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STUDY OF THE LIMITING BEHAVIOR OF DELAYED RANDOM SUMS UNDER NON-IDENTICAL DISTRIBUTIONS SETUP AND A CHOVER TYPE LIL

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Abstract. We consider delayed sums of the type $S_{n+a_n} - S_n$ where a_n is possibly a positive integer valued random variable satisfying certain conditions and S_n is the sum of independent random variables X_n with distribution functions $F_n \in \{G_1, G_2\}$. We study the limiting behavior of the delayed sums and prove laws of the iterated logarithm of Chover type. These results extend the results in Vasudeva and Divanji (1992) and Chen (2008).

Key words and phrases. Stable distribution, domain of normal attraction, Chover type law of the iterated logarithm, delayed random sum.

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1. Introduction and notations

We consider a sequence of independent random variables (rvs) $\{X_n\}$ with corresponding distribution functions $\{F_n\}$ where for each n, $F_n \in \{G_1, G_2\}$. Throughout the paper we assume, for each j, that G_j is in the domain of normal attraction of a symmetric stable law with exponent α_j, $0 < \alpha_j < 2$.

If G_j is in the domain of normal attraction of a non-normal stable law with the characteristic function $\varphi_j(t) = \exp(-\lambda_j |t|^{\alpha_j})$, $0 < \alpha_1 \le \alpha_2 < 2, \lambda_j > 0, j = 1, 2$, then it is known that

$$1 - G_j(x) = \frac{c_{j1} + \theta_j(x)}{x^{\alpha_j}}, \quad G_j(-x) = \frac{c_{j2} + \beta_j(-x)}{x^{\alpha_j}}, \quad x > 0, \tag{1.1}$$

where $\theta_j(x)$, $\beta_j(-x) \to 0$ as $x \to \infty$, $c_{j1} > 0$ and $c_{j2} > 0$. Set $S_n = \sum_{k=1}^{n} X_k$ and consider the sampling scheme $\{\tau_1(n), \tau_2(n)\}$ where $\tau_j(k) - \tau_j(k-1) = 1$ if $F_k = G_j$, and zero otherwise. Clearly, $\tau_1(n) + \tau_2(n) = n$. Assume that each $\tau_j(n) \to \infty$ as $n \to \infty$ such that $\limsup_{n\to\infty} \tau_j(n)/\tau_j(2n) < 1$. We shall consider the case with $0 < \alpha_1 < \alpha_2 < 2$ first. We shall discuss $\alpha_1 = \alpha_2$ case at the end.

For later use, we introduce the notation $U_{\tau_1(n)}$, the sum of those X_k in $\{X_1, X_2, \cdots, X_n\}$ with distribution function G_1 and $V_{\tau_2(n)}$, the sum of those X_k in $\{X_1, X_2, \cdots, X_n\}$ with distribution function G_2. If G_1 is in the domain of normal attraction of the stable (α_1) law and G_2 is in the domain of normal attraction of the stable (α_2) law, then the limit distributions of $\frac{U_{\tau_1(n)} - d_1(\tau_1(n))}{B_1(\tau_1(n))}$ and $\frac{V_{\tau_2(n)} - d_2(\tau_2(n))}{B_2(\tau_2(n))}$ are the stable (α_1) and the stable (α_2) laws respectively for appropriate choices of real numbers $d_1(\tau_1(n))$ and $d_2(\tau_2(n))$ and non-negative numbers $B_1(\tau_1(n))$ and $B_2(\tau_2(n))$. One can choose $d_1(\tau_1(n)) = 0 = d_2(\tau_2(n))$ if $\alpha_1 \ne 1, \alpha_2 \ne 1$, and $d_1(n) \sim n \log n$ $(d_2(n) \sim n \log n)$ if $\alpha_1 = 1$ $(\alpha_2 = 1)$, and in the case $\alpha_1 = 1 = \alpha_2$, we may take $A_n = d_1(\tau_1(n)) + d_2(\tau_2(n))$. Here, we follow the notation $f_n \sim g_n$ if $f_n/g_n \to C$, as $n \to \infty$, where $0 < C < \infty$. Henceforth we assume that the limit distribution of $\frac{(S_n - A_n)}{B_n}$ exists for real numbers A_n and $B_n > 0$. Thus, if the limit distribution is a composition of the two stable laws, then $\tau_1(n) \sim n^{\alpha_1/\alpha_2}$ and $\tau_2(n) \sim n$. If the limit distribution is stable (α_2), $\tau_2(n) \sim n$ because $\frac{(\tau_1(n))^{\alpha_2}}{(\tau_2(n))^{\alpha_1}} = \left(\frac{\tau_1(n)}{\tau_2(n)}\right)^{\alpha_1} [\tau_1(n)]^{\alpha_2 - \alpha_1} \to 0$. If the limit distribution is stable (α_1), then

$n\ (\tau_1(n))^{-\ \alpha_2/\alpha_1} \to 0$. Unfortunately, no more specific conclusions about the behavior of τ_j can be figured out when the limit distribution is stable (α_1) or stable (α_2) as in the case where the limit distribution is a composition.

Chover [3] was the first to prove a law of the iterated logarithm (LIL) for symmetric stable laws with exponent $\alpha < 2$ where he considered the limiting behavior of $\left|\frac{S_n}{n^{1/\alpha}}\right|^{1/\log\log n}$, and Heyde [9] extended Chover's result to certain rvs with common distribution in the domain of normal attraction of a symmetric stable law with exponent $\alpha \neq 1, 2$.

Consider the delayed sums $T_n = S_{n+a_n} - S_n$ where $\{a_n \to \infty\}$ is a sequence of positive integers. Lai [11] proved the LIL to the delayed sums. Zinchenko [16] extended Chover's LIL to delayed sums of independent identically distributed (iid) symmetric stable ($0 < \alpha < 2$) rvs. Vasudeva and Divanji [15] extended the result of Chover to the non-identical distribution setup assuming $G_j, j = 1, 2$ to be positive stable laws with exponents $0 < \alpha_1 < \alpha_2 < 1$. They assumed that the limit distribution of S_n, properly normed, exists and is a composition of the two stable laws. Chen [2] proved some general results on the limiting behavior of S_n and derived an extension of the result of [15] to the case of symmetric stable laws G_j with exponents $0 < \alpha_1 < \alpha_2 < 2$, thereby relaxing the assumption of positive stable laws $G_j, j = 1, 2$. Henceforth we drop the term symmetric and just refer to the limit distributions as stable laws.

