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THE WOLD DECOMPOSITION OF HILBERTIAN PERIODICALLY CORRELATED PROCESSES

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Abstract. The Wold decomposition of stationary processes is widely applied in time series prediction and provides interesting insights into the structure of stationary stochastic processes. In 1971, Kallianpur and Mandrekar, using the notion of resolution of identity and unitary operators, presented the Wold decomposition for weakly stationary stochastic processes with values in infinite dimensional separable Hilbert spaces. This paper aims to expand the idea of Wold decomposition to Hilbertian periodically correlated processes, applying the concept of $\mathcal{L}$-closed subspaces.

Key words and phrases. H-Valued Random Variables, $\mathcal{L}$-Closed Subspaces, Moving Average Representation, Periodically Correlated Processes, Wold Decomposition.

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1. Introduction

In random processes, prediction is a momentous problem. Based on this issue, processes can be categorized as deterministic and non-deterministic, see [4, 14]. Although the deterministic processes are of great importance as models for cyclic and seasonal behavior, they are not of interest in prediction theory. However, prediction of nondeterministic time series is challenging in many problems.

Using Wold decomposition, introduced by Wold [19], a stationary process can be written as the orthonormal sum of a purely nondeterministic process and a deterministic process. This decomposition is the foundation of time domain approach in time series analysis, digital signal processing, etc, and has a close affinity to spectral analysis of time series (see [8, 15]). Moreover, it can be applied to provide the best linear prediction in terms of moving average parameters and innovation processes [12]).

Stochastic processes with values in infinite dimensional Hilbert spaces attract the attention of researchers through last years. These processes are theoretically the building blocks of functional data analysis. Kallianpur and Mandrekar [7] have studied the Hilbertian weakly stationary stochastic processes and established the Wold decomposition and moving average representation for such processes, using the notion of resolution of identity and unitary operators. In 2007, Bosq has presented the Wold decomposition for Hilbertian processes, applying the idea of $\mathcal{L}$-closed subspaces.

Although stationary processes are applied drastically in various fields, they can not always be the best model for time series data. Some processes, specially in economics and signal processing, exhibit nonstationary behavior. Among different models for such processes, periodically correlated or cyclostationary random processes are studied by various researchers. These processes, which show some periodic rhythm in the structure, are introduced by Gladyshev (1961). The prediction problem for periodically correlated sequences has been addressed by Gladyshev [5] who presented conditions for regularity. Pourahmadi and Salehi [13] and Miamee and Salehi [10] have studied the Wold decomposition for real-valued periodically correlated processes. Moreover, Pagano [11] has discussed the connection between periodically correlated sequences and periodic moving average models.

The aim of this paper is to extend the idea of Wold decomposition to H-valued periodically correlated processes. For this purpose, Section 2 provides the required notations and definitions. Section 3 is devoted to the proof of the main theorem of this paper, which focuses on Wold decomposition, and in the last section, we will apply this decomposition to present the moving average representation for H-valued periodically correlated processes.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $L^2(\Omega, \mathcal{F}, P)$ be the space of real-valued square integrable random variables on $(\Omega, \mathcal{F}, P)$. Consider H to be a separable real Hilbert space, which is endowed with the inner product $\langle \cdot, \cdot \rangle_H$ and the norm $\|\cdot\|_H$. Also, let $\mathcal{B}$ denotes the Borel field on H. An $\mathcal{F}/\mathcal{B}$ measurable mapping $\xi : \Omega \to H$ is called an H-valued random variable and it is termed strongly second order (HSSO) if $E\|\xi\|_H^2 < \infty$, see [18]. The space of HSSO random variables will be denoted by $L_H^2(\Omega, \mathcal{F}, P)$.

The mean value of the HSSO random variable ξ, $\mathbb{E}(\xi)$, is an element of H and is defined as the Bochner integral of ξ over Ω (see [1]), that is,

$$\mathbb{E}(\xi) = (B) \int_\Omega \xi dP. \tag{1}$$

The following properties of the expectation are directly followed by its definition:

i. If ξ_1 and ξ_2 are H-valued random variables and $\mathbb{E}(\xi_1)$ and $\mathbb{E}(\xi_2)$ exist, then $\mathbb{E}(\xi_1 + \xi_2)$ exists as well and $\mathbb{E}(\xi_1 + \xi_2) = \mathbb{E}(\xi_1) + \mathbb{E}(\xi_2)$.

ii. Let ξ be an H-valued random variable and $\mathbb{E}(\xi)$ exists. If ℓ is a continuous linear operator from H into H, then $\mathbb{E}(\ell(\xi))$ also exists and $\mathbb{E}(\ell(\xi)) = \ell(\mathbb{E}(\xi))$.

