Теорія ймовірностей
та математична статистика
Вип. 2(101)/2019, с. 203–211

Teoriya Ĭmovirnosteĭ
ta Matematychna Statystyka
No. 2(101)/2019, pp. 203–211

UDC 519.21

DIFFERENTIAL AND INTEGRAL EQUATIONS FOR JUMP RANDOM MOTIONS

A. O. POGORUI, R. M. RODRÍGUEZ-DAGNINO

Abstract. In this paper we obtain a differential equation for the characteristic function of random jump motion on the line, where the direction alternations and random jumps occur according to the renewal epochs of the Erlang distribution. We also study random jump motion in higher dimensions and we obtain a renewal-type equation for the characteristic function of the process. In the 3-dimensional case we obtain the telegraph-type differential equation for jump random motion, where the direction alternations and random jumps occur according to the renewal epochs of the Erlang-2 distribution.

Key words and phrases. Telegraph process, random evolutions, semi-Markov processes, Erlang distribution, telegraph equation.

2010 *Mathematics Subject Classification.* Primary 60K35; Secondary 60K99, 60K15.

1. Introduction

In the years of 1951 and 1956 Goldstein and Kac solved the problem of a one-dimensional random motion driven by a Poisson process, i.e., defined by the movement of a particle at a constant velocity v traveling in some direction a random distance drawn from an exponential distribution. After that, the particle change to the opposite direction under the same stochastic conditions. The problem can be seen as a random motion governed by a switching Poisson process with alternating directions and having exponentially distributed sojourn times. Goldstein and Kac found that the solution of this problem satisfies the one-dimensional telegraph-type equation, which is similar to the Heaviside telegraph equation for wave propagation in transmission lines. Namely,

$$\frac{\partial^2 f(t,x)}{\partial t^2} + 2\lambda \frac{\partial}{\partial t} f(t,x) = v^2 \frac{\partial^2 f(t,x)}{\partial x^2}, \tag{1}$$

and it is called the Goldstein–Kac telegraph equation. Many variations of this basic idea have been approached by many researchers worldwide, and in this paper we are devoted to the important case of the jump telegraph process on a line and in higher dimensions. The one-dimensional jump telegraph process, which is a generalization of the telegraph process, was introduced and studied (including its applications in financial market theory) by Ratanov [1–3], and Ratanov and Melnikov [4]. Some limit theorems for the jump telegraph process on a line were studied by Di Crescenzo and Martinucci [5]. The jump telegraph process with random jumps was introduced and studied by Di Crescenzo and Iuliano [6]. Some previous results regarding Erlang distributions in the telegraph process context can be found in [7–9], random motions in multidimensional spaces [10, 11], connections between random motion and partial differential equations in physics [12], and recent results on jump-telegraph processes governed by alternating fractional Poisson process [13]. Other recent results find the telegraph equation as a proper model of physical absorption problems with partially reflecting boundaries, of run–and–tumble particles in one space dimension [14].

In this paper we obtain differential equations for the one-dimensional jump telegraph process in case of Erlang distributed switching times for velocities, and arbitrary distributed random jumps. We also study the jump telegraph process in $\mathbb{R}^n$, $n > 1$, and we obtain the Volterra integral equation of convolution type for the characteristic function of the process. As a particular case, we consider a differential equation for the jump process in $\mathbb{R}^3$ and study its solution.

2. The Erlang jump telegraph process on a line

Let us consider the random motion of a particle on the line described in the following manner: the particle moves according to one of two velocities $c > 0$ or $-v < 0$. Starting from the point $x_0 \in \mathbb{R}$ the particle moves with velocity c during the random time $\tau_1 = \xi_{\lambda_1} + \ldots + \xi_{\lambda_n}$, $n \geq 1$, where ξ_{λ_i} is exponentially distributed with rate $\lambda_i > 0$. Then, the particle jumps a random length η_1 and continues its motion with velocity $-v$ during the random time $\tau_2 = \xi_{\mu_1} + \ldots + \xi_{\mu_m}$, $m \geq 1$, where ξ_{μ_i} is exponential distributed with rate $\mu_i > 0$. Furthermore, the particle jumps a random length η_2 and moves with velocity c a random time τ_3, and so on. It is assumed that random variables $\{\eta_1, \eta_2, \xi_{\lambda_i}, \xi_{\mu_j}, i = 1, \ldots, n, j = 1, \ldots, m\}$ are mutually independent. Thus, during the random times $\tau_1 \overset{d}{=} \tau_3 \overset{d}{=} \tau_5 \cdots$, the particle has velocity c whereas during the random times $\tau_2 \overset{d}{=} \tau_4 \overset{d}{=} \tau_6 \cdots$ has velocity $-v$.