The main aims of this paper are:

(i) to extend the results of [2] to the case where each F_n is in the domain of normal attraction of the stable law G_1 or G_2 according to the sampling scheme described above and satisfying certain conditions. We shall not restrict to the case when the limit distribution of S_n, properly normed, is a composition of the two stable laws; that is, the limit distribution may be stable (α_1) or stable (α_2),

(ii) to extend the results of [2] to the case where the lags a_n are positive rvs independent of the summands X_k in the context described in (i).

In Section 2 we state the results of [2]. Further, we prove an extension of Lemma 2.1 in [1]. In Section 3, we discuss the delayed sum problem and, in Section 4, we consider similar problems with random a_n.

2. Statements of Chen's results

Chen [2] investigated the almost sure limiting behavior of the partial sums S_n and proved Chover's LIL type results for the delayed sums T_n under the assumption that G_j are symmetric stable. For the sampling scheme $\{\tau_1(n), \tau_2(n)\}$, a necessary and sufficient condition for $\frac{(S_n - A_n)}{B_n}$, with $A_n \in R$ and $B_n > 0$, to converge in distribution to a proper rv is that the ratio $\frac{(\tau_1(n))^{1/\alpha_1}}{(\tau_2(n))^{1/\alpha_2}} \to \xi$, where $\xi \geq 0$. If $\xi = 0$, the limit distribution is the stable (α_2); if $0 < \xi < \infty$, the limit distribution is a composition of the stable laws with exponents α_1 and α_2, and if $\xi = \infty$, the limit distribution is the stable (α_1). In the case of $\xi = \infty$, we may take $B_n \sim B_1(\tau_1(n)) \sim (\tau_1(n))^{1/\alpha_1}$. In the case the limit distribution is the stable (α_2), we may take $B_n \sim B_2(\tau_2(n)) \sim (\tau_2(n))^{1/\alpha_2} \sim n^{1/\alpha_2}$. Further, when the limit distribution is a composition of the two stable laws $\tau_1(n) \sim [n^{\alpha_1/\alpha_2}]$, $\tau_2(n) \sim n$, and we may take $B_n \sim n^{1/\alpha_2}$. For the details, we refer to [13].

We now introduce some assumptions which are applied in different situations:
Assumption (C_1): $\limsup_{n\to\infty} a_n/\tau_1(n) < \infty$.
Assumption (C_2): $\limsup_{n\to\infty} a_n/n < \infty$.
Assumption (C_3): For some $\mu > \frac{\alpha_2 - \alpha_1}{\alpha_2}$, $\tau_1(n) < n^{\alpha_1/\alpha_2} (\log n)^{-\mu}$.
Note that the assumption $\limsup_{n\to\infty} a_n/\tau_1(n) < \infty$ is slightly stronger than the assumption $\limsup_{n\to\infty} a_n/n < \infty$ which was applied by Chen [2] in the case $0 < \xi < \infty$.

We shall now recall Chen's results who, like Vasudeva and Divanji, assumed that the above limit distribution is a composition of the two stable laws with exponents α_1 and α_2 and proved the following.

Theorem 2.1. *[2] Let $f > 0$ be a nondecreasing function. Then with probability one*

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n\,(f(n))^{1/\alpha_1}} = \begin{cases} 0, \\ \infty \end{cases} \iff \int_1^\infty \frac{1}{xf(x)}dx \begin{cases} <\infty, \\ =\infty. \end{cases}$$

Corollary 2.2. *For every $\delta > 0$*

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.\,s.$$

and

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(\log n)^{1/\alpha_1}} = \infty \quad a.\,s.$$

In particular,

$$\limsup_{n\to\infty} \left|\frac{S_n}{B_n}\right|^{1/\log\log n} = e^{1/\alpha_1} \quad a.\,s.$$

Remark 2.3. (1) When the limit distribution of S_n is the stable (α_1), the proof of Theorem 2.1 and of Corollary 2.2 will go through with minor modifications. So the statements of Theorem 2.1 and Corollary 2.2 hold in this case too.
(2) When the limit distribution is the stable (α_2), under Assumption (C_3), the following result holds. For every $\delta > 0$

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(\log n)^{(1+\delta)/\alpha_2}} = 0 \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n(\log n)^{1/\alpha_2}} = \infty \quad \text{a.s.}$$

In particular,

$$\limsup_{n\to\infty} \left|\frac{S_n}{B_n}\right|^{1/\log\log n} = e^{1/\alpha_2} \quad \text{a.s.}$$

Theorem 2.4. *[2] Let $f > 0$ be a nondecreasing function and let $\{a_n\}$ satisfy Assumption (C_2). Then with probability one*

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n\,(f(n))^{1/\alpha_1}} = \begin{cases} 0, \\ \infty \end{cases} \iff \int_1^\infty \frac{1}{xf(x)}dx \begin{cases} <\infty, \\ =\infty. \end{cases}$$

Corollary 2.5. *Let $\{a_n\}$ satisfy Assumption (C_2). Then for every $\delta > 0$*

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.\,s.$$

and

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(\log n)^{1/\alpha_1}} = \infty \quad a.\,s.$$

In particular,

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_n}\right|^{1/\log\log n} = e^{1/\alpha_1} \quad a.\,s.$$

Remark 2.6. (1) In the case when the limit distribution of S_n, properly normed, is the stable (α_1) law we may take $B_n = B_1(\tau_1(n))$. The same results hold if Assumption (C_1) holds.
(2) In the case when the limit distribution of S_n, properly normed, is the stable (α_2) law we may take $B_n = B_2(\tau_2(n))$. Then the following result holds:
Let $\{a_n\}$ satisfy Assumption (C_1). Then for every $\delta > 0$

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(\log n)^{(1+\delta)/\alpha_2}} = 0 \quad \text{a. s.}$$

and

$$\limsup_{n\to\infty} \frac{|T_n|}{B_n(\log n)^{1/\alpha_2}} = \infty \quad \text{a. s.}$$

In particular,

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_n}\right|^{1/\log\log n} = e^{1/\alpha_2} \quad \text{a. s.}$$

Corollary 2.7. *Let $\{a_n\}$ be a subsequence of positive integers with $\limsup_{n\to\infty} a_n/n < \infty$ and let $\gamma_n = \log(n/a_n) + \log\log n$.*
(i) If $\lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = \infty$, then

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_2} \quad \textit{a. s.}$$

(ii) If $\lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = 0$, then

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_1} \quad \textit{a. s.}$$

(iii) If $\lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = s \in (0,\infty)$, then

$$\limsup_{n\to\infty} \left|\frac{T_n}{B_{a_n}}\right|^{1/\gamma_n} = e^{\frac{\alpha_1 s+\alpha_2}{(s+1)\alpha_1\alpha_2}} \quad \textit{a. s.}$$