Additionally, the variance and covariance operators of HSSO random variables are defined in terms of tensorial products. The tensorial product of x and $y \in H$, which is a random operator, is denoted by $x \otimes y$, and is defined as $(x \otimes y)h = \langle y, h \rangle_H x$, $\forall h \in H$, see [16]. Some important relationships, which are simple consequences of the definition of tensorial product, are as follows:

$$(x + y) \otimes z = x \otimes z + y \otimes z, \qquad x \otimes (y + z) = x \otimes y + x \otimes z. \tag{2}$$

$$\ell(x \otimes y) = (\ell x) \otimes y, \qquad (x \otimes y)\ell = x \otimes (\ell^* y), \tag{3}$$

where * stands for the adjoint of an operator. The variance and covariance operators for HSSO random variables ξ and $\eta \in H$ are defined respectively as:

$$C_\xi = \mathbb{E}[(\xi - \mathbb{E}(\xi)) \otimes (\xi - \mathbb{E}(\xi))], \tag{4}$$

$$C_{\xi,\eta} = \mathbb{E}[(\xi - \mathbb{E}(\xi)) \otimes (\eta - \mathbb{E}(\eta))]. \tag{5}$$

It can easily be proved that the operators C_ξ and $C_{\xi,\eta}$ are nuclear operators [2].

The orthogonality is an important feature of H-valued random variables and is defined by two different forms. The zero-mean random variables ξ and η are called weakly orthogonal if $E\langle \xi, \eta \rangle_H = 0$ and are said to be orthogonal if, for any $x, y \in H$, $E\langle \xi, x \rangle_H \langle \eta, y \rangle_H = 0$. It can easily be shown that orthogonality, which is equivalent to $C_{\xi,\eta} = 0$, is strictly stronger than weak orthogonality [2]. Further on, the notation $\xi \perp \eta$ will denote orthogonality.

Let $\mathcal{L} = \mathcal{L}(H, H)$ be the space of bounded linear operators from H to H, where the operatorial norm is denoted by $\|\cdot\|_0$. The notion of $\mathcal{L}$-closed subspaces (LCS) is defined as follows [2].

Definition. The space $\mathcal{G}$ is called an $\mathcal{L}$-closed subspace of $L_H^2(\Omega, \mathcal{F}, P)$, if it is a Hilbertian subspace of $L_H^2(\Omega, \mathcal{F}, P)$ and, for $\xi \in \mathcal{G}$ and $\ell \in \mathcal{L}$, $\ell(\xi) \in \mathcal{G}$.

If $\mathcal{G}$ contains only zero-mean H-valued random variables, it will be called zero-mean LCS. For simplicity of notation, let $sp_H(A)$, $A \subseteq H$, stand for

$$\left\{ \sum_{i \in J} \ell_i(a_i);\ J\ is\ finite,\ \ell_i \in \mathcal{L},\ a_i \in A \right\}.$$

It can be proved that for an arbitrary subset F of $L^2_H(\Omega, \mathcal{F}, P)$, the LCS $\mathcal{G}_F$, generated by F, is the closure of $sp_H(F) = \left\{ \sum_{i=1}^{k} \ell_i(\xi_i);\ \ell_i \in \mathcal{L},\ \xi_i \in F,\ k \geq 1 \right\}$, (see [2, Theorem 1.8] for the proof). Note that, conventionally, for $A \subseteq H$, $sp\{A\}$ stands for a space which is closed under the operation of forming linear combinations of elements in A. Also, $\overline{sp}\{A\}$ denotes the closed linear span of all elements of A in H-topology. Therefore, LCS is the general form of span closure of a set.

Let $\pi^{\mathcal{G}}$ stand for the orthogonal projection on $\mathcal{G}$. For any arbitrary zero-mean H-valued random variable, ξ, and zero mean LCS $\mathcal{G}$, it can be shown that $\pi^{\mathcal{G}}(\xi) \in \mathcal{G}$ and $C_{\xi - \pi^{\mathcal{G}}(\xi), \eta} = 0$, for $\eta \in \mathcal{G}$ [2]. However, the H-valued random variables does not essentially have mean zero. To deal with this situation, let us denote by χ_c the class of constant H-valued random variables. It can be demonstrated that $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_{\chi_c}$, where $\mathcal{G}_0$ is the LSC of zero-mean H-valued random variables [2]. Consequently, $\pi^{\mathcal{G}} = \pi^{\mathcal{G}_0} + \pi^{\mathcal{G}_{\chi_c}}$ and

$$\pi^{\mathcal{G}}(\xi) = \pi^{\mathcal{G}_0}(\xi - \mathbb{E}(\xi)) + \mathbb{E}(\xi). \tag{6}$$

Note that the orthogonal complement of an LCS is also an LCS [2, Theorem 1.10].