The motion of this particle can be described by using theory of random evolutions in the following form: Denote by $\psi(t)$, $t \geq 0$, the alternating semi-Markov process on the phase (or state) space $T = \{1, 2\}$, with sojourn time τ_i at state $i \in T$, and with transition probabilities matrix of the embedded Markov chain

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus, the alternating renewal process $\psi(t)$, $t \geq 0$ can be written in the equivalent expression

$$\psi(t) = \begin{cases} 1, & \text{if} \quad 0 \leq t < \tau_1, \\ 2, & \text{if} \quad \tau_1 \leq t < \tau_1 + \tau_2, \\ 1, & \text{if} \quad \tau_1 + \tau_2 \leq t < \tau_1 + \tau_2 + \tau_3, \\ 2, & \text{if} \quad \tau_1 + \tau_2 + \tau_3 \leq t < \tau_1 + \cdots + \tau_4, \\ \cdots \end{cases}$$

2.1. Description of the random evolution $y(t)$.

Denote by $y(t)$, $t \geq 0$, the position of the particle at time t.

Let us introduce the following function C_1 on T

$$C_1(y) = \begin{cases} c, & \text{if} \quad y = 1, \\ -v, & \text{if} \quad y = 2. \end{cases} \tag{2}$$

Thus, the process $y(t)$ is a random evolution in the semi-Markov medium $\psi(t)$ satisfying the following equation [15]

$$y(t) = x_0 + \int_0^t C_1(\psi(s))ds + \sum_{k=1}^{\nu(t)} \eta_k, \tag{3}$$

where $\nu(t)$ is the number of renewal events of $\psi(t)$, and η_k has the same distribution as η_1 for odd k, and η_k has the same distribution as η_2 for even k. In addition, the random variables $\eta_1, \eta_2, \ldots$ are mutually independent.

Lemma 1. $\mathsf{P}(\nu(t) < \infty) = 1$ *for each* $t \geq 0$.

Proof. It is enough to show that for each $t \geq 0$, $\mathsf{P}(\nu(t) \geq x) \to 0$ as $x \to +\infty$. Since $\mathsf{P}(\nu(t) \geq x)$ is monotonically decreasing with respect to x, then it is enough to prove that $\mathsf{P}(\nu(t) \geq 2r) \to 0$ as $\mathbb{N} \ni r \to \infty$.

Firstly, we should notice that if θ_k is exponential distributed with parameter λ, $\{\theta_k, k \in \mathbb{N}\}$, then for any $t > 0$ we have

$$\mathsf{P}\left(\sum_{k=1}^{r} \theta_k \le t\right) = 1 - \sum_{n=0}^{r-1} \frac{(\lambda t)^n}{n!}\, e^{-\lambda t} \to 0, \quad r \to \infty.$$

It is easily verified that

$$\begin{aligned} \mathsf{P}(\nu(t) \ge 2r) = \mathsf{P}\left(\sum_{k=1}^{2r} \tau_k \le t\right) \le \\ &\le \mathsf{P}(\tau_1 + \tau_3 + \dots + \tau_{2r-1} \le t) + \mathsf{P}(\tau_2 + \tau_4 + \dots + \tau_{2r} \le t) \le \\ &\le \sum_{j=1}^{n} \mathsf{P}\Big(\theta^{(1)}_{\lambda_j} + \theta^{(3)}_{\lambda_j} + \dots + \theta^{(2r-1)}_{\lambda_j} \le t\Big) + \\ &\quad + \sum_{j=1}^{m} \mathsf{P}\Big(\theta^{(2)}_{\mu_j} + \theta^{(4)}_{\mu_j} + \dots + \theta^{(2r)}_{\mu_j} \le t\Big) \to 0, \quad \text{as} \quad r \to \infty. \end{aligned}$$

