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EXACT VARIATIONS FOR STOCHASTIC HEAT EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS AND APPLICATION TO PARAMETER ESTIMATION

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ABSTRACT. We expand the quartic variations in time and the quadratic variations in space of the solution to a stochastic partial differential equation with piecewise constant coefficients. Both expansions allow us to deduce an estimation method of the parameters appearing in the equation.

Key words and phrases. Quartic and quadratic variations, stochastic partial differential equations, discontinuity, integration techniques, special functions, estimation of parameters.

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1. Introduction

A recent paper [15] introduced the following new stochastic partial differential equation

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x) + \dot{W}(t,x); \quad t>0,\ x\in\mathbb{R}, \\ u(0,.) := 0, \end{cases} \tag{1}$$

where $\mathcal{L}$ is the operator defined by:

$$\mathcal{L} = \frac{1}{2\rho(x)}\frac{d}{dx}\left(\rho(x)A(x)\frac{d}{dx}\right), \tag{2}$$

$$A(x) = a_1\mathbf{1}_{\{x\le 0\}} + a_2\mathbf{1}_{\{0<x\}} \quad \text{and} \quad \rho(x) = \rho_1\mathbf{1}_{\{x\le 0\}} + \rho_2\mathbf{1}_{\{0<x\}}, \tag{3}$$

a_i, ρ_i $(i=1,2)$ are strictly positive constants, and where $\dot{W}$ denotes the formal derivative of a space-time white noise. More precisely, W is a centered Gaussian field $W = \{W(t,C); t\in[0,T], C\in B_b(\mathbb{R})\}$ with covariance

$$\mathbb{E}(W(t,C)W(s,B)) = (t\wedge s)\lambda(C\cap B), \tag{4}$$

where λ denotes the Lebesgue measure. So W behaves as a Wiener process both in time and in space. The solution to equation (1) is a random field $\{u(t,x), t\ge 0, x\in\mathbb{R}\}$, where t represents the time variable and x is the space variable. Equation (1) can be considered as a stochastic counterpart of the deterministic partial differential equation

$$\frac{\partial u(t,x)}{\partial t} = \mathcal{L}u(t,x), \tag{5}$$

which appears in the mathematical modelling of heat propagations in heterogeneous media, consisting of two different materials. This already makes equation (1) interesting to be investigated. The fact that the coefficient A in (2) is piecewise constant reflects the heterogeneity of the medium in which the process under study propagates. Such diffusion phenomena is encountered in many fields, for example, in geophysics [5], ecology [1], biology [6] and so on.

An explicit expression of the fundamental solution of equation (5) was given in [13, 14] and [2]. Equation (1) could be a good model for diffusion phenomena in medium consisting of two kind of materials, undergoing stochastic perturbations. Equation (1) also represents a natural extension of the stochastic heat equation driven by space-time white noise, which has been widely studied in the literature (see [11] and the references therein). This can be considered as supplementary motivation for the investigation of such equation's solution.

In [15], the authors proved the existence of the solution $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ of equation (1), they presented explicit expressions for its covariance in time $\mathrm{Cov}\big(u(t,x), u(s,x)\big)$ and variance functions $\mathbf{V}\big(u(t,x)\big)$ for fixed $x \in \mathbb{R}$, they proved that the mild solution of (1) has a quasi-helix property, from which they deduced that its sample paths $t \longmapsto u(t,x)$ are Hölder continuous, but not differentiable.

We make here another step in the study of stochastic partial differential equations of the form (1). At first, we give an explicit expression of the covariance function $\mathrm{Cov}\big(u(t,x), u(s,y)\big)$, in the more general case where x and y are not necessarily equal. In the particular case where $x = y$ we regain the expression obtained in [15]. We study and we analyze, for fixed $x \in \mathbb{R}$, the quartic variations of the paths $t \longmapsto u(t,x)$, and for fixed $t > 0$, the quadratic variations of the functions $x \longmapsto u(t,x)$. Both obtained expansions are interesting since they allow us to estimate the parameters a_1 and a_2 appearing in (2) and (3). In [8], the authors dealt with the same problem, but in the particular case where the coefficient A is constant; that is $a_1 = a_2$. Our case is more general and its study is more complex. Indeed, the proofs require many integration techniques, calculation, special functions and analysis tools; they are particularly based on the use of the known expression of the fundamental solution of (5).

The paper is organized as follows. In the second section we specify the notion of solution to equation (1) and we give some of its basic characteristics. The next section is devoted to giving explicit expression of the covariance function of the solution to equation (1). Section 4 focuses on obtaining an explicit expansion of the temporal quartic variations. Such expansion allows us to deduce an estimation method for the parameters a_1 and a_2. In the last section, we present an explicit expansion of the spatial quadratic variations, that also allows us to estimate the parameters a_1 and a_2.

2. Preliminaries

The notion of solution to (1) is defined in the mild sense. We call a mild solution to (1) the stochastic process

$$u(t,x) = \int_0^t \int_{\mathbb{R}} G(t-s,x,y) W(ds,dy), \qquad t \in [0,T], x \in \mathbb{R}\,, \tag{6}$$

where W is the Gaussian noise with covariance given by (4), G denotes the fundamental solution of the operator $\mathcal{L}$ and the integral in (6) is a Wiener integral with respect to the Gaussian noise W.

In the following proposition we present the expression of the fundamental solution associated to the SPDE (1). For a proof see e.g. [13, 14] and [2].

Proposition 1. *The fundamental solution G of the partial differential equation (4) is given by*

$$G(t,x,y) = \Bigg[\frac{1}{\sqrt{2\pi t}}\left(\frac{\mathbf{1}_{\{y\leq 0\}}}{\sqrt{a_1}} + \frac{\mathbf{1}_{\{y>0\}}}{\sqrt{a_2}}\right) \times \Bigg\{\exp\left(-\frac{(f(x)-f(y))^2}{2t}\right) + \frac{\sqrt{a_1}+\sqrt{a_2}(\alpha-1)}{\sqrt{a_1}-\sqrt{a_2}(\alpha-1)}\,\mathrm{sign}(y)\,\exp\left(-\frac{(\mid f(x)\mid + \mid f(y)\mid)^2}{2t}\right)\Bigg\}\Bigg]\mathbf{1}_{0<t}, \tag{7}$$

where

$$f(y) = \frac{y}{\sqrt{a_1}}\mathbf{1}_{\{y\leq 0\}} + \frac{y}{\sqrt{a_2}}\mathbf{1}_{\{y>0\}} \ \ \textit{and} \ \ \alpha = 1 - \frac{\rho_1 a_1}{\rho_2 a_2}.$$

It is well known (see e. g. [3] and [4]) that the mild solution to (1) exists when the Wiener integral (6) is well-defined and this happens when the integrand G belongs to $\mathcal{H} = L^2([0,T]\times\mathbb{R})$, the canonical Hilbert space associated with the Gaussian process W. In fact, $\mathcal{H}$ is none other than the closure of the linear span generated by the indicator functions $\mathbf{1}_{[0,t]\times C}$, $t \in [0,T], C \in \mathcal{B}_b(\mathbb{R})$ with respect to the inner product

$$< 1_{[0,t]\times C}, 1_{[0,s]\times B} >_{\mathcal{H}} = (t\wedge s)\lambda(C\cap B).$$

The Wiener integral acts as an isometry between the Hilbert space $\mathcal{H}$ and $L^2(\Omega)$ in the sense that

$$\mathbf{E}\left(\int_0^T\int_{\mathbb{R}} \varphi(u,y)W(du,dy)\int_0^T\int_{\mathbb{R}} \psi(u,y)W(du,dy)\right) = \int_0^T\int_{\mathbb{R}} \varphi(u,y)\psi(u,y)du\,dy \quad (8)$$

for any functions φ, ψ satisfying

$$\int_0^T\int_{\mathbb{R}} |\varphi(u,y)\psi(u,y)|\,du\,dy < \infty.$$

The following theorem was obtained in [15].

Theorem 2. *The solution u of equation (1) exists, it is a centered Gaussian process and, for every $x\in\mathbb{R}, s,t\geq 0$:*

$$\mathbb{E}(u(t,x)u(s,x)) \quad = \quad \int_0^{t\wedge s} \frac{1}{\sqrt{2\pi A(x)(t+s-2u)}}\,F(x,s,t,\beta,u)\,du\,, \quad (9)$$

where

$$F(x,s,t,\beta,u) = 1+\gamma(x)\,\mathbf{erfc}\left(|x|\sqrt{\frac{(t+s-2u)}{2A(x)(t-u)(s-u)}}\right) + \mathrm{sign}(x)\,\beta\,\exp\left(\frac{-2\,x^2}{A(x)(t+s-2u)}\right), \quad (10)$$

$$A(x) = a_1\,\mathbf{1}_{\{x\leq 0\}} + a_2\,\mathbf{1}_{\{x>0\}},\ \beta = \frac{\sqrt{a_1}+\sqrt{a_2}(\alpha-1)}{\sqrt{a_1}-\sqrt{a_2}(\alpha-1)},\ \mathrm{sign}(y) = \begin{cases} +1 & \textit{if}\, y>0, \\ -1 & \textit{if}\, y\leq 0, \end{cases} \quad (11)$$

$$\gamma(x) = \frac{1}{2}\left(\sqrt{\frac{A(\mathrm{sign}(x))}{A(-(\mathrm{sign}(x))}}(1-\beta\,\mathrm{sign}(x))^2 - (1-\beta^2)\right), \quad (12)$$

and **erfc** *denotes the complementary error function defined by*

$$z\in\mathbb{R}\longmapsto \mathbf{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^{+\infty} e^{-u^2}du.$$

In the following we denote

$$\lambda = \frac{1}{2}\left(1+\beta-\sqrt{\frac{a_2}{a_1}}\,(1-\beta)\right).$$

3. Explicit calculation of covariance

For $x,y\in\mathbb{R}$, $s,t\in[0,T]$, $u\in[0,s\wedge t]$ we denote

$$\xi_1(t,x,s,y,u) = 1+\,\gamma(x)\;\mathbf{erfc}\left(\frac{|f(x)|(s-u)+|f(y)|(t-u)}{\sqrt{2(t+s-2u)(t-u)(s-u)}}\right),$$

and

$$\xi_2(t,x,s,y,u) = \exp\left(\frac{-(x-y)^2}{2(t+s-2u)}\right)\;\mathbf{erfc}\left(\frac{x(s-u)+y(t-u)}{\sqrt{2(t+s-2u)(s-u)(t-u)}}\right).$$

Theorem 3. *For every $t, s \in]0, T]$ and $x, y \in \mathbb{R}$ we have*

$$\mathbb{E}(u(t,x)u(s,y)) = \int_0^{t\wedge s} \frac{1}{\sqrt{2\pi(t+s-2u)}}\, \mathcal{H}(t,x,s,y,u)\, du\,, \tag{13}$$

where

$$\mathcal{H}(t,x,s,y,u) = \begin{cases} \dfrac{1}{\sqrt{A(x)}}\Bigg\{\exp\Bigg(-\dfrac{(f(x)-f(y))^2}{2(t+s-2u)}\Bigg)\xi_1(t,x,s,y,u) \\ \quad + \beta\ \text{sign}(x)\exp\Bigg(-\dfrac{(f(x)+f(y))^2}{2(t+s-2u)}\Bigg)\Bigg\}, & \textit{if } xy \geq 0, \\ \dfrac{1}{\sqrt{A(|\, x\,|)}}\Bigg\{(1+\beta)\ \exp\Bigg(\dfrac{-(f(x)-f(y))^2}{2(t+s-2u)}\Bigg) \\ \quad - \lambda\xi_2\Big(t, f(x), s, f(y), u\Big) + \beta\lambda\xi_2\Big(t, |f(x)|, s, |f(y)|, u\Big)\Bigg\}, & \textit{if not.} \end{cases}$$

Proof. We present the proof just in the case where $xy < 0$. The other case can be obtained by similar calculation. Using Wiener's isometry (8) we get

$$\begin{aligned} \mathbb{E}\Big(u(t,x)u(s,y)\Big) &= \\ &= \mathbb{E}\Bigg(\int_{(0,t)\times\mathbb{R}} G(t-u,x,v)\, W(dv,du) \times \int_{(0,s)\times\mathbb{R}} G(s-u,y,v)\, W(dv,du)\Bigg) \\ &= \int_0^{t\wedge s}\int_{\mathbb{R}} G(t-u,x,v)\, G(s-u,y,v)\, dv\, du \\ &= A(t,s,x,y) + B(t,s,x,y) \end{aligned} \tag{14}$$

with $\quad A(t,s,x,y) = \displaystyle\int_0^{t\wedge s}\int_0^{+\infty} G(t-u,x,v)\, G(s-u,y,v)\, dv\, du \quad$ and

$$B(t,s,x,y) = \int_0^{t\wedge s}\int_{-\infty}^{0} G(t-u,x,v)\, G(s-u,y,v)\, dv\, du.$$

To calculate $A(t,s,x,y)$ and $B(t,s,x,y)$ we discuss separately the cases $x > 0, y < 0$ and $x < 0, y > 0$.