The proofs of these results heavily depend on the fact that $\frac{U_{\tau_1(n)} - d_1(\tau_1(n))}{(\tau_1(n))^{1/\alpha_1}}$ and $\frac{V_{\tau_2(n)} - d_2(\tau_2(n))}{(\tau_2(n))^{1/\alpha_2}}$ are distributed as stable (α_1) and stable (α_2) respectively. This does not hold when G_j is not stable as in our case. To circumvent this difficulty, we use the Lemma 2.8 below. In the rest of the paper we use the letter C to denote a generic positive number which may be different at different places. Before we close this Section we shall prove an extension of the result in Lemma 2.1 in [1] for a sequence of independent rvs $\{Z_k\}$ with the common distribution function H in the domain of normal attraction of the stable law with characteristic function $\varphi(t) = \exp(-\lambda|t|^\alpha)$, $\lambda > 0, 0 < \alpha < 2$. We denote $W_n = Z_1 + Z_2 + \cdots + Z_n$. Then we have the following.

Lemma 2.8. *Let $f > 0$ be a nondecreasing function satisfying $\int_1^\infty \frac{1}{x\, f(x)} dx < \infty$. Then*

$$\lim_{n\to\infty} \frac{\max_{1\le k\le n}|W_k|}{B_n(\, f(n))^{1/\alpha}} = 0 \quad \textit{a. s.},$$

where $B_n = n^{1/\alpha}$.

Proof. We note that

$$\lim_{n\to\infty} \frac{\max_{1\le k\le n}|W_k|}{B_n(\, f(n))^{1/\alpha}} = 0 \quad \text{a. s.}$$

iff

$$\lim_{m\to\infty}\frac{|W_m|}{(mf(m))^{1/\alpha}}=0 \quad \text{a.s.}$$

(See Lemma 2 in [10]). Since, for each m, there exists an n such that $2^n\le m<2^{n+1}$ and

$$\frac{|W_m|}{(mf(m))^{1/\alpha}}\le\frac{\max_{2^n\le m<2^{n+1}}|W_m|}{(mf(m))^{1/\alpha}}\le\frac{\max_{2^n\le m<2^{n+1}}|W_m|}{(2^nf(2^n))^{1/\alpha}}$$

it is now sufficient to prove that for each $\epsilon>0$, $\mathsf{P}(E_n)=0$, where

$$E_n=\left\{\max_{2^n\le k<2^{n+1}}|W_k|>\epsilon(2^nf(2^n))^{1/\alpha}\right\}.$$

By the Lévy inequality, we have for all $n\ge1$, $\mathsf{P}(E_n)\le 2\,\mathsf{P}(D_n)$ where $D_n=\{|W_{2^{n+1}-1}|>$ $>\epsilon\,(2^n\,f(2^n))^{1/\alpha}\}$.

Recall that $W_{2^{n+1}-1}$ is the sum of iid rvs distributed as Z_1 which follows H. Since H is of the same type as G in (1.1), we have by the Corollary on page 279 in [7] (and the similar result for the left tail), for n large enough,

$$\mathsf{P}(D_n)=(2^{n+1}-1)[1-H(\epsilon\,(2^n\,f(2^n))^{1/\alpha})+H(-\epsilon\,(2^n\,f(2^n))^{1/\alpha})]$$

giving us

$$\mathsf{P}(D_n)=(2^{n+1}-1)\frac{C+\theta(\epsilon\,(2^n\,f(2^n))^{1/\alpha})+\beta(-\epsilon\,(2^n\,f(2^n))^{1/\alpha})}{\epsilon^\alpha\,2^n\,f(2^n)},$$

where $\theta(x),\beta(-x)\longrightarrow0$ as $x\longrightarrow\infty$. Hence for N sufficiently large

$$\sum_{n=N}^{\infty}\mathsf{P}(D_n)<\sum_{n=N}^{\infty}\frac{C}{f(2^n)}<\int_1^\infty\frac{1}{x\,f(x)}dx<\infty.$$

The result follows now by the Borel–Cantelli lemma. □

3. New results for delayed sums

We assume that the independent rvs $\{X_n\}$ have corresponding distribution functions $\{F_n\}$ where for each n, $F_n\in\{G_1,G_2\}$. In the following Lemma, we assume that G_j is in the domain of normal attraction of the stable law with characteristic function $\varphi_j(t)=$ $=\exp(-\lambda_j|t|^{\alpha_j})$, $0<\alpha_1<\alpha_2<2$, $\lambda_j>0, j=1,2$ and the limit distribution of S_n, properly normed, exists. Then we have the following result.

Lemma 3.1. *For any positive constant M and a non-decreasing function $f>0$ let*

$$\int_1^\infty\frac{1}{x\,f(x)}dx=\infty.$$

Then
(1) if $0<\xi\le\infty$, we have

$$\sum_{n=1}^{\infty}\mathsf{P}(|X_n|\ge M\,B_n(f(n))^{1/\alpha_1})=\infty,$$

where $B_n=B_1(\tau_1(n))$ and
(2) if $\xi=0$, we have

$$\sum_{n=1}^{\infty}\mathsf{P}(|X_n|\ge M\,B_n(f(n))^{1/\alpha_2})=\infty,$$

where $B_n=B_2(\tau_2(n))$.

Proof. Since G_j is in the domain of normal attraction of the stable (α_j) law $B_j(\tau_j(n)) \sim \sim (\tau_j(n))^{1/\alpha_j}$, $j = 1, 2$. Let us first consider the case where the limit distribution is either stable (α_1) or a composition of the two stable laws. Then

$$\frac{B_1(\tau_1(n))}{B_2(\tau_2(n))} \to \xi,\ 0 < \xi \le \infty$$

and B_n may be taken as $B_1(\tau_1(n))$. We recall that X_n follows G_1 if $\tau_1(n) - \tau_1(n-1) = 1$. Then we have

$$\sum_{n=1}^{\infty} \mathsf{P}\Big(|X_n| \ge M\ B_n(f(n))^{1/\alpha_1}\Big) = \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \mathsf{P}\Big(|X_n| \ge M\ B_n(f(n))^{1/\alpha_1}\Big) \ge$$
$$\ge \sum_{k=K_0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{C}{B_n^{\alpha_1} f(n)} \tag{3.1}$$

for K_0 large enough. This is obtained by omitting the terms that involve the rvs X_n that follow G_2 and then using the well-known relation (8.6) on page 313 in [7] for the tail probability of the distributions attracted to the stable (α_1) law.