Here we have used the fact that $\tau_{2l-1} = \sum_{j=1}^{n} \theta^{(2l-1)}_{\lambda_j}$, $\tau_{2l} = \sum_{j=1}^{m} \theta^{(2l)}_{\mu_j}$, $l \ge 1$. □

Now, let us show that the evolution $y(t)$ can be represented as an evolution driven by a Markov process. Denote by $\xi(t)$, $t \ge 0$, a Markov chain in the phase space $E = \{1, 2, \dots, n, n+1, \dots, n+m\}$ with the infinitesimal operator $Q = q[P - I]$, where $q = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n, \mu_1, \dots, \mu_m)$ and P is the following $(n+m) \times (n+m)$ transition probabilities matrix

$$P = \|p_{ij}\| = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We assume that $\mathsf{P}(\xi(0) = 1) = 1$.

Let us introduce the following function C on E

$$C(i) = \begin{cases} c, & \text{if} \quad i = 1, \dots, n, \\ -v, & \text{if} \quad i = n+1, \dots, n+m \end{cases}$$

and consider the equation

$$x(t) = x_0 + \int_0^t C(\xi(s))\, ds + \sum_{k=1}^{n(t)} \alpha_k, \ t \ge 0, \tag{4}$$

where $n(t)$ is the number of renewal events of $\xi(t)$, and $\alpha_{kn+(k-1)m}$, $k = 1, 2, \dots$ has the same distribution as η_1, and α_{km+kn}, $k = 1, 2, \dots$ has the same distribution as η_2, and $\alpha_k = 0$ in other cases. We suppose that there exist the probability density functions (pdf) $g_{\eta_i}(y)$, $i = 1, 2$, and we assume $g_{\alpha_i}(y) = \delta(y)$ for $\alpha_i = 0$, where $\delta(\cdot)$ is the delta function.

Eq. (4) describes a random evolution in Markov media $\xi(t)$, $t \ge 0$. In addition, it is easily verified that $\mathsf{P}(y(t) = x(t)) = 1$ for each $t \ge 0$. Thus, we reduce the semi-Markov case described by Eq. (3) to the Markov case described by Eq. (4).

In the rest of this paper we will study the process $x(t)$, $t \ge 0$, which is described by Eq. (4), and we will consider the two-component process $\zeta(t) = (x(t), \xi(t))$ in the phase space $Z = \mathbb{R} \times E$, where $\mathbb{R} = (-\infty, +\infty)$, $E = \{1, \dots, n, n+1, \dots, n+m\}$.

It is well-known [15, 16] that the process $\zeta(t) = (x(t), \xi(t))$ has the following infinitesimal operator

$$A\varphi(x,i) = C(x,i)\frac{\partial}{\partial x}\varphi(x,i) + \lambda_i(P\varphi(x,i) - \varphi(x,i)) + A_0\varphi(x,i),\ i \in E,$$

where $\varphi(x,i)$ is a differential function with respect to $x \in \mathbb{R}$ for all $i \in E$ and such that there exists $\int_{-\infty}^{\infty} g_{\alpha_i}(y)\varphi(x+y,j)\,dy$ for any $x \in \mathbb{R}$, $i \in E$, $P\varphi(x,i) = \sum_{j\in E} p_{ij}\varphi(x,j)$ and the operator A_0 is of the following form, for $i = n$ or $i = n+m$,

$$A_0\varphi(x,i) = \lambda_i\left(\sum_{j\in E} p_{ij}\int_{-\infty}^{\infty} g_{\alpha_i}(y)[\varphi(x+y,j)-\varphi(x,i)]\,dy\right)$$

and for other $i \in E$

$$A_0\varphi(x,i) = 0.$$

Let us write the operator A in more detail:

$$\begin{aligned}
A\varphi(x,1) &= c\frac{\partial}{\partial x}\varphi(x,1) + \lambda_1(\varphi(x,2) - \varphi(x,1)),\\
A\varphi(x,2) &= c\frac{\partial}{\partial x}\varphi(x,2) + \lambda_2(\varphi(x,3) - \varphi(x,2)),\\
&\ \ \vdots\\
A\varphi(x,n) &= c\frac{\partial}{\partial x}\varphi(x,n) + \lambda_n(\varphi(x,n+1) - 2\varphi(x,n)) +\\
&\quad + \lambda_n\int_{-\infty}^{\infty} g_{\eta_1}(y)\varphi(x+y,n+1)\,dy,\\
A\varphi(x,n+1) &= -v\frac{\partial}{\partial x}\varphi(x,n+1) + \mu_1(\varphi(x,n+2) - \varphi(x,n+1)),\\
&\ \ \vdots\\
A\varphi(x,n+m) &= -v\frac{\partial}{\partial x}\varphi(x,n+m) + \mu_m(\varphi(x,1) - 2\varphi(x,n+m)) +\\
&\quad + \mu_m\int_{-\infty}^{\infty} g_{\eta_2}(y)\varphi(x+y,1)\,dy.
\end{aligned}$$

Consider $f(t,x,k)dx = \mathsf{P}\{x \le x(t) \le x+dx, \xi(t) = k\}$, $k \in E$. The function $f(t,x,k)$ satisfies the Kolmogorov backward equation

$$\frac{\partial f(t,x,k)}{\partial t} = A\,f(t,x,k), \tag{5}$$

with the initial condition (similarly to the initial conditions stated in Chapter 2 of [17])

$$\sum_{k\in E} f(0,x,k) = \delta(x).$$

2.2. **Using characteristic functions to solve the PDEs.** Passing in Eq. (5) into characteristic functions $H_k(t,\omega)=\int_{-\infty}^{\infty} e^{i\omega x} f(t,x,k)dx$, we obtain

$$
\begin{aligned}
\frac{\partial H_1(t,\omega)}{\partial t} &= -i\omega c H_1(t,\omega)+\lambda_1(H_2(t,\omega)-H_1(t,\omega)),\\
\frac{\partial H_2(t,\omega)}{\partial t} &= -i\omega c H_2(t,\omega)+\lambda_2(H_3(t,\omega)-H_2(t,\omega)),\\
&\vdots\\
\frac{\partial H_n(t,\omega)}{\partial t} &= -i\omega c H_n(t,\omega)+\lambda_n(H_{n+1}(t,\omega)-2H_n(t,\omega))+\\
&\quad +\lambda_n \hat{g}_{\eta_1}(-\omega)H_{n+1}(t,\omega),\\
\frac{\partial H_{n+1}(t,\omega)}{\partial t} &= i\omega v H_{n+1}(t,\omega)+\mu_1(H_{n+2}(t,\omega)-H_{n+1}(t,\omega)),\\
&\vdots\\
\frac{\partial H_{n+m}(t,\omega)}{\partial t} &= i\omega v H_{n+m}(t,\omega)+\mu_m(H_1(t,\omega)-2H_{n+m}(t,\omega))+\\
&\quad +\mu_m \hat{g}_{\eta_2}(-\omega)H_1(t,\omega).
\end{aligned}
\tag{6}
$$

Let us write Eq. (6) in the matrix form

$$M\mathbf{H}(t,\omega)=0,$$

where $\mathbf{H}(t,\omega)=(H_1(t,\omega),H_2(t,\omega),\dots,H_{n+m}(t,\omega))^{\top}$ and the matrix M is as follows

$$
M=\begin{pmatrix}
\frac{\partial}{\partial t}+i\omega c+\lambda_1 & -\lambda_1 & 0 & \cdots & 0\\
0 & \frac{\partial}{\partial t}+i\omega c+\lambda_2 & -\lambda_2 & \cdots & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots\\
-\mu_m-\mu_m\hat{g}_{\eta_2}(-\omega) & \cdots & 0 & \cdots & \frac{\partial}{\partial t}-i\omega v+2\mu_m
\end{pmatrix}.
$$

Consider the function $H(t,\omega)=\sum_{i=1}^{n+m} H_i(t,\omega)$. It is easily seen that $H(t,\omega)$ is the characteristic function of $f(t,x)$, where $f(t,x)=\sum_{i=1}^{n+m} f(t,x,i)$. The function $f(t,x)$ is the pdf of the position $x(t)$, of the particle at time t.