If $x > 0$ and $y < 0$ then

$$\begin{aligned} A(t,s,x,y) &= \int_0^{t\wedge s}\int_0^{+\infty} \frac{1}{\sqrt{2\pi a_2(t-u)}}\Bigg\{\exp\Bigg(-\frac{\Big(\frac{x}{\sqrt{a_2}} - \frac{v}{\sqrt{a_2}}\Big)^2}{2(t-u)}\Bigg) \\ &\quad + \beta\exp\Bigg(-\frac{\Big(\frac{x}{\sqrt{a_2}} + \frac{v}{\sqrt{a_2}}\Big)^2}{2(t-u)}\Bigg)\Bigg\}\frac{1+\beta}{\sqrt{2\pi a_2(s-u)}}\exp\Bigg(-\frac{\Big(\frac{y}{\sqrt{a_1}} - \frac{v}{\sqrt{a_2}}\Big)^2}{2(s-u)}\Bigg) dvdu \\ &= \int_0^{t\wedge s} \frac{1+\beta}{2\pi a_2\sqrt{(t-u)(s-u)}}\Bigg\{A_1(t,s,x,y) + \beta\, A_2(t,s,x,y)\Bigg\}du \end{aligned}$$

with

$$A_1(t,s,x,y) = \int_0^{+\infty} \exp\left(-\frac{\left(\frac{x}{\sqrt{a_2}} - \frac{v}{\sqrt{a_2}}\right)^2}{2(t-u)}\right) \exp\left(-\frac{\left(\frac{y}{\sqrt{a_1}} - \frac{v}{\sqrt{a_2}}\right)^2}{2(s-u)}\right) dv$$

and

$$A_2(t,s,x,y) = \int_0^{+\infty} \exp\left(-\frac{\left(\frac{x}{\sqrt{a_2}} + \frac{v}{\sqrt{a_2}}\right)^2}{2(t-u)}\right) \exp\left(-\frac{\left(\frac{y}{\sqrt{a_1}} - \frac{v}{\sqrt{a_2}}\right)^2}{2(s-u)}\right) dv.$$

Making the change of variables $V = \frac{x}{\sqrt{a_2}} - \frac{v}{\sqrt{a_2}}$ then using Lemma 15 in Annex we get

$$\begin{aligned} A_1(t,s,x,y) &= \sqrt{a_2}\int_{-\infty}^{\frac{x}{\sqrt{a_2}}} \exp\left(-\frac{V^2}{2(t-u)}\right) \exp\left(-\frac{\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}} - V\right)^2}{2(s-u)}\right) dV \\ &= \sqrt{\frac{\pi\, a_2(t-u)(s-u)}{2(t+s-2u)}} \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}}\right)^2}{2(t+s-2u)}\right) \\ &\quad \times \mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a_2(t-u)(s-u)}}\right). \end{aligned}$$

Now by the change of variables $V = \frac{x+v}{\sqrt{a_2}}$ and using Lemma 16 in Annex we get

$$\begin{aligned} A_2(t,s,x,y) &= \sqrt{a_2}\int_{\frac{x}{\sqrt{a_2}}}^{+\infty} \exp\left(-\frac{V^2}{2(t-u)}\right) \exp\left(-\frac{\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}} - V\right)^2}{2(s-u)}\right) dV \\ &= \sqrt{\frac{\pi\, a_2(t-u)(s-u)}{2(t+s-2u)}} \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}}\right)^2}{2(t+s-2u)}\right) \\ &\quad \times \left[2 - \mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a_2(t-u)(s-u)}}\right)\right]. \end{aligned}$$

Thus,

$$A(t,s,x,y) = \int_0^{t\wedge s} \frac{(1+\beta)\,du}{2\sqrt{2\pi a_2(t+s-2u)}} \left\{ \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}}\right)^2}{2(t+s-2u)} \right) \right.$$

$$\times\, \mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a_2(t-u)(s-u)}} \right)$$

$$+\, \beta\ \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}}\right)^2}{2(t+s-2u)} \right)$$

$$\left. \times \left[2 - \mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a_2(t-u)(s-u)}} \right) \right] \right\}$$

$$= \int_0^{t\wedge s} \frac{(1+\beta)\,du}{2\sqrt{2\pi a_2(t+s-2u)}} \left\{ \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}}\right)^2}{2(t+s-2u)} \right) \right.$$

$$\times \left[2 - \mathbf{erfc}\left(\frac{\frac{x}{\sqrt{a_2}}(s-u) + \frac{y}{\sqrt{a_1}}(t-u)}{\sqrt{2(t+s-2u)(t-u)(s-u)}} \right) \right]$$

$$\left. +\ \beta \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}}\right)^2}{2(t+s-2u)} \right) \mathbf{erfc}\left(\frac{\frac{x}{\sqrt{a_2}}(s-u) - \frac{y}{\sqrt{a_1}}(t-u)}{\sqrt{2(t+s-2u)(t-u)(s-u)}} \right) \right\}, \tag{15}$$

where in the last equality we used the fact that

$$\mathbf{erfc}(x) + \mathbf{erfc}(-x) = 2 \text{ for every } x \in \mathbb{R}.$$

The same techniques allow us to get

$$B(t,s,x,y) = \int_0^{t\wedge s} \frac{1-\beta}{2\sqrt{2\pi a_1(t+s-2u)}} \left\{ \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}}\right)^2}{2(t+s-2u)} \right) \right.$$

$$\times \left[2 - \mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a_2(t-u)(s-u)}} \right) \right]$$

$$
- \beta \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}} \right)^2}{2(t+s-2u)} \right)
$$

$$
\times \left[2 - \mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}} \right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a_2(t-u)(s-u)}} \right) \right] \right\} du
$$

$$
= \int_0^{t\wedge s} \frac{(1-\beta)}{2\sqrt{2\pi a_2(t+s-2u)}} \sqrt{\frac{a_2}{a_1}} \times \left\{ \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}} \right)^2}{2(t+s-2u)} \right) \right.
$$

$$
\times \mathbf{erfc}\left(\frac{\frac{x}{\sqrt{a_2}}(s-u) + \frac{y}{\sqrt{a_1}}(t-u)}{\sqrt{2(t+s-2u)(t-u)(s-u)}} \right) - \beta \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}} \right)^2}{2(t+s-2u)} \right)
$$

$$
\left. \times \mathbf{erfc}\left(\frac{\frac{x}{\sqrt{a_2}}(s-u) - \frac{y}{\sqrt{a_1}}(t-u)}{\sqrt{2(t+s-2u)(t-u)(s-u)}} \right) \right\} du. \tag{16}
$$

Gathering expressions (14), (15) and (16) we get

$$
\mathbb{E}u(t,x)u(s,y) = \int_0^{t\wedge s} \frac{du}{\sqrt{2\pi a_2(t+s-2u)}} \left\{ (1+\beta)\exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}} \right)^2}{2(t+s-2u)} \right) \right.
$$

$$
- \frac{1}{2}\left(1+\beta - \sqrt{\frac{a_2}{a_1}}(1-\beta) \right) \left[\exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} - \frac{y}{\sqrt{a_1}} \right)^2}{2(t+s-2u)} \right) \right.
$$

$$
\times \mathbf{erfc}\left(\frac{\frac{x}{\sqrt{a_2}}(s-u) + \frac{y}{\sqrt{a_1}}(t-u)}{\sqrt{2(t+s-2u)(t-u)(s-u)}} \right)
$$

$$
\left. \left. - \beta \exp\left(\frac{-\left(\frac{x}{\sqrt{a_2}} + \frac{y}{\sqrt{a_1}} \right)^2}{2(t+s-2u)} \right) \mathbf{erfc}\left(\frac{\frac{x}{\sqrt{a_2}}(s-u) - \frac{y}{\sqrt{a_1}}(t-u)}{\sqrt{2(t+s-2u)(t-u)(s-u)}} \right) \right] \right\},
$$

that is equivalent to expression (13) in the case where $x > 0$ and $y < 0$.

If $x < 0$ and $y > 0$, the same calculation method gives:

$$
A(t,s,x,y) = \int_0^{t\wedge s} \frac{1+\beta}{2\sqrt{2\pi a_2(t+s-2u)}} \left\{ \exp\left(\frac{-\left(\frac{x}{\sqrt{a_1}} - \frac{y}{\sqrt{a_2}} \right)^2}{2(t+s-2u)} \right) \right.
$$

$$
\times \mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_1}} - \frac{y}{\sqrt{a_2}} \right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a_1(t-u)(s-u)}} \right)
$$

$$+\beta\exp\left(\frac{-\left(\frac{x}{\sqrt{a_1}}+\frac{y}{\sqrt{a_2}}\right)^2}{2(t+s-2u)}\right)$$

$$\times\,\mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_1}}+\frac{y}{\sqrt{a_2}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}}-\frac{x\sqrt{t+s-2u}}{\sqrt{2a_1(t-u)(s-u)}}\right)\Bigg\}du \tag{17}$$

and

$$B(t,s,x,y)=\int_0^{t\wedge s}\frac{1-\beta}{2\sqrt{2\pi a_1(t+s-2u)}}\Bigg\{\exp\left(\frac{-\left(\frac{x}{\sqrt{a_1}}-\frac{y}{\sqrt{a_2}}\right)^2}{2(t+s-2u)}\right)$$

$$\times\left[2-\mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_1}}-\frac{y}{\sqrt{a_2}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}}-\frac{x\sqrt{t+s-2u}}{\sqrt{2a_1(t-u)(s-u)}}\right)\right]$$

$$-\beta\exp\left(\frac{-\left(\frac{x}{\sqrt{a_1}}+\frac{y}{\sqrt{a_2}}\right)^2}{2(t+s-2u)}\right)$$

$$\times\,\mathbf{erfc}\left(\frac{\left(\frac{x}{\sqrt{a_1}}+\frac{y}{\sqrt{a_2}}\right)\sqrt{t-u}}{\sqrt{2(t+s-2u)(s-u)}}-\frac{x\sqrt{t+s-2u}}{\sqrt{2a_1(t-u)(s-u)}}\right)\Bigg\}\,du. \tag{18}$$

Thus, gathering expressions (14), (17) and (18), if $x<0$ and $y>0$ we also get Theorem 3. □

Remarks 4. If we put $x=y$ in (13) , we clearly regain expression (9) obtained in [15].

