$

   where

$$G = \left(\int_0^\infty y^{-2\beta} \exp\frac{2}{\sigma^2}\left(\frac{a \cdot y^{1-2\beta}}{1-2\beta} - \frac{b \cdot y^{2-2\beta}}{2-2\beta}\right) dy\right)^{-1}. \tag{32}$$

4. *The integral* $\int_0^\infty x^\mu p_\infty(x)dx$ *is finite for all* $\mu \in \mathbb{R}$.

Corollary 15

The ergodic theorem implies that for all $\mu \neq 0$

$$\frac{1}{T}\int_0^T r_t^{\mu} dt \to \int_0^{\infty} x^{\mu} p_{\infty}(x) dx$$
$$= G \int_0^{\infty} x^{\mu - 2\beta} \exp\left\{ \frac{2}{\sigma^2} \left(\frac{a \cdot x^{1-2\beta}}{1-2\beta} - \frac{b \cdot x^{2-2\beta}}{2-2\beta} \right) \right\} dx \quad a.s. \text{ as } T \to \infty.$$

Estimation of the vector parameter (a, b)

Let us start with the construction of the maximum likelihood estimator of the couple (a, b).

Lemma 16

The maximum likelihood estimator for the couple (a, b) *constructed by the continuous observations of* r *over the interval* $[0, T]$ *has the form*

$$\hat{a}_T = \frac{\int_0^T \frac{dr_t}{r_t^{2\beta}} \cdot \int_0^T r_t^{2-2\beta} dt - \int_0^T r_t^{1-2\beta} dr_t \cdot \int_0^T r_t^{1-2\beta} dt}{\int_0^T \frac{dt}{r_t^{2\beta}} \cdot \int_0^T r_t^{2-2\beta} dt - \left(\int_0^T r_t^{1-2\beta} dt\right)^2}; \tag{33}$$

$$\hat{b}_T = \frac{\int_0^T \frac{dr_t}{r_t^{2\beta}} \cdot \int_0^T r_t^{1-2\beta} dt - \int_0^T r_t^{1-2\beta} dr_t \cdot \int_0^T \frac{dt}{r_t^{2\beta}}}{\int_0^T \frac{dt}{r_t^{2\beta}} \cdot \int_0^T r_t^{2-2\beta} dt - \left(\int_0^T r_t^{1-2\beta} dt\right)^2}. \tag{34}$$

Theorem 17

The estimator $(\hat{a}_T, \hat{b}_T)$ is strongly consistent.

Theorem 18

The estimator $(\hat{a}_T, \hat{b}_T)$ is asymptotically normal:

$$\sqrt{T}\begin{pmatrix}\hat{a}_T - a\\ \hat{b}_T - b\end{pmatrix} \xrightarrow{d} \mathcal{N}\left(\mathbf{0}, \sigma^2\Sigma^{-1}\right), \quad T \to \infty,$$

where

$$\Sigma = \begin{pmatrix}\int_0^\infty x^{-2\beta} p_\infty(x)dx & -\int_0^\infty x^{1-2\beta} p_\infty(x)dx\\ -\int_0^\infty x^{1-2\beta} p_\infty(x)dx & \int_0^\infty x^{2-2\beta} p_\infty(x)dx\end{pmatrix}.$$

Estimation of b when a is known

The case of known a is considered similarly to the case of two unknown parameters, so we omit some details.

Theorem 19

Let a be known. The maximum likelihood estimator for b is

$$\check{b}_T = \frac{a \int_0^T r_t^{1-2\beta}\, dt - \int_0^T r_t^{1-2\beta}\, dr_t}{\int_0^T r_t^{2-2\beta}\, dt}$$

It is strongly consistent and asymptotically normal:

$$\sqrt{T}\left(\check{b}_T - b\right) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\int_0^\infty x^{2-2\beta} p_\infty(x)\, dx}\right).$$

Estimation of a when b is known

Theorem 20

Let b be known. The maximum likelihood estimator for a is

$$\check{a}_T = \frac{b\int_0^T r_t^{1-2\beta}\,dt + \int_0^T r_t^{-2\beta}\,dr_t}{\int_0^T r_t^{-2\beta}\,dt}$$

It is strongly consistent and asymptotically normal:

$$\sqrt{T}\,(\check{a}_T - a) \xrightarrow{d} \mathcal{N}\left(0, \frac{\sigma^2}{\int_0^\infty x^{-2\beta} p_\infty(x)\,dx}\right).$$

An alternative approach to drift parameters estimation

In this section we generalize an approach to drift parameters estimation proposed in [DMR(2021)] for the CIR process. We start with two auxiliary lemmas.