Considering the set of integers, $\mathbb{Z}$, a sequence of HSSO random variables, namely $\boldsymbol{\xi} = \{\xi_n,\ n \in \mathbb{Z}\}$, is called an HSSO discrete time stochastic process. The time domain of the stochastic process $\boldsymbol{\xi}$, $\mathcal{M}(\boldsymbol{\xi})$, is defined to be the LCS generated by $\{\xi_n :\ n \in \mathbb{Z}\}$. Moreover, the LCS generated by $\{\xi_n :\ n \leq t\}$ defines the past of $\boldsymbol{\xi}$ up to time t, and is denoted by $\mathcal{M}(\boldsymbol{\xi}, t)$. Also, $\cap_t \mathcal{M}(\boldsymbol{\xi}, t) := \mathcal{M}(\boldsymbol{\xi}, -\infty)$, which is a LCS, stands for the distant past of $\boldsymbol{\xi}$, see [2].

Definition. An H-valued stochastic process $\boldsymbol{\xi}$ is called deterministic or singular if $\mathcal{M}(\boldsymbol{\xi}, -\infty) = \mathcal{M}(\boldsymbol{\xi})$, otherwise, it is called nondeterministic. If $\mathcal{M}(\boldsymbol{\xi}, -\infty) = \{0\}$, where 0 is the zero of $L^2_H(\Omega, \mathcal{F}, P)$, the process is named purely nondeterministic or regular.

This paper focuses on Hilbertian periodically correlated processes, which are defined as follows.

Definition. An HSSO stochastic process $\boldsymbol{\xi}$ is periodically correlated (HPC-T) if there exists a positive integer T such that

$$\mathbb{E}(\xi_n) = \mathbb{E}(\xi_{n+T}), \tag{7}$$

$$C_{\xi_n, \xi_m} := C_{\xi}^{n,m} = C_{\xi}^{n+T, m+T}. \tag{8}$$

The smallest T in (7) and (8) is called the period of the process $\boldsymbol{\xi}$.

Note that, by setting $T = 1$, the results presented in this paper can be applied to stationary processes.

Remark 2.1. Kallianpur and Mandrekar [7] applied the idea of unitary operators for defining stationary H-valued random processes. For this purpose, they applied the notation $L^2(H)$ for the subspace of $L^2(\Omega, \mathcal{F}, P)$, generated by

$$\left\{ \langle \xi_n, h \rangle_H;\ \xi_n \in L^2_H(\Omega, \mathcal{F}, P),\ n \in \mathbb{Z},\ h \in H \right\}.$$

Besides, they use $L^2(H, t)$ to denote the subspace of $L^2(\Omega, \mathcal{F}, P)$, generated by

$$\left\{ \langle \xi_n, h \rangle_H;\ \xi_n \in L^2_H(\Omega, \mathcal{F}, P),\ n \leq t,\ h \in H \right\}.$$

Consequently, if $\cap_t L^2(H,t) = \{0\}$, where 0 is the zero of $L^2(H)$, they called an H-valued process purely nondeterministic. They called the process deterministic if $L^2(H,t) = L^2(H)$, for all t. Based on their approach, an HSSO stochastic process $\boldsymbol{\xi}$ is stationary if there exists an unitary operator $\mathbf{U}_{\boldsymbol{\xi}} : L^2(H) \to L^2(H)$ such that $\langle \xi_{t+1}, h\rangle_H = \mathbf{U}_{\boldsymbol{\xi}}\langle \xi_t, h\rangle_H$, $\forall h \in H$. Although the idea of unitary operators can be extended to periodically correlated processes, this paper will focus on $\mathcal{L}$-closed subspaces. See [6] for the application of unitary operators, when dealing with complex-valued periodically correlated processes.

3. Wold decomposition

In infinite dimensional time series analysis, applying Wold decomposition to any stationary time series results in writing down the process as the sum of two infinite dimensional processes, one deterministic and one nondeterministic. The following theorem represents the Wold decomposition for infinite dimensional periodically correlated processes.

Theorem 3.1 (The Wold Decomposition of $HPC-T$ Sequences)**.** *Any $HPC-T$ sequence ξ_t has a unique decomposition*

$$\xi_t = \zeta_t + \eta_t, \tag{9}$$

in which $\mathcal{M}(\boldsymbol{\xi},t) = \mathcal{M}(\boldsymbol{\zeta},t)\oplus\mathcal{M}(\boldsymbol{\eta},t)$, ζ_t is an $HPC-T$ deterministic process and η_t is an $HPC-T$ purely nondeterministic process.