In the particular case where $x=y$ and $s=t$, equation (13) leads to the variance expression:

$$\mathbb{E}u^2(t,x):=\Bigg\{\sqrt{\frac{t}{\pi A(x)}}+\gamma(x)\left[\sqrt{\frac{t}{\pi A(x)}}\,\mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{t\,A(x)}}\right)-\frac{|\,x\,|}{A(x)\pi}\mathbf{E}_1\left(\frac{x^2}{t\,A(x)}\right)\right]$$

$$+\beta\,\mathrm{sign}(x)\left[\sqrt{\frac{t}{\pi A(x)}}\exp\left(-\frac{x^2}{tA(x)}\right)-\frac{|\,x\,|}{A(x)}\,\mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right)\right]\Bigg\}\mathbf{1}_{\{\mathbf{x}\neq\mathbf{0}\}}$$

$$+\eta\sqrt{\frac{t}{\pi A(x)}}\mathbf{1}_{\{\mathbf{x}=\mathbf{0}\}} \tag{19}$$

with

$$\eta=\frac{1}{2}\left((1-\beta)^2+\sqrt{\frac{a_1}{a_2}}\,(1+\beta)^2\right),$$

where $\mathbf{E_1}$ is the exponential integral defined by $\mathbf{E}_1(x):=\int_x^\infty\frac{e^{-t}}{t}\,dt.$

We also note that in the particular case where $a_1=a_2=1$ and $\rho_1=\rho_2$, we have $\beta=0$, $\lambda=0$, $\eta=1$, $\gamma(x)=0$ for every $x\in\mathbb{R}$ and $\xi_1:=1$. Consequently, on the one hand, equation (13) is reduced to:

$$\mathbb{E}(u(t,x)u(s,y)) = \int_0^{t\wedge s} \frac{1}{\sqrt{2\pi(t+s-2u)}} \exp\left(-\frac{(x-y))^2}{2(t+s-2u)}\right) du$$

for every $x, y \in \mathbb{R}$, leading to exactly the same covariance expression of the solution to the heat equation obtained in [8]. On the other hand, equation (19) becomes

$$\mathbb{E}u^2(t,x) = \sqrt{\frac{t}{\pi}},$$

for every $x \in \mathbb{R}$, what also corresponds to the result obtained for the standard heat equation in [8] and [10].

4. Temporal variation

Let us fix $x \in \mathbb{R}$ and $0 < T_1 < T_2$. Define, for $\delta > 0$ and $s, t \in [T_1, T_2]$, the jointly Gaussian increments

$$\Delta(s,\delta) = u(s+\delta,x) - u(s,x), \quad \Delta(t,\delta) = u(t+\delta,x) - u(t,x).$$

Lemma 5. *For every $s \in [T_1, T_2]$ and small $\delta > 0$ we have*

$$\mathbb{E}\Delta^2(s,\delta) = \delta^{1/2}\sqrt{\frac{2\tau(x)}{\pi A(x)}} + O(\delta^2) \tag{20}$$

with

$$\tau(x) = \eta^2 \mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}} \tag{21}$$

and where $O(\delta^2)$ denotes a quantity that is bounded by $M\delta^2$ if δ is close enough to 0, with the same constant M for all $s \in [T_1, T_2]$.

Proof. We have

$$\begin{aligned}\mathbb{E}[\Delta^2(s,\delta)] &= \mathbb{E} \mid u(s+\delta,x) - u(s,x) \mid^2 \\ &= \mathbb{E}\Big(u^2(s+\delta,x)\Big) - 2\,\mathbb{E}\Big(u(s+\delta,x)u(s,x)\Big) + \mathbb{E}\Big(u^2(s,x)\Big).\end{aligned} \tag{22}$$

First case. If $x = 0$. Using the variance expression (19) and applying Taylor's formula we get

$$\mathbb{E}u^2(s,0) = \eta\sqrt{\frac{s}{\pi a_1}}$$

and

$$\mathbb{E}u^2(s+\delta,0) = \eta\sqrt{\frac{s+\delta}{\pi a_1}} = \eta\left[\sqrt{\frac{s}{\pi a_1}} + \frac{\delta}{2\sqrt{\pi a_1 s}}\right] + O(\delta^2).$$

Now by the covariance expression (13) and again applying Taylor's formula we get

$$\begin{aligned}\mathbb{E}\Big(u(s+\delta,0)u(s,0)\Big) &= \int_0^s \frac{1+\gamma(0)-\beta}{\sqrt{2\pi A(x)(2s+\delta-2u)}}\, du \\ &= \frac{\eta}{\sqrt{2\pi a_1}}\left[\sqrt{2s+\delta} - \sqrt{\delta}\right] \\ &= \frac{\eta}{\sqrt{2\pi a_1}}\left[-\sqrt{\delta} + \sqrt{2s} + \frac{\delta}{2\sqrt{2}\sqrt{s}}\right] + O(\delta^2).\end{aligned}$$

All this imply that

$$\mathbb{E}\Delta^2(s,x) = \eta\sqrt{\frac{2}{\pi A(x)}}\,\delta^{1/2} + O(\delta^2).$$

Second case. If $x \neq 0$.

$$\begin{aligned}
\mathbb{E}u^2(s+\delta,x) &= \underbrace{\sqrt{\frac{s+\delta}{\pi A(x)}}}_{F_1(\delta)} \\
&+ \gamma(x)\Bigg[\underbrace{\sqrt{\frac{s+\delta}{\pi A(x)}}\,\mathbf{erfc}\left(\frac{|x|}{\sqrt{(s+\delta)A(x)}}\right)}_{F_2(\delta)} - \underbrace{\frac{|x|}{A(x)\pi}\,\mathbf{E}_1\left(\frac{x^2}{(s+\delta)A(x)}\right)}_{F_3(\delta)}\Bigg] \\
&+ \beta\,\mathrm{sign}(x)\Bigg[\underbrace{\sqrt{\frac{s+\delta}{\pi A(x)}}\,\exp\left(-\frac{x^2}{(s+\delta)A(x)}\right)}_{F_4(\delta)} - \underbrace{\frac{|x|}{A(x)}\,\mathbf{erfc}\left(\frac{|x|}{\sqrt{(s+\delta)A(x)}}\right)}_{F_5(\delta)}\Bigg].
\end{aligned} \tag{23}$$

Applying Taylor's formula as $\delta \to 0$ for each $F_i(\delta)$; $i \in \{1,...,5\}$ we get

$$\begin{aligned}
F_1(\delta) &= \sqrt{\frac{s}{\pi A(x)}} + \frac{\delta}{2\sqrt{\pi A(x)\,s}} + O(\delta^2), \\
F_2(\delta) &= \sqrt{\frac{s}{\pi A(x)}}\,\mathbf{erfc}\left(\frac{|x|}{\sqrt{A(x)s}}\right) + \frac{\delta}{2\sqrt{\pi A(x)\,s}}\,\mathbf{erfc}\left(\frac{|x|}{\sqrt{A(x)\,s}}\right) \\
&\quad + \frac{|x|\,\delta}{s\,A(x)\pi}\,\exp\left(-\frac{x^2}{s\,A(x)}\right) + O(\delta^2), \\
F_3(\delta) &= \frac{|x|}{A(x)\pi}\,\mathbf{E}_1\left(\frac{x^2}{s\,A(x)}\right) + \frac{|x|\delta}{s\,A(x)\,\pi}\,\exp\left(\frac{-x^2}{s\,A(x)}\right) + O(\delta^2), \\
F_4(\delta) &= \sqrt{\frac{s}{\pi A(x)}}\,\exp\left(-\frac{x^2}{s\,A(x)}\right) + \frac{x^2\,\delta}{A(x)^{3/2}\sqrt{\pi}}\,s^{-3/2}\,\exp\left(-\frac{x^2}{s\,A(x)}\right) \\
&\quad + \frac{\delta}{2\sqrt{s\,\pi A(x)}}\exp\left(-\frac{x^2}{s\,A(x)}\right) + O(\delta^2)
\end{aligned}$$

and

$$F_5(\delta) = \frac{|x|}{A(x)}\,\mathbf{erfc}\left(\frac{|x|}{\sqrt{s\,A(x)}}\right) + \frac{x^2\delta\,s^{-3/2}}{A(x)^{3/2}\sqrt{\pi}}\exp\left(-\frac{x^2}{s\,A(x)}\right) + O(\delta^2).$$

To get the expansion, for example of $F_3(\delta)$, we have used the fact that $\mathbf{E}_1$ is twice continuously differentiable on $(0,+\infty)$, the Taylor's formula, and $\mathbf{E}_1'(x) = -\dfrac{e^{-x}}{x}$ for $x > 0$.

Consequently, we get

$$\begin{aligned}
\mathbb{E}u^2(s+\delta,x) &= \sqrt{\frac{s}{\pi A(x)}} + \frac{\delta}{2\sqrt{\pi A(x)\,s}} \\
&\quad + \gamma(x)\Bigg[\sqrt{\frac{s}{\pi A(x)}}\,\mathbf{erfc}\left(\frac{|x|}{\sqrt{A(x)s}}\right) + \frac{\delta}{2\sqrt{\pi A(x)\,s}}\,\mathbf{erfc}\left(\frac{|x|}{\sqrt{A(x)s}}\right) \\
&\quad - \frac{|x|}{A(x)\,\pi}\,\mathbf{E}_1\left(\frac{x^2}{s\,A(x)}\right)\Bigg]
\end{aligned}$$

$$+ \beta \,\mathrm{sign}(x)\Bigg[\sqrt{\frac{s}{\pi A(x)}}\exp\Big(-\frac{x^2}{sA(x)}\Big) + \frac{\delta}{2\sqrt{\pi A(x) s}}\exp\Big(-\frac{x^2}{sA(x)}\Big) - \frac{|x|}{A(x)}\ \mathbf{erfc}\Big(\frac{|x|}{\sqrt{sA(x)}}\Big)\Bigg] + O(\delta^2). \tag{24}$$

We also have

$$\mathbb{E}u(s+\delta,x)u(s,x) = \underbrace{\int_0^s \frac{1}{\sqrt{2A(x)\pi(2s+\delta-2u)}}\,du}_{I_1(\delta)}$$
$$+ \gamma(x)\underbrace{\int_0^s \frac{1}{\sqrt{2A(x)\pi(2s+\delta-2u)}}\ \mathbf{erfc}\Big(|\,x\,|\sqrt{\frac{2s+\delta-2u}{2A(x)(s+\delta-u)(s-u)}}\Big)du}_{I_2(\delta)}$$
$$+ \beta\ \mathrm{sign}(x)\underbrace{\int_0^s \frac{1}{\sqrt{2A(x)\pi(2s+\delta-2u)}}\exp\Big(\frac{-2\,x^2}{A(x)(2s+\delta-2u)}\Big)du}_{I_3(\delta)}.$$

An easy calculation allows us to get

$$I_1(\delta) = \frac{1}{\sqrt{2\pi A(x)}}\Big(-\sqrt{\delta} + (2s+\delta)^{1/2}\Big)$$
$$= \frac{1}{\sqrt{2\pi\, A(x)}}\Big(-\sqrt{\delta} + \sqrt{2s} + \frac{\delta}{2\sqrt{2}\sqrt{s}} + O(\delta^2)\Big).$$

Then, on the one hand we have

$$I_2(0) = \int_0^s \frac{1}{2\sqrt{A(x)\pi(s-u)}}\ \mathbf{erfc}\Big(\frac{|\,x\,|}{A(x)\sqrt{s-u}}\Big)du$$
$$= \sqrt{\frac{s}{\pi\, A(x)}}\ \mathbf{erfc}\Big(\frac{|\,x\,|}{\sqrt{s\, A(x)}}\Big) - \frac{|\,x\,|}{\pi\, A(x)}\ \mathbf{E}_1\Big(\frac{x^2}{s\, A(x)}\Big).$$

On the other hand, after permutation of the symbols derivative and integral we get

$$I_2'(0) = \int_0^s -\frac{1}{2^3\sqrt{A(x)\pi}}\,(s-u)^{-3/2}\ \mathbf{erfc}\Big(\frac{|\,x\,|}{\sqrt{A(x)(s-u)}}\Big)$$
$$+ \frac{|\,x\,|}{4\pi\, A(x)}(s-u)^{-2}\exp\Big(\frac{-x^2}{A(x)(s-u)}\Big)du$$
$$= \frac{1}{4\sqrt{\pi\, A(x) s}}\ \mathbf{erfc}\Big(\frac{|\,x\,|}{\sqrt{A(x)s}}\Big),$$

where the last equality is obtained after an integration by parts.