Lemma 21

One has the following convergence

$$\frac{1}{T}\int_0^T r_t dt \to \frac{a}{b} \quad a.s., \text{ as } T \to \infty. \tag{35}$$

Lemma 22

One has the following convergence

$$\frac{1}{T}\int_0^T r_t^{3-2\beta} dt - \frac{a}{bT}\int_0^T r_t^{2-2\beta} dt \to \frac{\sigma^2(1-\beta)a}{b^2} \quad a.s., \text{ as } T \to \infty. \tag{36}$$

The convergences (35) and (36) enable us to construct the estimators of the drift parameters by solving the corresponding system of equations. We obtain the following estimators:

$$\tilde{a}_T = \frac{\sigma^2(1-\beta)\left(\int_0^T r_t dt\right)^2}{T\int_0^T r_t^{3-2\beta}dt - \int_0^T r_t dt \cdot \int_0^T r_t^{2-2\beta}dt};$$

$$\tilde{b}_T = \frac{\sigma^2(1-\beta)T\int_0^T r_t dt}{T\int_0^T r_t^{3-2\beta}dt - \int_0^T r_t dt \cdot \int_0^T r_t^{2-2\beta}dt}.$$

Theorem 23

$(\tilde{a}_T, \tilde{b}_T)$ *is a strongly consistent estimator of the parameter* (a, b)

Estimation of β and σ

It turns out that if we observe the whole path $\{r_t, t \in [0, T]\}$, then the parameters β and σ can be obtained explicitly from the observations by considering realized quadratic variations. First, we assume that the parameter σ is known. The next proposition gives the value of β.

Proposition 24

For any $t \in [0, T]$,

$$\beta = \lim_{h \to 0} \frac{\log\left(\frac{[r]_{t+h} - [r]_t}{\sigma^2 h}\right)}{2 \log r_t} \quad a.\,s., \tag{37}$$

where $[r]_t = \lim_{n\to\infty} \sum_{k=1}^{2^n} \left(r_{\frac{kt}{2^n}} - r_{\frac{(k-1)t}{2^n}}\right)^2$.

Now let σ be unknown. In this case we can consider quadratic variations at two different times t and s, this allows us to exclude the unknown parameter σ and to obtain the following result.

Proposition 25

For any $s, t \in [0, T]$, $s \neq t$,

$$\beta = \lim_{h \to 0} \frac{\log\left(\frac{[r]_{t+h} - [r]_t}{[r]_{s+h} - [r]_s}\right)}{2 \log(r_t / r_s)} \quad a.\,s. \tag{38}$$

Finally, after β is known, the parameter σ can be evaluated using one of the following relations.

Proposition 26

For any $t \in [0, T]$,

$$\sigma^2 = \frac{[r]_t}{\int_0^t r_s^{2\beta}\, ds} = \lim_{h \to 0} \frac{[r]_{t+h} - [r]_t}{h r_t^{2\beta}} \quad a.\,s. \tag{39}$$

Remark 2

Let us give some practical recommendations concerning evaluation of β and σ from data. Note that the right-hand side of (37) can be calculated using the values of the process r in the neighborhood of any $t \in [0, T]$. To avoid large estimation errors, the value of r_t should not be close to 1 for chosen t. In order to obtain better numerical results we recommend to take several distinct points $t_1, \dots, t_m$ and use the following quantity instead of (37):

$$\hat{\beta}_1(h) = \frac{\sum_{i=1}^m \left|\log\left(\frac{[r]_{t_i+h} - [r]_{t_i}}{\sigma^2 h}\right)\right|}{2\sum_{i=1}^m \left|\log r_{t_i}\right|}.$$

Compared to an average value of the estimators obtained from (37) at points $\{t_i\}$, the estimator $\hat{\beta}_1(h)$ performs much better in the situation when some values of r_{t_i} may be close to 1.

Remark 3

Similarly, the following estimators may be used instead of (38) and (39):

$$\hat{\beta}_2(h) = \frac{\sum_{i=1}^m \left|\log\left(\frac{[r]_{t_i+h}-[r]_{t_i}}{[r]_{s_i+h}-[r]_{s_i}}\right)\right|}{2\sum_{i=1}^m |\log(r_{t_i}/r_{s_i})|}, \qquad \hat{\sigma}^2(h) = \frac{\sum_{i=1}^m([r]_{t_i+h}-[r]_{t_i})}{h\sum_{i=1}^m r_{t_i}^{2\beta}}.$$

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