Since $B_n = B_1(\tau_1(n)) \sim (\tau_1(n))^{1/\alpha_1}$, we have

$$\sum_{n=1}^{\infty} \mathsf{P}\Big(|X_n| \ge M\ B_n(f(n))^{1/\alpha_1}\Big) \ge C \sum_{k=K_1}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{1}{\tau_1(n)\ f(n)} \ge$$
$$\ge C \sum_{k=K_1}^{\infty} \big[\tau_1(2^{k+1}-1) - \tau_1(2^k-1)\big] \frac{1}{\tau_1(2^{k+1})\ f(2^{k+1})} \ge$$
$$\ge C \sum_{k=K_1}^{\infty} \big[\tau_1(2^{k+1}) - 1 - \tau_1(2^k)\big] \frac{1}{\tau_1(2^{k+1})\ f(2^{k+1})} \ge$$
$$\ge C \sum_{k=K_1+1}^{\infty} \frac{1}{f(2^k)} \tag{3.2}$$

for $K_1 > K_0$ because of the assumption that $\limsup_{n\to\infty} \frac{\tau_1(n)}{\tau_1(2n)} < 1$. Next, we note that

$$\infty = \int_1^{\infty} \frac{1}{x\ f(x)} dx = \sum_{k=0}^{\infty} \int_{x=2^k}^{2^{k+1}-1} \frac{1}{x\ f(x)} dx \le$$
$$\le C \sum_{k=0}^{\infty} \frac{1}{f(2^k)} \le C \sum_{k=K_1+1}^{\infty} \frac{1}{f(2^k)}.$$

This, together with (3.2), completes the proof of the Lemma in the case $0 < \xi \le \infty$. In the case $\xi = 0$ the proof can be constructed along the same lines by recalling that $B_n = B_2(\tau_2(n))$ and considering the terms for which X_n follows G_2 in the summation instead of those for which X_n follows G_1 while deriving inequality (3.1). □

Our next result shows that Theorem 2.1 holds when G_j is in the domain of normal attraction of the stable (α_j) law for $j = 1, 2$ with $0 < \alpha_1 < \alpha_2 < 2$.

Theorem 3.2. *Let $f > 0$ be a non-decreasing function and $0 < \xi \le \infty$. Then with probability one*

$$\limsup_{n\to\infty} \frac{|S_n|}{B_1(\tau_1(n))\ (f(n))^{1/\alpha_1}} = \begin{cases} 0, \\ \infty \end{cases} \quad \Longleftrightarrow \quad \int_1^{\infty} \frac{1}{xf(x)} dx \ \begin{cases} < \infty, \\ = \infty. \end{cases}$$

If $\xi = 0$ *and* $\limsup_{n\to\infty} \frac{(\log n)^{1+\eta}}{f(n)} < \infty$, *then for some* $\eta > 0$

$$\limsup_{n\to\infty} \frac{|S_n|}{B_2(\tau_2(n))\ (f(n))^{1/\alpha_2}} = 0 \quad a.\,s.$$

under the Assumption (C_3).
Further, if $\int_1^\infty \frac{1}{xf(x)}dx = \infty$, *then*

$$\limsup_{n\to\infty} \frac{|S_n|}{B_2(\tau_2(n))\ (f(n))^{1/\alpha_2}} = \infty \quad a.\,s.$$

Proof. Assume that $\int_1^\infty \frac{1}{x\,f(x)}dx < \infty$. Clearly, $\frac{\log n}{f(n)} \to 0$ and hence $f(n) \to \infty$ as $n \to \infty$. By the symmetrization argument (see Lemma 3.2.1 in [14]) we can prove the result assuming X_ns to be symmetric. Now, by Lemma 2.8,

$$\limsup_{n\to\infty} \frac{|U_{\tau_1(n)}|}{B_1(\tau_1(n))(f(\tau_1(n)))^{1/\alpha_1}} \le \limsup_{n\to\infty} \frac{\max_{1\le k\le\tau_1(n)} |U_k|}{B_1(\tau_1(n))(f(\tau_1(n)))^{1/\alpha_1}} = 0 \quad \text{a.\,s.} \tag{3.3}$$

Similarly,

$$\limsup_{n\to\infty} \frac{|V_{\tau_2(n)}|}{B_2(\tau_2(n))(f(\tau_2(n)))^{1/\alpha_2}} = 0 \quad \text{a.\,s.} \tag{3.4}$$

Suppose $0 < \xi \le \infty$. Then $B_n = B_1(\tau_1(n))$. Proceeding as in [2], we have from (3.3) and (3.4) and the facts $f(\tau_j(n)) \le f(n)$ and $0 < \alpha_1 < \alpha_2 < 2$

$$\begin{aligned}\limsup_{n\to\infty} \frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} &\le \limsup_{n\to\infty} \frac{|U_{\tau_1(n)}|}{B_n(f(n))^{1/\alpha_1}} + \limsup_{n\to\infty} \frac{|V_{\tau_1(n)}|}{B_n(f(n))^{1/\alpha_1}} = \\ &= \limsup_{n\to\infty} \frac{B_1(\tau_1(n))(f(\tau_1(n)))^{1/\alpha_1}}{B_n(f(n))^{1/\alpha_1}} \frac{|U_{\tau_1(n)}|}{B_1(\tau_1(n))(f(\tau_1(n)))^{1/\alpha_1}} + \\ &\quad + \limsup_{n\to\infty} \frac{B_2(\tau_2(n))(f(\tau_2(n)))^{1/\alpha_2}}{B_n(f(n))^{1/\alpha_1}} \frac{|V_{\tau_2(n)}|}{B_2(\tau_2(n))(f(\tau_2(n)))^{1/\alpha_2}} = \\ &= 0 \quad \text{a.\,s.}\end{aligned}$$

Here we use the facts that $B_n = B_1(\tau_1(n))$ and $\frac{B_2(\tau_2(n))}{B_n} \to 0$ if $\xi = \infty$, and $\frac{B_2(\tau_2(n))}{B_n} \to$ $\to 1/\xi$ if $0 < \xi < \infty$.