Therefore, by Taylor's formula,

$$I_2(\delta) = \sqrt{\frac{s}{\pi\, A(x)}}\ \mathbf{erfc}\Big(\frac{|\,x\,|}{\sqrt{s\, A(x)}}\Big) - \frac{|\,x\,|}{\pi\, A(x)}\ \mathbf{E}_1\Big(\frac{x^2}{s\, A(x)}\Big)$$
$$+ \frac{1}{4\sqrt{\pi\, A(x) s}}\ \mathbf{erfc}\Big(\frac{|\,x\,|}{\sqrt{A(x)s}}\Big)\delta + O(\delta^2).$$

Following the same steps as before we also get

$$I_3(\delta) = \sqrt{\frac{s}{\pi A(x)}} \exp\left(\frac{-x^2}{s A(x)}\right) - \frac{|x|}{A(x)} \ \mathbf{erfc}\left(\frac{|x|}{\sqrt{s A(x)}}\right) + \frac{s^{-1/2}\,\delta}{4\sqrt{\pi A(x)}} \exp\left(\frac{-x^2}{A(x)\, s}\right) + O(\delta^2)$$

and consequently

$$\begin{aligned}
\mathbb{E}(u(s+\delta,x)u(s,x)) &= \frac{-\delta^{1/2}}{\sqrt{2\pi A(x)}} + \sqrt{\frac{s}{\pi A(x)}} + \frac{\delta}{4\sqrt{\pi A(x)\, s}} \\
&+ \gamma(x)\left[\sqrt{\frac{s}{\pi A(x)}} \ \mathbf{erfc}\left(\frac{|x|}{\sqrt{s A(x)}}\right) - \frac{|x|}{\pi A(x)} \ \mathbf{E}_1\left(\frac{x^2}{s A(x)}\right)\right. \\
&+ \left.\frac{\delta}{4\sqrt{\pi A(x) s}} \ \mathbf{erfc}\left(\frac{|x|}{\sqrt{A(x) s}}\right)\right] \\
&+ \beta \ \mathrm{sign}(x)\left[\sqrt{\frac{s}{\pi A(x)}} \exp\left(\frac{-x^2}{s A(x)}\right) - \frac{|x|}{A(x)} \ \mathbf{erfc}\left(\frac{|x|}{\sqrt{s A(x)}}\right)\right. \\
&+ \left.\frac{\delta}{4\sqrt{s \pi A(x)}} \exp\left(\frac{-x^2}{A(x)\, s}\right)\right] + O(\delta^2). \qquad (25)
\end{aligned}$$

Gathering equalities (22), (23), (24) and (25) we get

$$\mathbb{E}\Delta^2(s,\delta) = \delta^{1/2}\sqrt{\frac{2}{\pi A(x)}} + O(\delta^2). \qquad \square$$

Lemma 6. *For fixed* $x \in \mathbb{R}$ *and times* $0 < T_1 < T_2$, *if* $t, s \in [T_1, T_2]$ *and* $0 < \delta \leq t - s$ *then we have*

$$\mathbb{E}\big[\Delta(s,\delta)\Delta(t,\delta)\big] = O\big(\delta^2\big) + O\Big((t-s)^{-3/2}\delta^2\Big), \qquad (26)$$

where $O((t-s)^{-3/2}\delta^2)$ *denotes a term bounded by* $M(t-s)^{-3/2}\delta^2$, *with the same constant* M *for all* $s \in [T_1, T_2]$.

Proof. From the covariance expression (13),

$$\begin{aligned}
\mathbb{E}\big[\Delta(s,\delta)\Delta(t,\delta)\big] &= \mathbb{E}\big[\big(u(s+\delta,x) - u(s,x)\big)\big(u(t+\delta,x) - u(t,x)\big)\big] \\
&= \mathbb{E}u(s+\delta,x)u(t+\delta,x) - \mathbb{E}u(s+\delta,x)u(t,x) \\
&\quad - \mathbb{E}u(s,x)u(t+\delta,x) + \mathbb{E}u(s,x)u(t,x) \\
&= H_1(\delta) + \gamma(x)\, H_2(\delta) + \beta \ \mathrm{sign}(x)\, H_3(\delta), \qquad (27)
\end{aligned}$$

where

$$\begin{aligned}
H_1(\delta) = \frac{1}{\sqrt{2\pi A(x)}}\Bigg(&\int_0^{s+\delta}\left[\frac{1}{\sqrt{t+s+2\delta-2u}} - \frac{1}{\sqrt{t+s+\delta-2u}}\right]du \\
&+ \int_0^{s}\left[\frac{1}{\sqrt{t+s-2u}} - \frac{1}{\sqrt{t+s+\delta-2u}}\right]du\Bigg),
\end{aligned}$$

$$
\begin{aligned}
H_2(\delta) = \frac{1}{\sqrt{2\,\pi\,A(x)}}\Bigg(\int_0^{s+\delta}\Bigg[&\frac{1}{\sqrt{(t+s+2\delta-2u)}}\\
&\times\ \mathbf{erfc}\Bigg(|\,x\,|\sqrt{\frac{t+s+2\delta-2u}{2A(x)(s+\delta-u)(t+\delta-u)}}\Bigg)\\
&-\frac{1}{\sqrt{(t+s+\delta-2u)}}\ \mathbf{erfc}\Bigg(|\,x\,|\sqrt{\frac{t+s+\delta-2u}{2A(x)(s+\delta-u)(t-u)}}\Bigg)\Bigg]du\\
&+\int_0^{s}\Bigg[\frac{1}{\sqrt{(t+s-2u)}}\ \mathbf{erfc}\Bigg(|\,x\,|\sqrt{\frac{t+s-2u}{2A(x)(s-u)(t-u)}}\Bigg)\\
&-\frac{1}{\sqrt{(t+s+\delta-2u)}}\ \mathbf{erfc}\Bigg(|\,x\,|\sqrt{\frac{t+s+\delta-2u}{2A(x)(s-u)(t+\delta-u)}}\Bigg)\Bigg]du\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
H_3(\delta) = \frac{1}{\sqrt{2\pi A(x)}}\Bigg(\int_0^{s+\delta}\Bigg[&\frac{1}{\sqrt{(t+s+2\delta-2u)}}\ \exp\Bigg(\frac{-2\,x^2}{A(x)(t+s+2\delta-2u)}\Bigg)\\
&-\frac{1}{\sqrt{t+s+\delta-2u}}\ \exp\Bigg(\frac{-2\,x^2}{A(x)(t+s+\delta-2u)}\Bigg)\Bigg]du\\
&+\int_0^{s}\Bigg[\frac{1}{\sqrt{(t+s-2u)}}\ \exp\Bigg(\frac{-2\,x^2}{A(x)(t+s-2u)}\Bigg)\\
&-\frac{1}{\sqrt{t+s+\delta-2u}}\ \exp\Bigg(\frac{-2\,x^2}{A(x)(t+s+\delta-2u)}\Bigg)\Bigg]du\Bigg).
\end{aligned}
$$

An integration calculation with Taylor's expansion for the function $x \longmapsto \sqrt{1+x}$ as $x \to 0$ allow to get

$$
\begin{aligned}
H_1(\delta) &= \frac{1}{\sqrt{2\,\pi\,A(x)}}\Bigg(\sqrt{t+s+2\delta} - 2\sqrt{t+s+\delta} + \sqrt{t+s}\\
&\quad + \sqrt{t-s+\delta} - 2\sqrt{t-s} + \sqrt{t-s-\delta}\Bigg)\\
&= O(\delta^2) + O(\delta^2(t-s)^{-3/2}). \qquad (28)
\end{aligned}
$$

In order to expand $H_2(\delta)$, we first write $H_2(\delta) = H_{2,1}(\delta) + H_{2,2}(\delta)$, where

$$
\begin{aligned}
H_{2,1}(\delta) = \frac{1}{\sqrt{2\,\pi\,A(x)}}\int_0^{s+\delta}\Bigg[&\frac{1}{\sqrt{(t+s+2\delta-2u)}}\\
&\times\ \mathbf{erfc}\Bigg(|\,x\,|\sqrt{\frac{t+s+2\delta-2u}{2A(x)(s+\delta-u)(t+\delta-u)}}\Bigg)\\
&-\frac{1}{\sqrt{(t+s+\delta-2u)}}\ \mathbf{erfc}\Bigg(|\,x\,|\sqrt{\frac{t+s+\delta-2u}{2A(x)(s+\delta-u)(t-u)}}\Bigg)\Bigg]du
\end{aligned}
$$

and

$$H_{2,2}(\delta) = \frac{1}{\sqrt{2\,\pi\,A(x)}} \int_0^s \Bigg[\frac{1}{\sqrt{(t+s-2u)}}\ \mathbf{erfc}\Bigg(|\,x\,| \sqrt{\frac{t+s-2u}{2A(x)(s-u)(t-u)}}\ \Bigg) - \frac{1}{\sqrt{(t+s+\delta-2u)}}\ \mathbf{erfc}\Bigg(|\,x\,| \sqrt{\frac{t+s+\delta-2u}{2A(x)(s-u)(t+\delta-u)}}\ \Bigg)\Bigg] du.$$

On the one hand we have $H_{2,1}(0) = H_{2,2}(0) = 0$. On the other hand, by the change of variable $w = \delta - u$, we get

$$H_{2,1}(\delta) = \frac{1}{\sqrt{2\pi A(x)}} \Bigg[\int_{-s}^{\delta} \frac{1}{\sqrt{(t+s+2w)}}\, \mathbf{erfc}\Bigg(|\,x\,| \sqrt{\frac{t+s+2w}{2A(x)(s+w)(t+w)}} \Bigg) dw - \int_{-s}^{\delta} \frac{1}{\sqrt{(t+s-\delta+2w)}}\, \mathbf{erfc}\Bigg(|\,x\,| \sqrt{\frac{t+s-\delta+2w}{2A(x)(s+w)(t-\delta+w)}} \Bigg) dw \Bigg].$$

By virtue of [7, Theorem 3, page 425], we get that H_2 is continuously twice differentiable and in particular

$$H'_{2,1}(0) = \frac{1}{\sqrt{2\,\pi\,A(x)}} \int_0^s \Big[-\frac{1}{2}(t+s-2u)^{-3/2}\ \mathbf{erfc}\Bigg(|\,x\,| \sqrt{\frac{t+s-2u}{2A(x)(s-u)(t-u)}}\ \Bigg) + \frac{|\,x\,|}{\sqrt{2\pi\,A(x)}} \frac{(t+s-2u)^{-1}}{(s-u)^{-1/2}(t-u)^{3/2}}\ \exp\Bigg(\frac{-x^2(t+s-2u)}{2A(x)(s-u)(t-u)} \Bigg)\Bigg] du.$$