Proof. Consider (6) and for simplicity of notation, let ξ_t, $t\in\mathbb{Z}$, be a sequence of zero-mean PC time series. As the first step, let us prove that $\mathcal{M}(\boldsymbol{\xi},t) = \mathcal{M}(\boldsymbol{\zeta},t)\oplus\mathcal{M}(\boldsymbol{\eta},t)$. For each $t\in\mathbb{Z}$, let ζ_t be the projection of ξ_t on $\mathcal{M}(\boldsymbol{\xi},-\infty)$, denoted by $\zeta_t := \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\xi_t \in \mathcal{M}(\boldsymbol{\xi},-\infty)$, and $\eta_t = \xi_t - \zeta_t = \xi_t - \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\xi_t \perp \mathcal{M}(\boldsymbol{\xi},-\infty)$. Hence, $\mathcal{M}(\boldsymbol{\zeta},t) \perp \mathcal{M}(\boldsymbol{\eta},t)$. For each $t\in\mathbb{Z}$, $\zeta_t \in \mathcal{M}(\boldsymbol{\xi},-\infty) \subseteq \mathcal{M}(\boldsymbol{\xi},t)$, and $\eta_t = \xi_t - \zeta_t \in \mathcal{M}(\boldsymbol{\eta},t) \subseteq \mathcal{M}(\boldsymbol{\xi},t)$. Hence, $\mathcal{M}(\boldsymbol{\zeta},t)\oplus\mathcal{M}(\boldsymbol{\eta},t) \subseteq \mathcal{M}(\boldsymbol{\xi},t)$. To complete the proof, it is enough to demonstrate that $\mathcal{M}(\boldsymbol{\xi},t) \subseteq \mathcal{M}(\boldsymbol{\zeta},t)\oplus\mathcal{M}(\boldsymbol{\eta},t)$. For this purpose, let $w\in\mathcal{M}(\boldsymbol{\xi},t)$. Define $u := \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}w \in \mathcal{M}(\boldsymbol{\zeta},t)$ and $v := w-u \in \mathcal{M}(\boldsymbol{\eta},t)$. Clearly, $w = u+v$, which shows that $\mathcal{M}(\boldsymbol{\xi},t) \subseteq \mathcal{M}(\boldsymbol{\zeta},t)\oplus\mathcal{M}(\boldsymbol{\eta},t)$.

To indicate ζ_t is a deterministic process, it will be shown that $\mathcal{M}(\boldsymbol{\zeta},-\infty) = \mathcal{M}(\boldsymbol{\zeta})$, or equivalently, $\mathcal{M}(\boldsymbol{\zeta},-\infty) = \mathcal{M}(\boldsymbol{\zeta},t)$, $\forall t\in\mathbb{Z}$. In the first part of the proof, it is demonstrated that, for each t, $\mathcal{M}(\boldsymbol{\zeta},t) \subseteq \mathcal{M}(\boldsymbol{\xi},-\infty)$. The proof will be continued by showing that $\mathcal{M}(\boldsymbol{\xi},-\infty) \subseteq \mathcal{M}(\boldsymbol{\zeta},t)$.

For fixed t, suppose that there is a nonzero $u\in\mathcal{M}(\boldsymbol{\xi},-\infty)\ominus\mathcal{M}(\boldsymbol{\zeta},t)$. For each $s\le t$, $u\perp\zeta_s$. Moreover,considering the fact that $\eta_s \perp \mathcal{M}(\boldsymbol{\xi},-\infty)$, we conclude that $u\perp\eta_s$. Consequently, $u\perp\xi_s = \zeta_s+\eta_s$, hence $u\perp\mathcal{M}(\boldsymbol{\xi},t)$, therefore, $u\perp\mathcal{M}(\boldsymbol{\xi},-\infty)$. This implies that $u=0$, which is a contradiction. Hence, $\mathcal{M}(\boldsymbol{\xi},-\infty)\subseteq\mathcal{M}(\boldsymbol{\zeta},t)$, which proves that ζ_t is a deterministic process. Moreover, $\mathcal{M}(\boldsymbol{\eta},-\infty) = \mathcal{M}(\boldsymbol{\xi},-\infty)\ominus\mathcal{M}(\boldsymbol{\zeta},-\infty)$. Since ζ_t is a deterministic process, it is easy to demonstrate that $\mathcal{M}(\boldsymbol{\eta},-\infty) = \{0\}$ therefore, η_t is a purely nondeterministic process.