Theorem 1. *The function $H(t,\omega)$ satisfies the following differential equation*

$$
\begin{aligned}
&\prod_{k=1}^{n-1}\left(\frac{\partial}{\partial t}+i\omega c+\lambda_k\right)\prod_{l=1}^{m-1}\left(\frac{\partial}{\partial t}-i\omega v+\mu_l\right)\times\\
&\qquad\times\left(\frac{\partial}{\partial t}+i\omega c+2\lambda_n\right)\left(\frac{\partial}{\partial t}-i\omega v+2\mu_m\right)H(t,\omega)-\\
&\qquad\qquad-(1+\hat{g}_{\eta_1}(-\omega))(1+\hat{g}_{\eta_2}(-\omega))\prod_{i=1}^{n}\lambda_i\prod_{j=1}^{m}\mu_j H(t,\omega)=0.
\end{aligned}
\tag{7}
$$

Proof. It is well-known [8] that $H(t,\omega)$ satisfies the equation

$$(\det(M))H(t,\omega)=0, \tag{8}$$

where $\det(M)$ is the determinant of the matrix M. It is easily verified that

$$
\begin{aligned}
\det(M)=&\prod_{k=1}^{n}\left(\frac{\partial}{\partial t}+i\omega c+\lambda_k\right)\prod_{l=1}^{m}\left(\frac{\partial}{\partial t}-i\omega v+\mu_l\right)\times\\
&\times\left(\frac{\partial}{\partial t}+i\omega c+2\lambda_n\right)\left(\frac{\partial}{\partial t}-i\omega v+2\mu_m\right)-
\end{aligned}
$$

$$- (1 + \hat{g}_{\eta_1}(-\omega))\big(1 + \hat{g}_{\eta_2}(-\omega)\big) \prod_{i=1}^{n} \lambda_i \prod_{j=1}^{m} \mu_j. \qquad \square$$

Consider the case where $c = v$, $n = m$, $\lambda_i = \mu_i = \lambda$. In this case Eq. (7) has the following form

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} + \omega^2 c^2 + \lambda^2\right)^{n-1} \left(\frac{\partial^2}{\partial t^2} + 4\lambda\frac{\partial}{\partial t} + \omega^2 c^2 + 4\lambda^2\right) H(t,\omega) - \\ - (1 + \hat{g}_{\eta_1}(-\omega))(1 + \hat{g}_{\eta_2}(-\omega))\lambda^{2n} H(t,\omega) = 0. \quad (9)$$

Without loss of generality, we assume that $\mathsf{P}(\xi(0) = 1) = 1$, i.e., the particle starts moving with velocity c. Therefore, $H(0,\omega) = H_1(0,\omega) = 1$. It follows from the first and the last equations of the set of Eqs. (6) that (all derivatives are taken with respect to t)

$$H'(0,\omega) = H_1'(0,\omega) + H_{n+m}'(0,\omega) = -i\omega c + \lambda \hat{g}_{\eta_2}(-\omega).$$

Similarly we can calculate $H^{(n)}(0,\omega)$ for any $n \in \mathbb{N}$.

Now, the pdf $f(t,x)$ can be obtained through the inverse Fourier transform applied to Eq. (9). Hence,

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - c^2\frac{\partial^2}{\partial x^2} + \lambda^2\right)^{n-1} \left(\frac{\partial^2}{\partial t^2} + 4\lambda\frac{\partial}{\partial t} - c^2\frac{\partial^2}{\partial x^2} + 4\lambda^2\right) f(t,x) - \lambda^{2n} f(t,x) + \\ + \lambda^{2n} \int_{-\infty}^{\infty} (g_{\eta_1}(-x+u) + g_{\eta_2}(-x+u) - g_{\eta_1+\eta_2}(-x+u)) f(t,u)\, du = 0,$$

with initial conditions $f(0,x) = \delta(x)$, $f'(0,x) = -c\delta'(x) + \lambda g_{\eta_2}(-x)$ and so on, where $\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - c^2\frac{\partial^2}{\partial x^2} + \lambda^2\right)^m$ is a formal m-times product of the differential operator $\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} - c^2\frac{\partial^2}{\partial x^2} + \lambda^2$, and it is assumed that $\left(\frac{\partial^2}{\partial t^2}\right)^m = \frac{\partial^{2m}}{\partial t^{2m}}$.