Thus,

$$H_{2,1}(\delta) = \frac{\delta}{\sqrt{2\pi A(x)}} \int_0^s \Big[-\frac{1}{2}(t+s-2u)^{-3/2}\, \mathbf{erfc}\Bigg(|\,x\,| \sqrt{\frac{t+s-2u}{2A(x)(s-u)(t-u)}}\ \Bigg) + \frac{|\,x\,|}{\sqrt{2\pi\,A(x)}} \frac{(t+s-2u)^{-1}}{(s-u)^{-1/2}(t-u)^{3/2}}\ \exp\Bigg(\frac{-x^2(t+s-2u)}{2A(x)(s-u)(t-u)} \Bigg)\Bigg] du + O(\delta^2).$$

By the same technique we show that

$$H_{2,2}(\delta) = \frac{\delta}{\sqrt{2\,\pi\,A(x)}} \int_0^s \Big[\frac{1}{2}(t+s-2u)^{-3/2}\ \mathbf{erfc}\Bigg(|\,x\,| \sqrt{\frac{t+s-2u}{2A(x)(s-u)(t-u)}}\ \Bigg) - \frac{|\,x\,|}{\sqrt{2\pi\,A(x)}} \frac{(t+s-2u)^{-1}}{(s-u)^{-1/2}(t-u)^{3/2}}\ \exp\Bigg(\frac{-x^2(t+s-2u)}{2A(x)(s-u)(t-u)} \Bigg)\Bigg] du + O(\delta^2)$$

and consequently,

$$H_2(\delta) = O(\delta^2). \tag{29}$$

Following the same steps as above we get

$$H_3(\delta) = O(\delta^2). \tag{30}$$

Combining equalities (27),(28), (29) and (30) the proof of Lemma 6 is achieved. □

Corollary 7. *For fixed $x \in \mathbb{R}$ and times $0 < T_1 < T_2$, if $t, s \in [T_1, T_2]$ and $0 < \delta \le t - s$ then we have*

$$\mathbb{E}\big[\Delta^4(s,\delta)\Delta^4(t,\delta)\big] = \frac{36\delta^2}{A(x)^2\,\pi^2}\Big[\eta^4 \mathbf{1}_{\{\mathbf{x}=\mathbf{0}\}} + \mathbf{1}_{\{\mathbf{x}\neq\mathbf{0}\}}\Big] + O(\delta^{7/2}) + O\Big((t-s)^{-3}\delta^5\Big), \tag{31}$$

where the asymptotic holds uniformly over s, t.

Proof. According to Isserlis theorem [12], for any centered Gaussian random variables X and Y,

$$\mathbb{E}(X^4Y^4) = 9\big(\mathbb{E}(X^2)\,\mathbb{E}(Y^2)\big)^2 + 24\,(\mathbb{E}(XY))^4 + 72\,\mathbb{E}(X^2)\,\mathbb{E}(Y^2)\,(\mathbb{E}(XY))^2.$$

Therefore, using Lemmas 5 and 6 we get

$$\begin{aligned}
\mathbb{E}\Delta^4(t,\delta)\Delta^4(s,\delta) &= 9\Big(\mathbb{E}(\Delta^2(t,\delta))\mathbb{E}(\Delta^2(s,\delta))\Big)^2 \\
&\quad + 24\Big(\mathbb{E}(\Delta(t,\delta)\Delta(s,\delta))\Big)^4 + 72\,\mathbb{E}(\Delta^2(s,\delta))\mathbb{E}(\Delta^2(t,\delta))\Big(\mathbb{E}(\Delta(t,\delta)\Delta(s,\delta))\Big)^2 \\
&= 9\Big(\frac{\sqrt{2}}{\sqrt{A(x)\pi}}\delta^{1/2}\Big[\eta\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}}\Big] + O(\delta^2)\Big)^4 \\
&\quad + 24\Big(O(\delta^2) + O\big((t-s)^{-3/2}\delta^2\big)\Big)^4 \\
&\quad + 72\,\Big(\frac{\sqrt{2}}{\sqrt{A(x)\,\pi}}\,\delta^{1/2}\Big[\eta\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}}\Big] + O(\delta^2)\Big)^2\Big(O(\delta^2) + O\big((t-s)^{-3/2}\delta^2\big)\Big)^2 \\
&= 9\Big(\frac{2\delta}{A(x)\,\pi}\Big[\eta^2\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}}\Big] + O(\delta^4) + O(\delta^{5/2})\Big)^2 \\
&\quad + 24\Big(O(\delta^4) + O\big((t-s)^{-3/2}\delta^4\big) + O\big((t-s)^{-3}\delta^4\big)\Big)^2 \\
&\quad + 72\,\Big(\frac{2\delta}{A(x)\,\pi}\Big[\eta^2\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}}\Big] + O(\delta^4) + O(\delta^{5/2})\Big)\Big(O(\delta^4) + O\big((t-s)^{-3/2}\delta^4\big) \\
&\quad + O\big((t-s)^{-3}\delta^4\big)\Big).
\end{aligned}$$

This, with the fact that for every fractional numbers $0 \le s_1 \le r_1$ and $0 \le r_2 \le s_2$, $O((t-s)^{-r_2}\delta^{r_1})$ is also an $O((t-s)^{-s_2}\delta^{s_1})$, yield

$$\begin{aligned}
\mathbb{E}\Big(\Delta^4(t,\delta)\Delta^4(s,\delta)\Big) &= 9\Big(\frac{4\delta^2}{A(x)^2\pi^2}\Big[\eta^4\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}}\Big] + O(\delta^{7/2})\Big) \\
&\quad + 24\Big(O(\delta^8) + O\big((t-s)^{\frac{-3}{2}}\delta^8\big)\Big) \\
&\quad + 72\Big(\frac{2\delta}{A(x)\pi}\Big[\eta^2\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}}\Big] + O(\delta^{5/2})\Big)\Big(O(\delta^4) + O\big((t-s)^{-3}\delta^4\big)\Big) \\
&= \frac{36\delta^2}{A(x)^2\,\pi^2}\Big[\eta^4\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}}\Big] + O(\delta^{7/2}) + O\Big((t-s)^{-3}\delta^5\Big).
\end{aligned}$$

□

Theorem 8. *Fix $0 < T_1 < T_2$ and $x \in \mathbb{R}$. Define for $j = 0, 1, \dots, n$ a time grid by $t_j = T_1 + j\delta$, where $\delta = \frac{1}{n}(T_2 - T_1)$. Then the following limit holds in mean square:*

$$\lim_{n\longmapsto\infty}\sum_{j=1}^{n}\Big(u(t_j,x) - u(t_{j-1},x)\Big)^4 = \frac{6\tau(x)}{\pi\,A(x)}\,(T_2 - T_1) \tag{32}$$

with

$$\tau(x) = \eta^2\mathbf{1}_{\{\mathbf{x=0}\}} + \mathbf{1}_{\{\mathbf{x\neq 0}\}} \tag{33}$$

Proof. We have

$$\begin{aligned}
&\mathbb{E}\Bigg(\sum_{j=1}^{n}\Delta^4(t_{j-1},\delta) - \frac{6\,\delta\,n}{A(x)\,\pi}\tau(x)\Bigg)^2 \\
&\qquad = \mathbb{E}\Bigg(\sum_{j=1}^{n}\Big(\Delta^4(t_{j-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\tau(x)\Big)\Bigg)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \mathbb{E}\Big(\Delta^4(t_{j-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\tau(x)\Big)^2 \\
&\quad + 2\sum_{i=1}^{n}\sum_{i<j}^{n} \mathbb{E}\Bigg(\Big(\Delta^4(t_{i-1},\delta) - \frac{6\delta}{A(x)\pi}\tau(x)\Big)\Big(\Delta^4(t_{j-1},\delta) - \frac{6\delta}{A(x)\,\pi}\tau(x)\Big)\Bigg) \\
&= \sum_{j=1}^{n} \mathbb{E}\Bigg(\Delta^8(t_{j-1},\delta) - \frac{12\,\delta}{A(x)\,\pi}\Delta^4(t_{j-1},\delta)\,\tau(x) + \frac{36\,\delta^2}{A(x)^2\,\pi^2}\,\tau(x)^2\Bigg) \\
&\quad + 2\sum_{i=1}^{n}\sum_{i<j}^{n} \mathbb{E}\Bigg(\Big(\Delta^4(t_{i-1},\delta) - \frac{6\delta}{A(x)\pi}\tau(x)\Big)\Big(\Delta^4(t_{j-1},\delta) - \frac{6\delta}{A(x)\,\pi}\tau(x)\Big)\Bigg).
\end{aligned}
$$

Since, $\Delta(t_{j-1},\delta)$ is a centered Gaussian random variable, we have

$$
\mathbb{E}(\Delta^4(t_{j-1},\delta)) = 3(\mathbb{E}(\Delta^2(t_{j-1},\delta)))^2 \;\text{ and }\; \mathbb{E}(\Delta^8(t_{j-1},\delta)) = 105\Big(\mathbb{E}(\Delta^2(t_{j-1},\delta))\Big)^4.
$$

This with Lemma 5 allow us to get

$$
\mathbb{E}(\Delta^4(t_{j-1},\delta)) = 3\left(\frac{\sqrt{2}\,\delta^{1/2}}{\sqrt{A(x)\,\pi}}\,\tau(x) + O(\delta^2)\right)^2 = \frac{6\,\delta}{A(x)\,\pi}\tau(x) + O(\delta^{5/2}) \tag{34}
$$

and

$$
\mathbb{E}\Delta^8(t_{j-1},\delta) = 105\left(\frac{\delta^{1/2}\sqrt{2}}{\sqrt{A(x)\,\pi}}\,\tau(x) + O(\delta^2)\right)^4 = O(\delta^2). \tag{35}
$$

Therefore

$$
\begin{aligned}
&\sum_{j=1}^{n} \mathbb{E}\Big(\Delta^8(t_{j-1},\delta) - \frac{12\,\delta}{A(x)\,\pi}\,\tau(x)\Delta^4(t_{j-1},\delta) + \frac{36\,\delta^2}{A(x)^2\pi^2}\,\tau^2(x)\Big) \\
&\qquad = \sum_{j=1}^{n} O(\delta^2) - \frac{72\,\delta^2}{A(x)^2\pi^2}\,\tau(x) - O(\delta^{5/2}) + \frac{36\,\delta^2}{A(x)^2\pi^2}\,\tau^2(x) \\
&\qquad \le \sum_{j=1}^{n} Cte\delta^2 \\
&\qquad \le Cte\frac{1}{n} \longrightarrow 0 \;\text{ as } n \longrightarrow \infty,
\end{aligned}
$$

where we have used that $\delta = \dfrac{T_2 - T_1}{n}$, and Cte denoted a generic constant.