The last part of the proof is devoted to the periodicity of ζ_t and η_t. As mentioned previously, the process $\boldsymbol{\xi}$ is considered to have zero mean. Therefore, using the properties of expectation and tensorial product and periodic behavior of $\boldsymbol{\xi}$, stated in Section 2, it can be shown that for all $t\in\mathbb{Z}$:1

$$\begin{aligned}\mathbb{E}(\zeta_t) &= \mathbb{E}\Big(\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\xi_t\Big) = \\ &= \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\mathbb{E}(\xi_t) = \\ &= 0.\end{aligned}$$

For $x \in H$, we have:

$$\begin{aligned} C_{\zeta}^{t,t'}(x) &= \mathbb{E}(\zeta_t \otimes \zeta_{t'})(x) = \\ &= \mathbb{E}\Big(\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\xi_t \otimes \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\xi_{t'}\Big)(x) = \\ &= \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\mathbb{E}(\xi_t \otimes \xi_{t'})\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}(x) = \\ &= \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}C_{\xi}^{t,t'}\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}(x) = \\ &= \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}C_{\xi}^{t+T,t'+T}\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}(x) = \\ &= C_{\zeta}^{t+T,t'+T}(x). \end{aligned}$$

Similarly, it can be demonstrate that $\mathbb{E}(\eta_t) = \mathbb{E}(\xi_t) - \mathbb{E}(\zeta_t) = 0$. For all $x \in H$,

$$\begin{aligned} C_{\eta}^{t,t'}(x) &= \mathbb{E}(\xi_t - \zeta_t \otimes \xi_{t'} - \zeta_{t'})(x) = \\ &= \mathbb{E}(\xi_t \otimes \xi_{t'})(x) - \mathbb{E}(\xi_t \otimes \zeta_{t'})(x) - \mathbb{E}(\zeta_t \otimes \xi_{t'})(x) + \mathbb{E}(\zeta_t \otimes \zeta_{t'})(x) = \\ &= \mathbb{E}(\xi_t \otimes \xi_{t'})(x) - \mathbb{E}(\xi_t \otimes \xi_{t'})\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}(x) - \\ &\quad - \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}\mathbb{E}(\xi_t \otimes \xi_{t'})(x) + \mathbb{E}(\zeta_t \otimes \zeta_{t'})(x) = \\ &= C_{\xi}^{t,t'}(x) - C_{\xi}^{t,t'}\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}(x) - \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}(x)C_{\xi}^{t,t'} + C_{\zeta}^{t,t'} = \\ &= C_{\xi}^{t+T,t'+T}(x) - C_{\xi}^{t+T,t'+T}\pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}(x) - \\ &\quad - \pi^{\mathcal{M}(\boldsymbol{\xi},-\infty)}C_{\xi}^{t+T,t'+T}(x) + C_{\zeta}^{t+T,t'+T} = \\ &= C_{\eta}^{t+T,t'+T}(x). \end{aligned}$$

□

3.1. **Moving Average Representation for** $HPC - T$ **Processes.** As stated in the previous section, any $HPC - T$ process can be decomposed into a deterministic process and a purely nondeterministic one. This section is devoted to present a moving average representation for the purely nondeterministic part. In this case, the Wold decomposition can be used for prediction problem in terms of moving average parameters and the innovation process, see [12].

Consider the nondecreasing spaces $\mathcal{M}(\boldsymbol{\xi}, t)$. If there exists some t_0 where $\mathcal{M}(\boldsymbol{\xi}, t_0 - 1)$ is a proper subspace of $\mathcal{M}(\boldsymbol{\xi}, t_0)$, i.e., $\mathcal{M}(\boldsymbol{\xi}, t_0 - 1) \subset \mathcal{M}(\boldsymbol{\xi}, t_0)$, then innovation space of the sequence ξ_t at time t is defined as:

$$\mathcal{I}(\boldsymbol{\xi}, t) = \mathcal{M}(\boldsymbol{\xi}, t) \ominus \mathcal{M}(\boldsymbol{\xi}, t-1) = \{\xi \in \mathcal{M}(\boldsymbol{\xi}, t) : \ \xi \perp \mathcal{M}(\boldsymbol{\xi}, t-1)\}. \tag{10}$$