Now, we consider the case $\xi = 0$ where $B_n = B_2(\tau_2(n))$. Since $\int_1^\infty \frac{1}{x(\log x)^{(1+\eta)}}dx < \infty$, by Lemma 2.8, we have with probability one

$$\limsup_{n\to\infty} \frac{|U_{\tau_1(n)}|}{B_1(\tau_1(n))\ (\log\tau_1(n))^{(1+\eta)/\alpha_1}} \le \lim_{n\longrightarrow\infty} \frac{\max_{1\le k\le\tau_1(n)} |U_k|}{B_1(\tau_1(n))\ (\log\tau_1(n))^{(1+\eta)/\alpha_1}} = 0$$

and

$$\limsup_{n\to\infty} \frac{|V_{\tau_2(n))}|}{B_2(\tau_2(n))\ (\log\tau_2(n))^{(1+\eta)/\alpha_2}} \le \lim_{n\longrightarrow\infty} \frac{\max_{1\le k\le\tau_2(n)} |V_k|}{B_2(\tau_2(n))\ (\log\tau_2(n))^{(1+\eta)/\alpha_2}} = 0.$$

As in the previous case, we have with probability one

$$\begin{aligned}\limsup_{n\to\infty} &\frac{|S_n|}{B_2(\tau_2(n))\ (\log n)^{(1+\eta)/\alpha_2}} \le \\ &\le \limsup_{n\to\infty} \frac{B_1(\tau_1(n))(\log\tau_1(n))^{(1+\eta)/\alpha_1}}{B_2(\tau_2(n))(\log n)^{(1+\eta)/\alpha_2}} \frac{|U_{\tau_1(n)}|}{B_1(\tau_1(n))(\log\tau_1(n))^{(1+\eta)/\alpha_1}} + \\ &\quad + \limsup_{n\to\infty} \frac{(\log\tau_2(n))^{(1+\eta)/\alpha_2}}{(\log n)^{(1+\eta)/\alpha_2}} \frac{|V_{\tau_2(n)}|}{B_2(\tau_2(n))(\log\tau_2(n))^{(1+\eta)/\alpha_2}} = 0\end{aligned}$$

if

$$\limsup_{n\to\infty} \frac{B_1(\tau_1(n))(\log\tau_1(n))^{(1+\eta)/\alpha_1}}{B_2(\tau_2(n))(\log n)^{(1+\eta)/\alpha_2}} < \infty. \tag{3.5}$$

But in view of the Assumption (C_3)

$$\frac{B_1(\tau_1(n))(\log\tau_1(n))^{(1+\eta)/\alpha_1}}{B_2(\tau_2(n))(\log n)^{(1+\eta)/\alpha_2}} \le (\log n)^{-\mu/\alpha_1}\frac{((\alpha_1/\alpha_2)\log n - \mu\log\log n)^{(1+\eta)/\alpha_1}}{(\log n)^{(1+\eta)/\alpha_2}} \sim$$
$$\sim (\log n)^{[(1+\eta)(\alpha_2-\alpha_1)-\mu\alpha_2]/\alpha_1\alpha_2} \to 0$$

for μ sufficiently large. Since (3.5) holds, we have

$$\limsup_{n\to\infty}\frac{|S_n|}{B_2(\tau_2(n))\,(\log n)^{(1+\eta)/\alpha_2}} = 0 \quad \text{a. s.}$$

The required result follows because of the condition on f.

We now turn to the divergence part. Assume that $\int_1^\infty \frac{1}{x\,f(x)}dx = \infty$. Suppose $0 < \xi \le \infty$. Then $B_n = B_1(\tau_1(n))$. By Lemma 3.1, we then have, for any $M > 0$

$$\sum_{n=1}^{\infty}\mathsf{P}\Big(|X_n| \ge M\,B_n(f(n))^{1/\alpha_1}\Big) = \infty \tag{3.6}$$

which by the Borel–Cantelli lemma implies

$$\limsup_{n\to\infty}\frac{|X_n|}{B_n(f(n))^{1/\alpha_1}} = \infty \quad \text{a. s.} \tag{3.7}$$

Note that

$$\limsup_{n\to\infty}\frac{|X_n|}{B_n(f(n))^{1/\alpha_1}} \le \limsup_{n\to\infty}\frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} +$$
$$+\limsup_{n\to\infty}\frac{B_{n-1}\,(f(n-1))^{1/\alpha_1}}{B_n\,(f(n))^{1/\alpha_1}}\,\frac{|S_{n-1}|}{B_{n-1}\,(f(n-1))^{1/\alpha_1}} \le$$
$$\le 2\limsup_{n\to\infty}\frac{|S_n|}{B_n(f(n))^{1/\alpha_1}},$$

and hence from (3.7) we have

$$\limsup_{n\to\infty}\frac{|S_n|}{B_n(f(n))^{1/\alpha_1}} = \infty \quad \text{a. s.}$$

Finally, in the case $\xi = 0$, similar steps give the result with α_1 replaced by α_2 at appropriate places since $B_n = B_2(\tau_2(n))$. □

Remark 3.3. For $f(x) = \log x$,

$$\int_1^\infty \frac{1}{x(f(x))^\eta}dx$$

is finite or infinite accordingly as $\eta > 1$ or ≤ 1. Hence by Lemma 3.1 in [12], we note that Corollary 2.2 will hold in the case when $G_j, j = 1, 2$, are in the domains of normal attraction of the corresponding stable laws. Thus, Theorem 2.1 and Corollary 2.2 follow from the above Theorem.

We now give an extension of Theorem 2.4 to the situation where $G_j, j = 1, 2$, are in the domains of normal attraction of the stable laws with characteristic functions $\varphi_j(t) = \exp(-\lambda_j|t|^{\alpha_j})$, $0 < \alpha_1 < \alpha_2 < 2, j = 1, 2$.

Theorem 3.4. *Let $f > 0$ be a non-decreasing function and let $\{a_n\}$ be a subsequence of positive integers satisfying Assumption (C_1) in the case $0 < \xi \le \infty$. Then with probability one*

$$\limsup_{n\to\infty}\frac{|T_n|}{B_n\,(f(n))^{1/\alpha_1}} = \begin{cases} 0, \\ \infty \end{cases} \iff \int_1^\infty \frac{1}{xf(x)}dx \begin{cases} <\infty, \\ =\infty. \end{cases}$$

In the case $\xi = 0$*, let* $\{a_n\}$ *be a subsequence of positive integers satisfying Assumptions* (C_2) *and* (C_3)*. If* $\limsup_{n\to\infty} \frac{(\log n)^{1+\eta}}{f(n)} < \infty$ *for some* $\eta > 0$*, then*

$$\limsup_{n\to\infty} \frac{|T_n|}{B_2(\tau_2(n))\ (f(n))^{1/\alpha_2}} = 0 \quad a.\,s.$$

Further, if $\int_1^\infty \frac{1}{xf(x)}dx = \infty$*, then*

$$\limsup_{n\to\infty} \frac{|T_n|}{B_2(\tau_2(n))\ (f(n))^{1/\alpha_2}} = \infty \quad a.\,s.$$

We omit the proof as it builds exactly along the same lines of Theorem 1.2 in [2] and by using Lemma 3.1 in the divergence part. Further, it is a particular case of Theorem 4.2 proved in the next Section.