Using again the expansions (34) and (35), and also Lemma 6, we get

$$
\begin{aligned}
&2\sum_{i=1}^{n}\sum_{i<j}^{n} \mathbb{E}\Bigg[\Big(\Delta^4(t_{i-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\,\tau(x)\Big)\Big(\Delta^4(t_{j-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\,\tau(x)\Big)\Bigg] \\
&= 2\sum_{i=1}^{n}\sum_{i<j}^{n} \mathbb{E}\Bigg[\Big(\Delta^4(t_{i-1},\delta)\Delta^4(t_{j-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\,\tau(x)\Big(\Delta^4(t_{j-1},\delta) + \Delta^4(t_{i-1},\delta)\Big) \\
&\quad + \frac{36\,\delta^2}{A(x)^2\pi^2}\,\tau^2(x)\Big)\Bigg]
\end{aligned}
$$

$$= 2\sum_{i=1}^{n}\sum_{i<j}^{n}\Bigg[\mathbb{E}\Big(\Delta^4(t_{i-1},\delta)\Delta^4(t_{j-1},\delta)\Big) - \frac{6\,\delta}{A(x)\pi}\tau(x)\,\mathbb{E}\Big(\Delta^4(t_{j-1}\delta)+\Delta^4(t_{i-1},\delta)\Big)$$
$$+ \frac{36\,\delta^2}{A(x)^2\pi^2}\,\tau^2(x)\Bigg]$$
$$= 2\sum_{i=1}^{n}\sum_{i<j}^{n}\Big[O(\delta^{7/2}) + O\Big((j-i)^{-3}\delta^2\Big)\Big].$$

Then, there exist two strictly positive constants C and C' such that:

$$2\sum_{i=1}^{n}\sum_{i<j}^{n}\mathbb{E}\Bigg[\Big(\Delta^4(t_{i-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\,\tau(x)\Big)\Big(\Delta^4(t_{j-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\,\tau(x)\Big)\Bigg]$$
$$\leq C\sum_{i=1}^{n}\sum_{i<j}^{n}\delta^{7/2} + C'\sum_{i=1}^{n}\sum_{i<j}^{n}(j-i)^{-3}\delta^2.$$

The first sum tends to 0 as n tends to ∞. Indeed,

$$\sum_{i=1}^{n}\sum_{i<j}^{n}\delta^{7/2} \leq \frac{Cte}{n^{7/2}}\sum_{i=1}^{n}\sum_{j=i+1}^{n}1 = \frac{Cte}{n^{7/2}}\sum_{i=1}^{n}n-i = \frac{Cte}{n^{7/2}}\left[n^2 - \frac{n(n+1)}{2}\right] \to 0\,\text{as}\,n\to\infty.$$

Concerning the second sum, we have

$$\sum_{i=1}^{n}\sum_{i<j}^{n}(j-i)^{-3}\delta^2 = \frac{Cte}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}(j-i)^{-3} = \frac{Cte}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n-i}k^{-3}.$$

Therefore,

$$\sum_{i=1}^{n}\sum_{i<j}^{n}(j-i)^{-3}\delta^2 \leq \frac{Cte}{n^2}\sum_{i=1}^{n}\sum_{k=1}^{n}k^{-3} = \frac{Cte}{n}\sum_{k=1}^{n}k^{-3} = \frac{Cte}{n}\left[1+\frac{1}{2^3}+\sum_{k=3}^{n}k^{-3}\right].$$

Now since

$$\sum_{k=3}^{n}k^{-3} \leq \sum_{k=3}^{n}\frac{1}{k(k-1)(k-2)} = \frac{1}{2}\sum_{k=3}^{n}\frac{1}{k} - \sum_{k=3}^{n}\frac{1}{k-1} + \frac{1}{2}\sum_{k=3}^{n}\frac{1}{k-2}$$
$$= \frac{1}{2}\sum_{k=3}^{n}\left[\frac{1}{k} - \frac{1}{k-1}\right] - \frac{1}{2}\sum_{k=3}^{n}\left[\frac{1}{k-1} - \frac{1}{k-2}\right]$$
$$= \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n-1}\right] + \frac{1}{4},$$

we have

$$\sum_{i=1}^{n}\sum_{i<j}^{n}(j-i)^{-3}\delta^2 \leq \frac{Cte}{2n}\left[\frac{11}{4} + \frac{1}{n} - \frac{1}{n-1}\right] \to 0 \text{ as } n\to\infty.$$

Consequently,

$$2\sum_{i=1}^{n}\sum_{i<j}^{n}\mathbb{E}\Bigg[\Big(\Delta^4(t_{i-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\,\tau(x)\Big)\Big(\Delta^4(t_{j-1},\delta) - \frac{6\,\delta}{A(x)\,\pi}\,\tau(x)\Big)\Bigg] \to 0\,\text{as}\,n\to\infty. \quad \square$$

As consequence of Theorem 8 we get the following method of estimation of the parameter $A(x)$ defined in (3).

Corollary 9. *Fix* $0 < T_1 < T_2$ *and* $x \in \mathbb{R}$. *Define for* $j = 0, 1, \ldots, n$ *a time grid by* $t_j = T_1 + j\delta$, *where* $\delta = \frac{1}{n}(T_2 - T_1)$. *Then if we denote*

$$\hat{A}_n(x) = \frac{6(T_2 - T_1)}{\pi \sum_{j=1}^{n} \Big(u(t_j, x) - u(t_{j-1}, x) \Big)^4} \tau(x),$$

then $\lim_{n\to\infty} \hat{A}_n(x) = A(x)$ *in probability.*

Proof. Let us first remark that because $\eta > 0$, we have $\tau(x) > 0$ and then, $\hat{A}_n(x) \neq 0$, for every $n \in \mathbb{N} \setminus \{0\}$. We also have $A(x) > 0$. So, by the virtue of Theorem 8, the sequence $(\hat{A}_n(x))^{-1}$ converges in mean square, and consequently in probability, to $(A(x))^{-1}$. Corollary 9 is then obtained accordingly. □

5. Spatial variation

We fix $t > 0$ and we define the spatial increments: $\Delta(x, \delta) = u(t, x+\delta) - u(t, x)$, for $\delta > 0$.

Lemma 10. *For every* $x \in \mathbb{R}$ *and small* $\delta > 0$ *we have*

$$\mathbb{E}[\Delta^2(x, \delta)] = \theta_1(x) + \frac{\theta_2(x)}{A(x)}\delta + O(\delta^2), \tag{36}$$

where

$$\theta_1(x) = -2\beta \sqrt{\frac{t}{\pi A(x)}} \mathbf{1}_{\{\mathbf{x}=\mathbf{0}\}}, \tag{37}$$

$$\theta_2(x) = \mathbf{1}_{\{\mathbf{x}\neq\mathbf{0}\}} + \Big\{ 1 + (\eta + \beta - 1) \left(\frac{1}{2} - \frac{1}{\pi} \log(2) \right) \Big\} \mathbf{1}_{\{\mathbf{x}=\mathbf{0}\}} \tag{38}$$

and the expression $O(\delta^2)$ *denotes a quantity bounded by* $M\delta^2$, *for small* δ, *with the same constant* M *for all* $x \in \mathbb{R}$.

Proof. The desired variance can be written as:

$$\mathbb{E}[\Delta^2(x, \delta)] = \mathbb{E} \mid u(t, x+\delta) - u(t, x) \mid^2 = \mathbb{E}u^2(t, x+\delta) - 2\,\mathbb{E}u(t, x+\delta)u(t, x) + \mathbb{E}u^2(t, x). \tag{39}$$

We distinguish two cases: $x = 0$ and $x \neq 0$.

If $x = 0$, on one side, from the mild solution's variance expression (19) we have

$$\mathbb{E}[u^2(t, 0)] = \eta \sqrt{\frac{t}{\pi A(0)}}$$

and

$$\mathbb{E}[u^2(t, \delta)] = \sqrt{\frac{t}{\pi A(\delta)}} + \gamma(\delta) \left[\sqrt{\frac{t}{\pi A(\delta)}} \,\mathbf{erfc}\left(\frac{\delta}{\sqrt{tA(\delta)}} \right) - \frac{\delta}{\pi A(\delta)} \mathbf{E}_1 \left(\frac{\delta^2}{tA(\delta)} \right) \right] + \beta g(\delta),$$

where

$$g(\delta) = \sqrt{\frac{t}{\pi A(\delta)}} \exp\left(\frac{-\delta^2}{tA(\delta)} \right) - \frac{\delta}{A(\delta)} \,\mathbf{erfc}\left(\frac{\delta}{\sqrt{tA(\delta)}} \right).$$

On the other side, from the solution's covariance expression (13) we have

$$\mathbb{E}\Big(u(t, \delta) u(t, 0) \Big) := (1 + \beta)\, H_1(t, \delta) + H_2(t, \delta) \tag{40}$$

with

$$H_1(t,\delta) = \int_0^t \frac{1}{2\sqrt{\pi A(\delta)(t-u)}} \exp\left(\frac{-\delta^2}{4A(\delta)(t-u)}\right) du$$

and

$$H_2(t,\delta) = \gamma(\delta) \int_0^t \frac{1}{2\sqrt{\pi A(\delta)(t-u)}} \exp\left(\frac{-\delta^2}{4A(\delta)(t-u)}\right) \mathbf{erfc}\left(\frac{\delta}{2\sqrt{A(\delta)(t-u)}}\right) du.$$

By the change of variables $v = \dfrac{\delta}{2\sqrt{A(\delta)(t-u)}}$, then by an integration by parts we get

$$\begin{aligned} H_1(t,\delta) &= \frac{\delta}{2\,A(\delta)\sqrt{\pi}} \int_{\frac{\delta}{2\sqrt{t\,A(\delta)}}}^{+\infty} \frac{1}{v^2} \quad \exp\left(-v^2\right) dv \\ &= \frac{\delta}{2\,A(\delta)\sqrt{\pi}} \int_{\frac{\delta}{2\sqrt{t\,A(\delta)}}}^{+\infty} \left(\frac{-1}{v}\right)' \quad \exp\left(-v^2\right) dv \\ &= g\left(\frac{\delta}{2}\right). \end{aligned} \tag{41}$$

As for $H_2(t,\delta)$, by the change of variables $w = \dfrac{\delta}{2\sqrt{A(\delta)(t-u)}}$, by an integration by parts, then by using the fact that the function $w \longmapsto -\dfrac{\sqrt{\pi}}{4}\,\mathbf{erfc}^2(w)$ is an antiderivative of $w \longmapsto \exp(-w^2)\,\mathbf{erfc}(w)$, we get

$$\begin{aligned} H_2(t,\delta) &= \frac{\delta\,\gamma(\delta)}{2A(\delta)\sqrt{\pi}} \int_{\frac{\delta}{2\sqrt{t\,A(\delta)}}}^{+\infty} \frac{1}{w^2} \exp\left(-w^2\right) \mathbf{erfc}(w)\,dw \\ &= \frac{\delta\,\gamma(\delta)}{2A(\delta)\sqrt{\pi}} \int_{\frac{\delta}{2\sqrt{t\,A(\delta)}}}^{+\infty} \left(\frac{-1}{w}\right)' \exp\left(-w^2\right) \mathbf{erfc}(w)\,dw \\ &= \gamma(\delta)\sqrt{\frac{t}{\pi A(\delta)}} \exp\left(\frac{-\delta^2}{4tA(\delta)}\right) \mathbf{erfc}\left(\frac{\delta}{2\sqrt{tA(\delta)}}\right) \\ &\quad - \frac{\delta\gamma(\delta)}{4A(\delta)} \mathbf{erfc}^2\left(\frac{\delta}{2\sqrt{tA(\delta)}}\right) - \frac{\delta\gamma(\delta)}{2A(\delta)\pi} \mathbf{E}_1\left(\frac{\delta^2}{2tA(\delta)}\right). \end{aligned} \tag{42}$$

Expressions (40), (41) and (42) imply that

$$\begin{aligned} \mathbb{E}\Big(u(t,\delta)u(t,0)\Big) = (1+\beta)g\left(\frac{\delta}{2}\right) &+ \gamma(\delta)\sqrt{\frac{t}{\pi A(\delta)}} \exp\left(\frac{-\delta^2}{4tA(\delta)}\right) \mathbf{erfc}\left(\frac{\delta}{2\sqrt{tA(\delta)}}\right) \\ &- \frac{\delta\gamma(\delta)}{4A(\delta)} \mathbf{erfc}^2\left(\frac{\delta}{2\sqrt{tA(\delta)}}\right) - \frac{\delta\gamma(\delta)}{2A(\delta)\pi} \mathbf{E}_1\left(\frac{\delta^2}{2tA(\delta)}\right) \end{aligned}$$

and consequently,

$$\begin{aligned} \mathbb{E}[\Delta^2(0,\delta)] &= \eta\sqrt{\frac{t}{\pi A(0)}} + \sqrt{\frac{t}{\pi A(\delta)}} + \beta\,g(\delta) - 2(1+\beta)g\left(\frac{\delta}{2}\right) \\ &+ \gamma(\delta)\sqrt{\frac{t}{\pi A(\delta)}}\left\{\mathbf{erfc}\left(\frac{\delta}{\sqrt{tA(\delta)}}\right) - 2\exp\left(\frac{-\delta^2}{4tA(\delta)}\right)\mathbf{erfc}\left(\frac{\delta}{2\sqrt{tA(\delta)}}\right)\right\} \\ &+ \frac{\delta\,\gamma(\delta)}{A(\delta)}\left\{\frac{1}{\pi}\,\mathbf{E_1}\left(\frac{\delta^2}{2tA(\delta)}\right) - \frac{1}{\pi}\,\mathbf{E_1}\left(\frac{\delta^2}{tA(\delta)}\right) + \frac{1}{2}\,\mathbf{erfc}^2\left(\frac{\delta}{2\sqrt{tA(\delta)}}\right)\right\}. \end{aligned}$$