3. Examples

3.1. **Exponential case.** In case where $n = 1$ Eq. (9) can be written in the following form

$$\left[\frac{\partial^2}{\partial t^2} + 4\lambda\frac{\partial}{\partial t} + \omega^2 c^2 + 4\lambda^2 - (1 + \hat{g}_{\eta_1}(-\omega))(1 + \hat{g}_{\eta_2}(-\omega))\right] H(t,\omega) = 0, \qquad (10)$$

with the initial conditions $H(0,\omega) = 1$, $H'(0,\omega) = -i\omega c + \lambda\hat{g}_{\eta_2}(-\omega)$.

By solving Eq.(10), we obtain the characteristic function

$$H(t,\omega) = e^{-2\lambda t}\left(\cosh(A(\omega)t) + \frac{2\lambda - i\omega c + \lambda\hat{g}_{\eta_2}(-\omega)}{A(\omega)} \sinh(A(\omega)t)\right),$$

where $A(\omega) = \sqrt{\lambda^2(1 + \hat{g}_{\eta_1}(-\omega) + \hat{g}_{\eta_2}(-\omega) + \hat{g}_{\eta_1}(-\omega)\hat{g}_{\eta_2}(-\omega)) - \omega^2 c^2}$.

3.2. **Erlang-2 case.** In the case $n = 2$ Eq. (9) can be written in the following form

$$\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} + \omega^2 c^2 + \lambda^2\right)\left(\frac{\partial^2}{\partial t^2} + 4\lambda\frac{\partial}{\partial t} + \omega^2 c^2 + 4\lambda^2\right) H(t,\omega) - \\ - \lambda^4 (1 + \hat{g}_{\eta_1}(-\omega))(1 + \hat{g}_{\eta_2}(-\omega)) H(t,\omega) = 0,$$

with the initial conditions $H(0,\omega) = H_1(0,\omega) = 1$;

$$\frac{\partial H(0,\omega)}{\partial t} = \frac{\partial H_1(0,\omega)}{\partial t} + \frac{\partial H_4(0,\omega)}{\partial t} = -i\omega c + \lambda\hat{g}_{\eta_2}(-\omega),$$

$$\frac{\partial^2 H(0,\omega)}{\partial t^2} = \frac{\partial^2 H_1(0,\omega)}{\partial t^2} + \frac{\partial^2 H_3(0,\omega)}{\partial t^2} + \frac{\partial^2 H_4(0,\omega)}{\partial t^2} =$$

$$= -\omega^2 c^2 - \lambda^2 + \lambda^2 \hat{g}_{\eta_2}(-\omega) - 2i\omega c\lambda \hat{g}_{\eta_2}(-\omega),$$

$$\frac{\partial^3 H(0,\omega)}{\partial t^3} = \frac{\partial^3 H_1(0,\omega)}{\partial t^3} + \frac{\partial^3 H_2(0,\omega)}{\partial t^3} + \frac{\partial^3 H_3(0,\omega)}{\partial t^3} + \frac{\partial^3 H_4(0,\omega)}{\partial t^3} =$$
$$= 6ic\lambda^2\omega\hat{g}_{\eta_2}(-\omega) - 3c^2\lambda\omega^2\hat{g}_{\eta_2}(-\omega) + 3ic\lambda^2\omega +$$
$$+ \lambda^3(\hat{g}_{\eta_1}(-\omega) + 5\hat{g}_{\eta_2}(-\omega) + 3) + ic^3\omega^3.$$

Thus, the characteristic function is given by

$$H(t,\omega) = e^{-\frac{3\lambda}{2}t}(C_1\cosh(A(\omega)t) + C_2\cosh(B(\omega)t) +$$
$$+ C_3\sinh(A(\omega)t) + C_4\sinh(B(\omega)t)),$$

where

$$A(\omega) = \sqrt{L - 4\lambda\sqrt{\lambda^2(1 + \hat{g}_{\eta_1}(-\omega) + \hat{g}_{\eta_2}(-\omega) + \hat{g}_{\eta_1}(-\omega)\hat{g}_{\eta_2}(-\omega)) + 2\omega^2 c^2}},$$

$$B(\omega) = \sqrt{L + 4\lambda\sqrt{\lambda^2(1 + \hat{g}_{\eta_1}(-\omega) + \hat{g}_{\eta_2}(-\omega) + \hat{g}_{\eta_1}(-\omega)\hat{g}_{\eta_2}(-\omega)) + 2\omega^2 c^2}},$$

where $L = \lambda^2 - 4\omega^2 c^2$, and the coefficients C_i, $i = 1, 2, 3, 4$ can be calculated by using initial conditions.