Remark 3.5. Corollaries 2.5 and 2.7 can be easily deduced under the weaker assumption that G_j is in the domain of normal attraction of the stable (α_j) law when the limit distribution of S_n, properly normed, is a composition of the two stable laws.

4. Delayed random sums

We shall now consider the situation where each a_n may be a positive integer valued rv. Very little work is done in this setup. There is, however, a large body of work related to random sums and random indexed statistics. The importance of this area of research is seen in reliability, insurance, financial mathematics and statistical quality control. We envisage that the delayed random sum theory will have applications in studies concerning control charts with censored samples where the sample size on each occasion will be a random number. To the best of our knowledge, there are only two papers dealing with this kind of problem, viz., Divanji and Raviprakash [6] and Divanji [5]. Both the papers deal with positive valued rvs which are identically distributed under rather strange assumptions. The usual method of investigation in limit theorems with random index is to convert them to limit theorems for non-random index and apply existing results. This is usually done via what is known as Anscombe's condition or Gnedenko's Transfer theorem. But these techniques of conversion from random index to non-random index do not seem to work in almost sure limit theory except when the original random variables are positive valued. However, the method proposed by Chen [2] helps us dealing with random index in the LIL discussed in this Section. We impose slightly stronger conditions on the rvs a_n than those in the previous Sections. Our first result below is a direct application of Theorem 2.2 in [8].
Let us introduce the following assumptions:
Assumption (C_1^*): $\limsup_{n\to\infty} a_n/\tau_1(n) < \infty$ a. s.
Assumption (C_2^*): $\limsup_{n\to\infty} a_n/n < \infty$ a. s.
In the rest of the paper, let $f > 0$ be a non-decreasing function and let $\{a_n\}$ be a sequence of positive integer valued rvs such that for each n, a_n is independent of the rvs $\{X_k\}$.

Theorem 4.1. *Under Assumption* (C_1^*)*, if* $0 < \xi \le \infty$*, we have with probability one*

$$\limsup_{n\to\infty} \frac{|S_{n+a_n}|}{B_n\ (f(n))^{1/\alpha_1}} = 0 \quad or\ \infty$$

accordingly as

$$\int_1^\infty \frac{1}{xf(x)}dx < \infty \quad or\ = \infty.$$

In the case $\xi = 0$, let $\{a_n\}$ satisfy the Assumption (C_2^*) and Assumption (C_3) hold. If $\limsup_{n\to\infty} \frac{(\log n)^{1+\eta}}{f(n)} < \infty$ for some $\eta > 0$, then with probability one

$$\limsup_{n\to\infty} \frac{|S_{n+a_n}|}{B_n\ (f(n))^{1/\alpha_2}} = 0$$

and

$$\limsup_{n\to\infty} \frac{|S_{n+a_n}|}{B_n\ (f(n))^{1/\alpha_2}} = \infty$$

if

$$\int_1^\infty \frac{1}{xf(x)} dx = \infty.$$

Proof. In the case $\xi > 0$, we take $B_n = B_1(\tau_1(n))$. Recall from Theorem 3.2 that

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n\ (f(n))^{1/\alpha_1}} = 0$$

if

$$\int_1^\infty \frac{1}{xf(x)} dx < \infty.$$

Since $\mathsf{P}(n + a_n \to \infty) = 1$, from Theorem 2.2 in [8], we now get with probability one

$$\limsup_{n\to\infty} \frac{|S_{n+a_n}|}{B_{n+a_n}\ (f(n+a_n))^{1/\alpha_1}} = 0.$$

Further, in this case, by Lemma 2.1 of [2], without loss of generality, we may assume that $\limsup_{x\to\infty} f(2x)/f(x) < \infty$. This in turn implies $\limsup_{n\to\infty} f(n+a_n)/f(n) < \infty$ because f is non-decreasing and Assumption (C_2^*) holds.

Under Assumption (C_1^*) we observe that with probability one

$$\limsup_{n\to\infty} \frac{B_{n+a_n}}{B_n} < \infty.$$

Writing

$$\frac{|S_{n+a_n}|}{B_n(f(n))^{1/\alpha_1}} = \frac{|S_{n+a_n}|}{B_{n+a_n}(f(n+a_n))^{1/\alpha_1}} \frac{B_{n+a_n}}{B_n} \left(\frac{f(n+a_n)}{f(n)}\right)^{1/\alpha_1}$$

steps similar to those in Theorem 3.2 give the stated result in the case $\xi = 0$.
To prove the divergence part, recall from Theorem 3.2 that with probability one

$$\limsup_{n\to\infty} \frac{|S_n|}{B_n\ (f(n))^{1/\alpha_1}} = \infty$$

with $B_n = B_1(\tau_1(n))$ in the case $0 < \xi \le \infty$ if

$$\int_1^\infty \frac{1}{xf(x)} dx = \infty.$$

Now, by Theorem 2.2 in [8], we have with probability one

$$\limsup_{n\to\infty} \frac{|S_{n+a_n}|}{B_{n+a_n}\ (f(n+a_n))^{1/\alpha_1}} = \infty.$$

The required result is immediate since with probability one

$$\frac{|S_{n+a_n}|}{B_n\ (f(n))^{1/\alpha_1}} \ge \frac{|S_{n+a_n}|}{B_{n+a_n}\ (f(n+a_n))^{1/\alpha_1}}.$$

In the case $\xi = 0$, $B_n = B_2(\tau_2(n))$ and similar steps give the result. □

Our next result is similar to Theorem 3.4 for delayed random sums. Our proof resembles that of Theorem 2.4 but some modifications are required.