On the basis of some differentiability characteristics of the functions g, **erfc** and $\mathbf{E}_1$, Taylor's formula allows to get

$$\begin{aligned}\mathbb{E}[\Delta^2(0,\delta)] &= -2\beta\sqrt{\frac{t}{\pi A(0)}} + \frac{\delta}{A(0)}\left\{1+(\eta+\beta-1)\left(\frac{1}{2}-\frac{1}{\pi}\log(2)\right)\right\} + O(\delta^2)\\ &= \theta_1(0) + \frac{\theta_2(0)}{A(0)}\delta + O(\delta^2).\end{aligned}$$

If $x \neq 0$, on the one hand we have

$$\begin{aligned}\mathbb{E}[u^2(t,x)] = \sqrt{\frac{t}{\pi\, A(x)}} &+ \gamma(x)\left[\sqrt{\frac{t}{\pi\, A(x)}}\,\mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right) - \frac{|\,x\,|}{\pi A(x)}\mathbf{E}_1\left(\frac{x^2}{tA(x)}\right)\right]\\ &+ \beta\ \mathrm{sign}(x)\left[\sqrt{\frac{t}{\pi\ A(x)}}\ \exp\left(\frac{-x^2}{tA(x)}\right) - \frac{|\,x\,|}{A(x)}\ \mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right)\right]. \qquad (43)\end{aligned}$$

On the other hand, using the same calculation techniques as those employed in the case $x = 0$, we get

$$\begin{aligned}\mathbb{E}[u^2(t,x+\delta)] = \sqrt{\frac{t}{\pi A(x)}} &+ \gamma(x)\left[\sqrt{\frac{t}{\pi A(x)}}\,\mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right) - \frac{|\,x\,|}{\pi A(x)}\mathbf{E}_1\left(\frac{x^2}{tA(x)}\right)\right]\\ &+ \beta\ \mathrm{sign}(x)\left[\sqrt{\frac{t}{\pi\, A(x)}}\ \exp\left(\frac{-x^2}{tA(x)}\right) - \frac{|\,x\,|}{A(x)}\ \mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right)\right]\\ &- \frac{\beta\,\delta}{A(x)}\ \mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right) + O(\delta^2). \qquad (44)\end{aligned}$$

Concerning the term $\mathbb{E}u(t,x+\delta)u(t,x)$, we first write it in the form

$$\mathbb{E}[u(t,x+\delta)u(t,x)] = I_1(\delta) + \gamma(x)I_2(\delta) + \ \beta\ \mathrm{sign}(x)\, I_3(\delta)\,,$$

where

$$I_1(\delta) = \int_0^t \frac{1}{2\sqrt{\pi A(x)(t-u)}}\exp\left(\frac{-\delta^2}{4A(x)(t-u)}\right)du,$$

$$I_2(\delta) = \int_0^t \frac{1}{2\sqrt{\pi A(x)(t-u)}}\exp\left(\frac{-\delta^2}{4A(x)(t-u)}\right)\ \mathbf{erfc}\left(\frac{|\,2x+\delta\,|}{2\sqrt{A(x)(t-u)}}\right)du,$$

and

$$I_3(\delta) = \int_0^t \frac{1}{2\sqrt{\pi A(x)(t-u)}}\exp\left(\frac{-(2x+\delta)^2}{4A(x)(t-u)}\right)du.$$

Then on the basis of a study of the differentiability of I_1, I_2 and I_3, by applying Taylor's formula, and by some integration by parts and some suitable change of variables we get

$$I_1(\delta) = \sqrt{\frac{t}{\pi A(x)}} - \frac{\delta}{2A(x)} + O(\delta^2)\,,$$

$$I_2(\delta) = \sqrt{\frac{t}{\pi A(x)}}\ \mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right) - \mathbf{E}_1\left(\frac{x^2}{tA(x)}\right)\left[\frac{|\,x\,|}{\pi A(x)} + \frac{\delta\ \mathrm{sign}(x)}{2\pi A(x)}\right] + O(\delta^2)\,,$$

$$\begin{aligned}I_3(\delta) = \sqrt{\frac{t}{\pi A(x)}}\ \exp\left(\frac{-x^2}{tA(x)}\right) - \frac{|\,x\,|}{A(x)}\ \mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right) - \frac{\delta\ \mathrm{sign}(x)}{2\,A(x)}\ \mathbf{erfc}\left(\frac{|\,x\,|}{\sqrt{tA(x)}}\right)\\ + O(\delta^2),\end{aligned}$$

which yields:

$$\begin{aligned}\mathbb{E}u(t,x+\delta)u(t,x) &= \sqrt{\frac{t}{\pi A(x)}} - \frac{\delta}{2A(x)} \\ &+ \gamma(x)\left(\sqrt{\frac{t}{\pi A(x)}}\ \mathbf{erfc}\left(\frac{\mid x\mid}{\sqrt{tA(x)}}\right) - \mathbf{E}_1\left(\frac{x^2}{tA(x)}\right)\left[\frac{\mid x\mid}{\pi A(x)} + \frac{\delta\ \mathrm{sign}(x)}{2\pi A(x)}\right]\right) \\ &+ \beta\ \mathrm{sign}(x)\left(\sqrt{\frac{t}{\pi A(x)}}\ \exp\left(\frac{-x^2}{tA(x)}\right) - \frac{\mid x\mid}{A(x)}\ \mathbf{erfc}\left(\frac{\mid x\mid}{\sqrt{tA(x)}}\right)\right. \\ &\left. - \frac{\delta\ \mathrm{sign}(x)}{2\,A(x)}\ \mathbf{erfc}\left(\frac{\mid x\mid}{\sqrt{tA(x)}}\right)\right) + O(\delta^2). \end{aligned} \tag{45}$$

Combining (39) and (43)–(45) we get

$$\mathbb{E}[\Delta^2(x,\delta)] = \frac{\delta}{A(x)} + O(\delta^2). \qquad \square$$

Lemma 11. *For every $x, y \in \mathbb{R}$; $xy \neq 0$, and small $\delta < \inf\Big(y - x, \mid x\mid, \mid y\mid, \mid x+y\mid/2\Big)$ we have*

$$\mathbb{E}\Big(\Delta(x,\delta)\Delta(y,\delta)\Big) = O(\delta^2). \tag{46}$$

Proof. We distinguish two cases: $xy < 0$ and $xy > 0$, and we present the proof in the first case. The second one is similar. We have

$$\begin{aligned}&\mathbb{E}\big[\Delta(x,\delta)\Delta(y,\delta)\big] = \mathbb{E}\big[\big(u(t,x+\delta) - u(t,x)\big)\big(u(t,y+\delta) - u(t,y)\big)\big] \\ &= \mathbb{E}u(t,x+\delta)u(t,y+\delta) - \mathbb{E}u(t,x+\delta)u(t,y) - \mathbb{E}u(t,x)u(t,y+\delta) + \mathbb{E}u(t,x)u(t,y).\end{aligned}$$

Since $xy < 0$ and $0 < \delta < \inf(y - x, |x|, |y|)$, we surely have $y + \delta > y > 0 > x + \delta > x$, $A(x+\delta) = A(x)$ and $A(y+\delta) = A(y)$. Then, using the covariance expression (13) we get

$$\mathbb{E}\big[\Delta(x,\delta)\Delta(y,\delta)\big] = (1+\beta)\,T_1(\delta) - \lambda\,T_2(\delta) + \lambda\,T_3(\delta)$$

with

$$\begin{aligned}T_1(\delta) = \int_0^t \frac{1}{2\sqrt{\pi A(y)(t-u)}} &\left\{ \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4(t-u)}\right)\right. \\ &- \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4(t-u)}\right) - \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4(t-u)}\right) \\ &\left. + \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4(t-u)}\right) \right\} du\,,\end{aligned}$$

$$
\begin{aligned}
T_2(\delta) = \int_0^t \frac{1}{2\sqrt{\pi A(y)(t-u)}} \Bigg\{ & \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \\
& \times \mathbf{erfc}\left(\frac{\frac{y+\delta}{\sqrt{A(y)}} + \frac{x+\delta}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \\
& - \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \mathbf{erfc}\left(\frac{\frac{y+\delta}{\sqrt{A(y)}} + \frac{x}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \\
& - \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \mathbf{erfc}\left(\frac{\frac{y}{\sqrt{A(y)}} + \frac{x+\delta}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \\
& + \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \mathbf{erfc}\left(\frac{\frac{y}{\sqrt{A(y)}} + \frac{x}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \Bigg\} \, du
\end{aligned}
$$

and

$$
\begin{aligned}
T_3(\delta) = \int_0^t \frac{1}{2\sqrt{\pi A(y)(t-u)}} \Bigg\{ & \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} + \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \\
& \times \mathbf{erfc}\left(\frac{\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \\
& - \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} + \frac{x}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \mathbf{erfc}\left(\frac{\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \\
& - \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} + \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \mathbf{erfc}\left(\frac{\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \\
& + \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} + \frac{x}{\sqrt{A(x)}}\right)^2}{4(t-u)} \right) \mathbf{erfc}\left(\frac{\frac{y}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}}{2\sqrt{t-u}} \right) \Bigg\} \, du.
\end{aligned}
$$

An integration by parts and a simple calculation allow us to get

$$T_1(\delta) = \sqrt{\frac{t}{\pi A(y)}}\left\{\exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4t}\right) - \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4t}\right)\right.$$
$$\left. - \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4t}\right) + \exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4t}\right)\right\}$$
$$- \frac{1}{2\sqrt{A(y)}}\left\{\mathbf{erfc}\left(\frac{\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}}{2\sqrt{t}}\right) - \mathbf{erfc}\left(\frac{\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}}{2\sqrt{t}}\right)\right.$$
$$\left. - \mathbf{erfc}\left(\frac{\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}}{2\sqrt{t}}\right) + \mathbf{erfc}\left(\frac{\frac{y}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}}{2\sqrt{t}}\right)\right\}$$

and

$$T_1'(\delta) = \frac{1}{2\sqrt{\pi A(y)t}}\left\{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)\left(\frac{1}{\sqrt{A(y)}} - \frac{1}{\sqrt{A(x)}}\right)\right.$$
$$\times \exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4t}\right)$$
$$+ \left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)\frac{1}{\sqrt{A(y)}}\exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4t}\right)$$
$$- \left(\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)\frac{1}{\sqrt{A(x)}}\exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4t}\right)$$
$$+ \left(\frac{1}{\sqrt{A(y)}} - \frac{1}{\sqrt{A(x)}}\right)\exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4t}\right)$$
$$\left. - \frac{1}{\sqrt{A(y)}}\exp\left(\frac{-\left(\frac{y+\delta}{\sqrt{A(y)}} - \frac{x}{\sqrt{A(x)}}\right)^2}{4t}\right) + \frac{1}{\sqrt{A(x)}}\exp\left(\frac{-\left(\frac{y}{\sqrt{A(y)}} - \frac{x+\delta}{\sqrt{A(x)}}\right)^2}{4t}\right)\right\}.$$

We see that $T_1(0) = 0$, $T_1'(0) = 0$, and consequently, $T_1(\delta) = O(\delta^2)$.