Note that if such t_0 does not exist, i.e., $\mathcal{M}(\boldsymbol{\xi}, t) = \mathcal{M}(\boldsymbol{\xi}, t-1)$, for all t, the process $\boldsymbol{\xi}$ will be deterministic and $\mathcal{M}(\boldsymbol{\xi}, t) = \mathcal{M}(\boldsymbol{\xi}, -\infty)$. Equation (10) can be re-stated as $\mathcal{M}(\boldsymbol{\xi}, t) = \mathcal{I}(\boldsymbol{\xi}, t) \oplus \mathcal{M}(\boldsymbol{\xi}, t-1)$. More accurately,

$$\mathcal{M}(\boldsymbol{\xi}, t) = \oplus \sum_{j \le t} \mathcal{I}(\boldsymbol{\xi}, j) \oplus \mathcal{M}(\boldsymbol{\xi}, -\infty). \tag{11}$$

Moreover, $\mathcal{I}(\boldsymbol{\xi}, t) \perp \mathcal{M}(\boldsymbol{\xi}, -\infty)$, for every $t \in \mathbb{Z}$, and, by the Wold decomposition, it can be proved that $\mathcal{I}(\boldsymbol{\xi}, t) = \mathcal{M}(\boldsymbol{\eta}, t) \ominus \mathcal{M}(\boldsymbol{\eta}, t-1)$, see [6]. Let $d_{\boldsymbol{\xi}}(t)$ stand for $\dim \mathcal{I}(\boldsymbol{\xi}, t)$. Following the same steps as Hurd and Miamee [6], it can be illustrated that $d_{\boldsymbol{\xi}}(t) = 0$ or 1 and $d_{\boldsymbol{\xi}}(t+T) = d_{\boldsymbol{\xi}}(t)$.

The block innovation at time t, $\mathcal{I}^T(\boldsymbol{\xi}, t)$, is as follows:

$$\begin{aligned} \mathcal{I}^T(\boldsymbol{\xi}, t) &:= \mathcal{M}(\boldsymbol{\xi}, t) \ominus \mathcal{M}(\boldsymbol{\xi}, t-T) = \\ &= \mathcal{I}(\boldsymbol{\xi}, t) \oplus \mathcal{I}(\boldsymbol{\xi}, t-1) \oplus \cdots \oplus \mathcal{I}(\boldsymbol{\xi}, t-T+1). \end{aligned} \tag{12}$$

The rank or dimension of the block innovation is the number of times in one period that the innovation is not trivial, i. e.,

$$d^T_{\boldsymbol{\xi}}(t) := \sum_{s=0}^{T-1} d_{\boldsymbol{\xi}}(t-s). \tag{13}$$

It can be concluded from the periodicity of $d_{\boldsymbol{\xi}}(t)$, i.e., $d_{\boldsymbol{\xi}}(t) = d_{\boldsymbol{\xi}}(t+T)$, that $d^T_{\boldsymbol{\xi}}(t)$ is constant with respect to t. The process ξ_t is called full rank if $d^T_{\boldsymbol{\xi}}(t) = T$.

If ξ_t is $HPC-T$, then the past of $\boldsymbol{\eta}$ up to time t can be presented, in terms of an orthogonal sum of innovation spaces, as:

$$\mathcal{M}(\boldsymbol{\eta}, t) = \mathcal{I}(\boldsymbol{\xi}, t) \oplus \mathcal{I}(\boldsymbol{\xi}, t-1) \oplus \cdots = \oplus \sum_{j=0}^{\infty} \mathcal{I}(\boldsymbol{\xi}, t-j),$$

or, equivalently, as $\mathcal{M}(\boldsymbol{\eta}, t) = \oplus \sum_{j=0}^{\infty} \mathcal{I}^T(\boldsymbol{\xi}, t-jT)$. This formulation is applied to present the moving average representation of η_t.

Let D^+ stand for the time indices where ξ_t has positive innovation dimension, that is:

$$D^+ = \{t : \ d_{\boldsymbol{\xi}}(t) > 0\}. \tag{14}$$

For periodic process ξ_t, the set D^+ is periodic in the sense that if $t \in D^+$, then $t + kT \in D^+$ for every $k \in \mathbb{Z}$, see [6]. Moreover,

$$d^T_{\boldsymbol{\xi}}(t) = \operatorname{card}\big(D^+ \cap \{0, 1, \cdots, T-1\}\big).$$

This notation allows us to have the moving average representation even when ξ_t is not of full rank, i. e., $d^T_{\boldsymbol{\xi}}(t) < T$.