Theorem 4.2. *Let $T_n^* = S_{n+a_n} - S_n$. Suppose $0 < \xi \le \infty$. Then under Assumption (C_1^*) we have with probability one*

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (f(n))^{1/\alpha_1}} = \begin{cases} 0, \\ \infty \end{cases} \iff \int_1^\infty \frac{1}{xf(x)}dx \ \begin{cases} < \infty, \\ = \infty. \end{cases}$$

In the case $\xi = 0$, let $\{a_n\}$ satisfy Assumption (C_2^*) and Assumption (C_3) hold. If $\limsup_{n\to\infty} \frac{(\log n)^{1+\eta}}{f(n)} < \infty$ for some $\eta > 0$, then we have with probability one

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (f(n))^{1/\alpha_2}} = 0$$

and

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (f(n))^{1/\alpha_2}} = \infty$$

if

$$\int_1^\infty \frac{1}{xf(x)}dx = \infty.$$

Proof. Assume that $\int_1^\infty \frac{1}{x\ f(x)}dx\ < \infty$. Note that

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (f(n))^{1/\alpha_1}} \le \limsup_{n\to\infty} \frac{|S_{n+a_n}|}{B_n\ (f(n))^{1/\alpha_1}} + \limsup_{n\to\infty} \frac{|S_n|}{B_n\ (f(n))^{1/\alpha_1}} = 0 \quad \text{a.s.}$$

by Theorems 3.2 and 4.1. Similar steps give the result in the case $\xi = 0$. This completes the proof of the convergence part.

Next, assume that $\int_1^\infty \frac{1}{x\ f(x)}dx\ = \infty$. Let $0 < \xi \le \infty$. Then, by Lemma 3.1, for any $M > 0$

$$\sum_{n=1}^{\infty} \mathsf{P}(|X_n| \ge M\ B_n(f(n))^{1/\alpha_1}) = \infty. \tag{4.1}$$

Suppose

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (f(n))^{1/\alpha_1}} = \infty \quad \text{a.s.}$$

does not hold. Then by the Kolmogorov zero–one law, there exists a constant $C \in [0, \infty)$ such that

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (f(n))^{1/\alpha_1}} = C \quad \text{a.s.}$$

Choose a positive valued function $h(x) \to \infty$ as $x \to \infty$ that is given by Lemma 2.2 in [1] such that

$$\int_1^\infty \frac{1}{x\ f(x)\ h(x)}dx = \infty.$$

Then for that function h

$$\lim_{n\to\infty} \frac{T_n^*}{B_n\ (f(n)\ h(n))^{1/\alpha_1}} = 0 \quad \text{a.s.} \tag{4.2}$$

Further, since G_1 and G_2 are in the domains of normal attraction of the stable laws

$$\frac{X_{n+1}}{B_n\ (f(n)\ h(n))^{1/\alpha_1}} \to 0$$

in probability. Then from (4.2)

$$\lim_{n\to\infty} \frac{T_n^*\ -\ X_{n+1}}{B_n\ (f(n)\ h(n))^{1/\alpha_1}} = 0$$

in probability. Hence, using Lemma 3 of [4], we have

$$\frac{X_{n+1}}{B_n\ (f(n)\ h(n))^{1/\alpha_1}} \to 0 \quad \text{a.s.}$$

Then by the Borel–Cantelli lemma, for any $M > 0$, we have

$$\sum_{n=1}^{\infty} \mathsf{P}\Big(|X_n| \geq M\ B_n(f(n)h(n))^{1/\alpha_1}\Big) < \infty$$

contradicting the result of Lemma 3.1 (with $f(n)h(n)$ in place of $f(n)$). This completes the proof in the case $0 < \xi \leq \infty$. Similar steps with $B_n = B_2(\tau_2(n))$ give the result if $\xi = 0$. □

Corollary 4.3. *In the case $0 < \xi \leq \infty$, we have for every $\delta > 0$*

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (\log n)^{(1+\delta)/\alpha_1}} = 0 \quad a.\ s.$$

and

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (\log n)^{1/\alpha_1}} = \infty \quad a.\ s.$$

In particular,

$$\limsup_{n\to\infty} \left|\frac{T_n^*}{B_n}\right|^{1/\log\log n} = e^{1/\alpha_1} \quad a.\ s.$$

Further, in the case $\xi = 0$ we have for every $\delta > 0$

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (\log n)^{(1+\delta)/\alpha_2}} = 0 \quad a.\ s.$$

and

$$\limsup_{n\to\infty} \frac{|T_n^*|}{B_n\ (\log n)^{1/\alpha_2}} = \infty \quad a.\ s.$$

In particular,

$$\limsup_{n\to\infty} \left|\frac{T_n^*}{B_n}\right|^{1/\log\log n} = e^{1/\alpha_2} \quad a.\ s.$$

The last statement follows by Lemma 3.1 in [12].

We now state and prove our final results which are Chover type laws of the iterated logarithm. We recall that if the limit distribution of S_n is a composition of the two stable laws or the stable (α_2) law, then $\tau_2(n) \sim\ n$.

Theorem 4.4. *The following Chover type results hold.*

(A) Suppose that the limit distribution of S_n, properly normed, is a composition of the two stable laws. Let $\gamma_n = \log(n/a_n) + \log\log n$ and let Assumption (C_2^*) hold. Then with probability one

$$\limsup_{n\to\infty} \left|\frac{T_n^*}{B_{a_n}}\right|^{1/\gamma_n} = \begin{cases} e^{1/\alpha_2}, & \text{if } \lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = \infty \quad a.\ s., \\ e^{1/\alpha_1}, & \text{if } \lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = 0 \quad a.\ s., \\ \exp\Big(\frac{\alpha_1 s + \alpha_2}{\alpha_1\alpha_2\ (s+1)}\Big), & \text{if } \lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n} = s \in (0,\infty) \quad a.\ s. \end{cases}$$

(B) Suppose that the limit distribution of S_n, properly normed, is the stable (α_1) law. Let $\gamma_n^* = \log(\tau_1(n)/\tau_1(a_n)) + \log\log n$ and let Assumption (C_1^*) hold. Further, let $\lim_{n\to\infty} \frac{\log(\tau_1(n)/\tau_1(a_n))}{\log\log n}$ exist. Then with probability one

$$\limsup_{n\to\infty} \left|\frac{T_n^*}{B_{a_n}}\right|^{1/\gamma_n^*} = e^{1/\alpha_1}.$$

(C) Suppose that the limit distribution of S_n, properly normed, is the stable (α_2) law. Let $\gamma_n = \log(n/a_n) + \log\log n$ and let Assumption (C_2^*) hold. Further, let $\lim_{n\to\infty} \frac{\log(n/a_n)}{\log\log n}$ exist. Then with probability one

$$\limsup_{n\to\infty} \left|\frac{T_n^*}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_2}.$$