By the same technique we get

$$T_2(\delta) = O(\delta^2) \text{ and } T_3(\delta) = O(\delta^2). \qquad \square$$

Theorem 12. *Let us consider $t > 0$ and $A_1 < A_2$ such that $A_1 A_2 > 0$. Define, for every $i = 0, 1, \ldots, n$ a space grid by $x_i = A_1 + i\delta$, where $\delta = \frac{1}{n}(A_2 - A_1)$. Then, the following limit holds in mean square:*

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} [u(t, x_{i+1}) - u(t, x_i)]^2 = \frac{1}{A(x_0)} (A_2 - A_1).$$

Proof. Denoting

$$I_n = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \mathbb{E}\left[\left(\Delta^2(x_i,\delta) - \frac{\delta}{A(x_0)}\right)\left(\Delta^2(x_j,\delta) - \frac{\delta}{A(x_0)}\right)\right],$$

it suffices to prove that the sequence (I_n) converges to 0 as $n \to \infty$. We have

$$\begin{aligned} I_n &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\mathbb{E}\Delta^2(x_i,\delta)\Delta^2(x_j,\delta) - \frac{\delta}{A(x_0)}\mathbb{E}\Delta^2(x_i,\delta) - \frac{\delta}{A(x_0)}\mathbb{E}\Delta^2(x_j,\delta) + \frac{\delta^2}{A(x_0)^2}\right] \\ &= \sum_{i=0}^{n-1}\left[\mathbb{E}\Delta^4(x_i,\delta) - \frac{2\delta}{A(x_0)}\mathbb{E}\Delta^2(x_i,\delta) + \frac{\delta^2}{A(x_0)^2}\right] \\ &\quad + 2\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1}\left[\mathbb{E}\Delta^2(x_i,\delta)\Delta^2(x_j,\delta) - \frac{\delta}{A(x_0)}\mathbb{E}\Delta^2(x_i,\delta) - \frac{\delta}{A(x_0)}\mathbb{E}\Delta^2(x_j,\delta)\right. \\ &\quad \left. + \frac{\delta^2}{A(x_0)^2}\right]. \end{aligned} \tag{47}$$

On the one hand, since $\Delta(x_i, \delta)$ is centered Gaussian random variable, we have $\mathbb{E}\Delta^4(x_i,\delta) = 3(\mathbb{E}\Delta^2(x_i,\delta))^2$, for every $i \in \{0, \ldots, n\}$. This with Lemma 10 allow us to get

$$\mathbb{E}\Delta^4(x_i,\delta) = 3\left(\frac{\delta}{A(x_i)}\,\theta_2(x_i) + O(\delta^2)\right)^2 = O(\delta^2). \tag{48}$$

On the other hand, since $\mathbb{E}X^2Y^2 = \mathbb{E}X^2\,\mathbb{E}Y^2 + 2\,(\mathbb{E}[XY])^2$ for every zero-mean Gaussian random variables $\{X, Y\}$ (see e.g. Theorem 5.12 in [9]) we have

$$\begin{aligned} \mathbb{E}\Delta^2(x_i,\delta)\Delta^2(x_j,\delta) &= \mathbb{E}\Delta^2(x_i,\delta)\,\mathbb{E}\Delta^2(x_j,\delta) + 2\,(\mathbb{E}\Delta(x_i,\delta)\Delta(x_j,\delta))^2 \\ &= \left[\frac{\delta}{A(x_i)}\,\theta_2(x_i) + O(\delta^2)\right]\left[\frac{\delta}{A(x_j)}\,\theta_2(x_j) + O(\delta^2)\right] + O(\delta^4) \\ &= \frac{\delta^2}{A^2(x_0)} + O(\delta^3)\,, \end{aligned} \tag{49}$$

where in the second equality we used Lemmas 10 and 11, and in the third one we used the fact that $\text{sign}(x_i) = \text{sign}(x_0)$, $A(x_i) = A(x_0)$ and $\theta_2(x_i) = 1$, for every $i \in \{0, \ldots, n-1\}$.

Combining (47), (48) and (49), and using again Lemma 10, we get

$$\begin{aligned} I_n &= \sum_{i=0}^{n-1} O(\delta^2) + 2\sum_{i=0}^{n-1}\sum_{j=i+1}^{n-1} O(\delta^3) \\ &= n\,O(\delta^2) + \frac{n}{2}[n-1]\,O(\delta^3) \to 0 \ \text{ as } \ n \to \infty. \qquad \square \end{aligned}$$

As consequence of Theorem 12 we can estimate the parameters a_1 and a_2 defined in (3) via the following result.

Corollary 13. *Let us consider $t > 0$ and $0 < A_1 < A_2$ (respectively $A_1 < A_2 < 0$). Define, for every $i = 0, 1, \dots, n$ a space grid by $x_i = A_1 + i\delta$, where $\delta = \frac{1}{n}(A_2 - A_1)$. If we denote*

$$\hat{a}_n(t) = \frac{(A_2 - A_1)}{\sum_{j=0}^{n-1} \Big(u(t, x_{j+1}) - u(t, x_j)\Big)^2},$$

then $\lim_{n\to\infty} \hat{a}_n(t) = a_2$ (respectively $\lim_{n\to\infty} \hat{a}_n(t) = a_1$) in probability.

Proof. We consider the case where $0 < A_1 < A_2$. The proof in the other case is similar. Since $\hat{a}_n(x) \neq 0$, for every $n \in \mathbb{N} \setminus \{0\}$, we deduce from Theorem 12 that the sequence $(\hat{a}_n(x))^{-1}$ converges in mean square, and hence in probability, to $(A(x_0))^{-1} = a_2^{-1}$. The convergence in probability of $(\hat{a}_n(x))$ to a_2 is then obtained accordingly. □

Remarks 14. In Corollary 9 [respectively 13] we presented a weakly consistent estimators of the parameters a_1 and a_2 defined in (3). The method of estimation we used is based on the knowledge of the exact variations, in time [respectively in space], of the solution to SPDE (1). The benefit of this method is that it allows to find a_1 and a_2 from an arbitrarily small segment of a single path. In other words, our results are local in structure, being dependent only on the characteristics of the Gaussian noise and the small time asymptotics of the fundamental solution G. For stochastic differential equations, the parameters are typically not reflected in the local structure of paths, because when the drift changes, we get a process whose distribution in the trajectory space is absolutely continuous relative to the similar distribution of the original process. Therefore, observing the process on a small, but fixed segment, we will definitely not estimate the drift, but just construct the Maximum Likelihood estimate of the Radon-Nikodym density.

Finally, we remark that the asymptotic normality of the estimators of a_1 and a_2 we gave in this paper remains an open problem.

6. Annex

Lemma 15. *For every $t, s \geq 0$ and $x, \tilde{X}, a \in \mathbb{R}$*

$$\begin{aligned} I(t, s, x, \tilde{X}, a) &= \int_{-\infty}^{x} \exp\left(\frac{-v^2}{2a(t-u)}\right) \exp\left(\frac{-(\tilde{X} - v)^2}{2a(s-u)}\right) dv \\ &= \sqrt{\frac{\pi a(t-u)(s-u)}{2(t+s-2u)}} \exp\left(\frac{-\tilde{X}^2}{2a(t+s-2u)}\right) \\ &\quad \times \mathbf{erfc}\left(\frac{\tilde{X}\sqrt{t-u}}{\sqrt{2a(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a(t-u)(s-u)}}\right). \end{aligned}$$

Proof.

$$I(t, s, x, \tilde{X}, a) = \exp\left(\frac{-\tilde{X}^2}{2a(t+s-2u)}\right) I'(t, s, x, \tilde{X}, a),$$

where

$$\begin{aligned} I'(t, s, x, \tilde{X}, a) &= \int_{-\infty}^{x} \exp\left(\frac{\tilde{X}^2}{2a(t+s-2u)} - \frac{v^2}{2a(t-u)} - \frac{(\tilde{X} - v)^2}{2a(s-u)}\right) dv \\ &= \int_{-\infty}^{x} \exp\left(\frac{\tilde{X}^2(u-t)}{2a(t+s-2u)(s-u)} - \frac{v^2(t+s-2u)}{2a(t-u)(s-u)} + \frac{2\tilde{X}v}{2a(s-u)}\right) dv \end{aligned}$$

$$= \int_{-\infty}^{x} \exp\left(\frac{-1}{2a} \left(\frac{\tilde{X}^2(t-u)}{(t+s-2u)(s-u)} - \frac{2v\tilde{X}}{s-u} + \frac{v^2(t+s-2u)}{(t-u)(s-u)} \right) \right) dv$$

$$= \int_{-\infty}^{x} \exp\left(\frac{-1}{2a} \left(\frac{\tilde{X}\sqrt{t-u}}{\sqrt{(t+s-2u)(s-u)}} - \frac{v\sqrt{t+s-2u}}{\sqrt{(t-u)(s-u)}} \right)^2 \right) dv.$$

By the change of variables

$$W = \frac{\tilde{X}\sqrt{t-u}}{\sqrt{2a(t+s-2u)(s-u)}} - \frac{v\sqrt{t+s-2u}}{\sqrt{2a(t-u)(s-u)}}$$

we get Lemma 15. □

Using the same techniques that those we used in the proof of Lemma 15 we get the following result.

Lemma 16. *For every $t, s \geq 0$ and $x, \tilde{X}, a \in \mathbb{R}$*

$$J(t,s,x,\tilde{X},a) = \int_{x}^{+\infty} \exp\left(\frac{-v^2}{2a(t-u)} \right) \exp\left(\frac{-(\tilde{X}-v)^2}{2a(s-u)} \right) dv$$

$$= \sqrt{\frac{\pi\, a(t-u)(s-u)}{2(t+s-2u)}} \exp\left(\frac{-\tilde{X}^2}{2a(t+s-2u)} \right)$$

$$\times \left[2 - \mathbf{erfc}\left(\frac{\tilde{X}\sqrt{t-u}}{\sqrt{2a(t+s-2u)(s-u)}} - \frac{x\sqrt{t+s-2u}}{\sqrt{2a(t-u)(s-u)}} \right) \right].$$

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ТОЧНІ ВАРІАЦІЇ ДЛЯ СТОХАСТИЧНИХ РІВНЯНЬ ТЕПЛОПРОВІДНОСТІ З КУСКОВО-СТАЛИМИ КОЕФІЦІЄНТАМИ ТА ЇХНЄ ЗАСТОСУВАННЯ ДЛЯ ОЦІНЮВАННЯ ПАРАМЕТРІВ

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Анотація. Знайдено розклади варіацій четвертого порядку у часі та квадратичних варіацій у просторі для розв'язку стохастичного рівняння із частинними похідними з кусково-сталими коефіцієнтами. Обидва розклади дозволяють вивести метод оцінювання параметрів, що фігурують у рівнянні.

ТОЧНЫЕ ВАРИАЦИИ ДЛЯ СТОХАСТИЧЕСКИХ УРАВНЕНИЙ ТЕПЛОПРОВОДНОСТИ С КУСОЧНО-ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ И ИХ ПРИМЕНЕНИЕ ДЛЯ ОЦЕНКИ ПАРАМЕТРОВ

М. ЗИЛИ, Э. ЗУГАР

Аннотация. Найдены разложения вариаций четвертого порядка во времени и квадратичных вариаций в пространстве для решения стохастического уравнения в частных производных с кусочно-постоянными коэффициентами. Оба разложения позволяют вывести метод оценки параметров, фигурирующих в уравнении.