The next theorem provides the moving average representation for a purely nondeterministic $HPC-T$ sequence. For this purpose, consider the following assumptions:
Assumption 1: There exists a periodic set of indices D^+ with period T, and $d^T_{\xi}(t) = = \operatorname{card}(D^+ \cap \{0, 1, \cdots, T-1\})$.
Assumption 2: $a_{j,t}$ is a sequence of bounded linear operators such that

$$\sum_{j \geq 0 \ t-j \in D^+} \|a_{j,t}\|_0^2 < \infty,$$

and $a_{j,t+kT} = a_{j,t}$ for every j, k, t, with $t - j \in D^+$.

Theorem 3.2 (Moving Average Representation of $HPC-T$ Sequences)**.** *Let ξ_t be an HSSO stochastic process. This process is a purely nondeterministic $HPC-T$ sequence of rank $d^T_{\xi}(t)$ if and only if Assumption 1 holds and there exists a set of orthonormal innovation processes, $\mathcal{I} = \{\eta_m,\ m \in D^+\}$, such that, for every t,*

$$\xi_t = \sum_{j \geq 0 \ t-j \in D^+} a_{j,t}(\eta_{t-j}), \tag{15}$$

where $a_{j,t}$ satisfies Assumption 2.

Proof. If Part: Let ξ_t be given by (15). Since η_m's are orthogonal and $a_{j,t}$ are square summable, it can be concluded that ξ_t is a HSSO stochastic process. Based on the properties of expectation and periodic structure of D^+ and $a_{j,t}$, it can easily be proved that $\mathbb{E}(\xi_t) = \mathbb{E}(\xi_{t+T})$. For $t \geq s$, using the properties of Bochner integral and tensorial product, we have:

$$C_{\xi}^{t,s} = \mathbb{E}(\xi_t \otimes \xi_s) = \mathbb{E}\left(\sum_{j \geq 0 \ t-j \in D^+} a_{j,t}(\eta_{t-j}) \otimes \sum_{j' \geq 0 \ s-j' \in D^+} a_{j',s}(\eta_{s-j'}) \right) =$$

$$= \sum_{j\geq 0\ t-j\in D^+}\ \sum_{j'\geq 0\ s-j'\in D^+} \mathbb{E}(a_{j,t}(\eta_{t-j}) \otimes a_{j',s}(\eta_{s-j'})) =$$
$$= \sum_{j\geq 0\ t-j\in D^+}\ \sum_{j'\geq 0\ s-j'\in D^+} a_{j,t}\mathbb{E}(\eta_{t-j} \otimes \eta_{s-j'})a^*_{j',s} =$$
$$= \sum_{j\geq 0\ t-j\in D^+}\ \sum_{j'\geq 0\ s-j'\in D^+} a_{j,t}C_{\eta}^{t-j,s-j'} a^*_{j's} =$$
$$= \sum_{j\geq 0\ t-j\in D^+} a_{j,t}C_{\eta}a^*_{s-t+j,s} =$$
$$= \sum_{j\geq 0\ t-j\in D^+} a_{j,t+T}C_{\eta}a^*_{s+T-t-T+j,s+T} =$$
$$= \mathbb{E}(\xi_{t+T} \otimes \xi_{s+T}) = C_{\xi}^{t+T,s+T}.$$

By orthogonality of η_m, we have:

$$\mathbb{E}(\eta_{t-j} \otimes \eta_{s-j'}) := C_{\eta}^{t-j,s-j'} = \begin{cases} C_{\eta}, & \text{if } t-j = s-j', \\ 0, & \text{if } t-j \neq s-j'. \end{cases}$$

Therefore, ξ_t is $HPC-T$. Moreover, $\mathcal{M}(\boldsymbol{\xi}, t) \subset \mathcal{M}(\boldsymbol{\eta}, t)$ and, consequently, $\cap_t\mathcal{M}(\boldsymbol{\xi}, t) \subset \subset \cap_t\mathcal{M}(\boldsymbol{\eta}, t) = \{0\}$, which demonstrates that ξ_t is a purely nondeterministic process. To prove $d^T_{\boldsymbol{\xi}}(t) = \mathrm{card}(D^+ \cap \{0, 1, \cdots, T-1\})$, note that, from (15) and for all $0 \leq t \leq T-1$, there exists $card(D^+ \cap \{0, 1, \cdots, T-1\})$ values of t for which ξ_t depends on η_t. For other values of t, the process depends only on the past innovations, i. e., $j > 0$. More precisely, for $0 \leq t \leq T-1$, ξ_t has $\mathrm{card}(D^+ \cap \{0, 1, \cdots, T-1\})$ non-zero innovations, which implies that ξ_t is of rank $\mathrm{card}(D^+ \cap \{0, 1, \cdots, T-1\})$.