Proof. Let us first consider the case in which the limit distribution of S_n , properly normed, is a composition of the two stable laws. Denote $s_n = \frac{\log(n/a_n)}{\log\log n}$ and let $\delta > 0$. We have from Corollary 4.3

$$\mathsf{P}\Big(|T_n^*| \geq B_n\ (\log n)^{(1+\delta)/\alpha_1} \quad \text{i. o.}\Big) = 0 \tag{4.3}$$

for all $\delta > 0$ and

$$\mathsf{P}\Big(|T_n^*| \geq B_n\ (\log n)^{1/\alpha_1} \quad \text{i. o.}\Big) = 1. \tag{4.4}$$

Since $B_n \sim n^{1/\alpha_2}$, $\frac{B_{a_n}}{B_n} \sim (\frac{a_n}{n})^{1/\alpha_2}$, the above relations are respectively equivalent to

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1}{\alpha_2}\log(n/a_n) \ + \ \frac{1+\delta}{\alpha_1}\log\log n \ \text{ i. o.}\right) \ = \ 0 \tag{4.5}$$

for all $\delta > 0$ and

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1}{\alpha_2}\log(n/a_n) \ + \ \frac{1}{\alpha_1}\log\log n \ \text{ i. o.}\right) \ = \ 1. \tag{4.6}$$

Then (4.5) and (4.6) can be rewritten as

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1}{\alpha_2}\frac{s_n}{1+s_n}\gamma_n \ + \ \frac{1+\delta}{\alpha_1}\frac{\gamma_n}{1+s_n} \ \text{ i. o.}\right) \ = \ 0 \tag{4.7}$$

and

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1}{\alpha_2}\frac{s_n}{1+s_n}\gamma_n \ + \ \frac{1}{\alpha_1}\frac{\gamma_n}{1+s_n} \ \text{ i. o.}\right) \ = \ 1. \tag{4.8}$$

(i) Assume that $\lim_{n\to\infty} s_n = \infty$ a. s. holds. Then the above two relations give us the result.

(ii) Suppose $s_n \to 0$ a. s. Then from (4.7) and (4.8) we note that

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1+\delta_1}{\alpha_1}\frac{\gamma_n}{1+s_n} \ \text{ i. o.}\right) \ = \ 0 \tag{4.9}$$

and

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1-\delta_1}{\alpha_1}\frac{\gamma_n}{1+s_n} \ \text{ i. o.}\right) \ = \ 1 \tag{4.10}$$

for all $\delta_1 > 0$ giving the result

$$\limsup_{n\to\infty} \left|\frac{T_n^*}{B_{a_n}}\right|^{1/\gamma_n} = e^{1/\alpha_1} \quad \text{a. s.}$$

(iii) Finally, assume that $s_n \to s$ a. s. where $0 < s < \infty$. Then from (4.7) and (4.8) we note that

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{\alpha_1\ s+\alpha_2}{\alpha_1\alpha_2(1+s)}(1+\delta_2)\,\gamma_n \ \text{ i. o.}\right) \ = \ 0 \tag{4.11}$$

and

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{\alpha_1\ s+\alpha_2}{\alpha_1\alpha_2(1+s)}(1-\delta_2)\,\gamma_n \ \text{ i. o.}\right) \ = \ 1 \tag{4.12}$$

for all $\delta_2 > 0$ giving the result

$$\limsup_{n\to\infty} \left|\frac{T_n^*}{B_{a_n}}\right|^{1/\gamma_n} = e^{\frac{\alpha_1\ s+\alpha_2}{\alpha_1\alpha_2\ (s+1)}} \quad \text{a. s.}$$

Next, we consider the case when the limit distribution of S_n, properly normed, is the stable (α_1) law. Then in place of (4.5) and (4.6) we have

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1}{\alpha_1}\log\frac{\tau_1(n)}{\tau_1(a_n)} + \frac{1+\delta}{\alpha_1}\log\log n \ \text{ i. o.}\right) = 0 \tag{4.13}$$

and

$$\mathsf{P}\left(\log\left|\frac{T_n^*}{B_{a_n}}\right| \geq \frac{1}{\alpha_1}\log\frac{\tau_1(n)}{\tau_1(a_n)} + \frac{1}{\alpha_1}\log\log n \ \text{ i. o.}\right) = 1 \tag{4.14}$$

and the rest of the steps to get the result in (B) are similar and hence omitted. Result (C) is proved along the similar lines using the second half of Corollary 4.3 since $\xi = 0$. ☐

Remark 4.5. When $\alpha_1 = \alpha_2 = \alpha$, if the limit distribution of S_n, properly normed, exists, it will be stable (α). In this case, $B_n \sim n^{1/\alpha}$. All the results will hold with $\alpha_1 = \alpha_2 = \alpha$ and $\limsup_{n\to\infty}\left|\frac{T_n^*}{n^{1/\alpha}}\right|^{1/\gamma_n} = e^{1/\alpha}$ a. s.

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ДОСЛІДЖЕННЯ ГРАНИЧНОЇ ПОВЕДІНКИ ВИПАДКОВИХ СУМ ІЗ ЗАПІЗНЕННЯМ У ВИПАДКУ НЕОДНАКОВИХ РОЗПОДІЛІВ ТА ЗАКОНУ ПОВТОРНОГО ЛОГАРИФМА У ФОРМІ ЧОВЕРА

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Анотація. Розглянуто суми із запізненням вигляду $S_{n+a_n} - S_n$, де a_n — можливо додатна цілочисельна випадкова величина, що задовольняє певні умови, а S_n — сума незалежних випадкових величин X_n з функціями розподілу $F_n \in \{G_1, G_2\}$. Досліджено граничну поведінку таких сум із запізненням і доведено закони повторного логарифма у формі Човера. Ці результати поширюють результати, отримані в роботах Vasudeva і Divanji (1992) та Chen (2008).

ИССЛЕДОВАНИЕ ПРЕДЕЛЬНОГО ПОВЕДЕНИЯ СЛУЧАЙНЫХ СУММ С ЗАПАЗДЫВАНИЕМ В СЛУЧАЕ НЕОДИНАКОВЫХ РАСПРЕДЕЛЕНИЙ И ЗАКОНА ПОВТОРНОГО ЛОГАРИФМА В ФОРМЕ ЧОВЕРА

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Аннотация. Рассмотрены суммы с запаздыванием вида $S_{n+a_n} - S_n$, где a_n — возможно положительная целочисленная случайная величина, удовлетворяющая определенным условиям, а S_n — сумма независимых случайных величин X_n с функциями распределения $F_n \in \{G_1, G_2\}$. Исследовано предельное поведение таких сумм с запаздыванием и доказаны законы повторного логарифма в форме Човера. Эти результаты распространяют результаты, полученные в работах Vasudeva и Divanji (1992) и Chen (2008).