Only If Part: Suppose that ξ_t is purely nondeterministic $HPC-T$, and of rank $d^T_{\boldsymbol{\xi}}$. As mentioned previously, the dimension of the innovation spaces $\mathcal{I}(\boldsymbol{\xi}, p)$ is zero or one and $\mathcal{I}(\boldsymbol{\xi}, p) \perp \mathcal{I}(\boldsymbol{\xi}, q)$ for $p \neq q$. Let $D^+ = \{t : d_{\boldsymbol{\xi}}(t) = \dim \mathcal{I}(\boldsymbol{\xi}, t) = 1\}$, and $\eta_p := \xi_p - \pi^{\mathcal{M}(\boldsymbol{\xi}, p-1)}(\xi_p)$ be the unit element of $\mathcal{I}(\boldsymbol{\xi}, p)$, $p \in D^+$. Based on (11) and the definition of LCS, any $X \in \mathcal{M}(\boldsymbol{\xi}, t)$ can be presented in terms of η_p, $p \in D^+$, as

$$X = \sum_{j\geq 0\ t-j\in D^+} \ell_j(\eta_{t-j}),$$

where $\ell_j \in \mathcal{L}$ and, since $\xi_t \in \mathcal{M}(\boldsymbol{\xi}, t)$, we have

$$\xi_t = \sum_{j\geq 0\ t-j\in D^+} a_{j,t}(\eta_{t-j}).$$

To determine $a_{j,t}$, note that for each t, ξ_t is a periodic HSSO process, i.e., $E\|\xi_t\|^2 < \infty$, which results in $\sum_{j\geq 0\ t-j\in D^+} \|a_{j,t}\|_0^2 < \infty$. Also, since the process ξ_t demonstrates a periodic behavior in the mean and covariance operator and D^+ has a periodic structure, it can be deduced that $a_{j,t}$ is a set of periodic operators, i.e., $a_{j,t+kT} = a_{j,t}$. □

The introduced moving average presentation can be applied in prediction theory in time series analysis. For more information on prediction theory see [12]. The following example demonstrates moving average representation for an H-valued periodically correlated autoregressive process.

Example (Moving Average Representation for $PCARH(1)$ Processes). An HSSO sequence $\boldsymbol{\xi} = \{\xi_t,\ t \in \mathbb{Z}\}$ is called an H-valued periodically correlated autoregressive process of order 1 with period T, or $PCARH(1)$ in abbreviation, if it satisfies

$$\xi_t = \phi_t\xi_{t-1} + \eta_t, \tag{16}$$

where $\{\eta_t,\ t \in \mathbb{Z}\}$ is a collection of orthogonal HSSO random variables. Besides, $\boldsymbol{\phi} = \{\phi_0, \phi_1, \cdots, \phi_{T-1}\}$ is a T-periodic sequence in $\mathcal{L}$, i. e., $\phi_t = \phi_{t+T}$, $t \in \mathbb{Z}$, see [17]. This process is called causal if it does not depend on the future values of η_t, i. e., if $\mathcal{M}(\boldsymbol{\xi}, t) \subset \mathcal{M}(\boldsymbol{\eta}, t)$.

If ξ_t is a causal solution to $PCARH(1)$, then

$$\xi_t = \sum_{j=0}^{\infty} a_{j,t}\eta_{t-j}, \tag{17}$$

where $a_{0,t} = I$, $a_{j,t} = \phi_t\phi_{t-1}\cdots\phi_{t-j+1}$, $j > 0$, and $\sum_{j\geq 0}\|a_{j,t}\|_0^2 < \infty$. More precisely,

$$\xi_{nT+i} = \sum_{k=0}^{\infty}\sum_{l=0}^{T-1}[a_{T,nT+i}]^k a_{l,nT+i}\eta_{(n-k)T_i-l}. \tag{18}$$

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РОЗКЛАД УОЛДА ГІЛЬБЕРТОВИХ ПЕРІОДИЧНО КОРЕЛЬОВАНИХ ПРОЦЕСІВ

А. ЗАМАНІ, З. САДЖАДНІА, М. ХАШЕМІ

Анотація. Розклад Уолда стаціонарних процесів широко застосовується у прогнозуванні часових рядів і дає цікаве розуміння структури стаціонарних випадкових процесів. У 1971 році Калліанпур і Мандрекар, використовуючи поняття розкладу одиниці та унітарних операторів, представили розклад Уолда для слабко стаціонарних випадкових процесів зі значеннями в нескінченновимірних гільбертових просторах. Ця стаття має на меті розширити ідею розкладу Уолда до гільбертових періодично корельованих процесів, застосовуючи поняття $\mathcal{L}$-замкнених підпросторів.