4. The jump telegraph process in $\mathbb{R}^n$

Let us consider the following renewal process $\xi(t) = \max\{m \geq 0 : \tau_m \leq t\}$, $t \geq 0$, where $\tau_m = \sum_{k=0}^{m} \theta_k$, $\tau_0 = 0$ and $\theta_k \geq 0$, $k = 1, 2, \ldots$, are i.i.d. random variables with cdf $G(t)$ and corresponding pdf $g(t) = \frac{d}{dt}G(t)$. Suppose that a particle starting from $(0, 0, \ldots, 0) \in \mathbb{R}^n$, at $t = 0$, continues its motion with an absolute velocity v along a direction $\overrightarrow{\eta}_0^{(n)}$, where $\overrightarrow{\eta}_0^{(n)} = (x_1, x_2, \cdots, x_n)$ is a random vector in $\mathbb{R}^n$ uniformly distributed on the unit sphere

$$\Omega_1^{n-1} = \{(x_1, x_2, \ldots, x_n) : x_1^2 + x_2^2 + \ldots + x_n^2 = 1\}.$$

At instant τ_1 the particle changes its direction to $\overrightarrow{\eta}_1^{(n)}$ and jumps by a random vector $\overrightarrow{\beta}_1$, where $\overrightarrow{\eta}_1^{(n)}$, $\overrightarrow{\eta}_0^{(n)}$ and $\overrightarrow{\beta}_1 \in \mathbb{R}^n$ are independent. Then, at time τ_2 the particle changes its direction to $\overrightarrow{\eta}_2^{(n)}$ and jumps by $\overrightarrow{\beta}_2$, where $\overrightarrow{\eta}_2^{(n)}$, $\overrightarrow{\eta}_1^{(n)}$, $\overrightarrow{\eta}_0^{(n)}$, $\overrightarrow{\beta}_1$ and $\overrightarrow{\beta}_2$ are independent, and so on. We assume that all $\overrightarrow{\eta}_i^{(n)}$, $i = 0, 1, 2, \ldots$ are identically distributed and all $\overrightarrow{\beta}_i$, $i = 1, 2, \ldots$ are identically distributed and radial (isotropic).

Denoting by $\overrightarrow{x}^{(n)}(t)$, $t \geq 0$, the particle position at time t, then we have

$$\overrightarrow{x}^{(n)}(t) = \sum_{j=1}^{\xi(t)} \left\{ v\overrightarrow{\eta}_{j-1}^{(n)}(\tau_j - \tau_{j-1}) + \overrightarrow{\beta}_j \right\} + v\overrightarrow{\eta}_{\xi(t)}^{(n)}\big(t - \tau_{\xi(t)}\big). \tag{11}$$

Here we have assumed $\sum_{j=1}^{0} = 0$.

Consider $\varphi(t,\alpha) = \mathsf{E}\left[e^{itv\left(\overrightarrow{\alpha}, \overrightarrow{\eta}_0^{(n)}\right)}\right]$, where $\alpha = \|\overrightarrow{\alpha}\|$. We should notice that the function $\varphi(t,\alpha)$ is well-known [9] and is of the following form

$$\varphi(t,\alpha) = 2^{\frac{n-2}{2}}\Gamma\Big(\frac{n}{2}\Big)\frac{J_{\frac{n-2}{2}}(\alpha t v)}{(\alpha t v)^{\frac{n-2}{2}}}. \tag{12}$$

Theorem 2. *The characteristic function* $H(t,\alpha) = \mathsf{E}\left[e^{itv\left(\overrightarrow{\alpha}, \overrightarrow{x}^{(n)}(t)\right)}\right]$, $t \geq 0$, *is a solution of the following integral Volterra equation of a convolution type*

$$H(t,\alpha) = (1 - G(t))\varphi(t,\alpha) + \varphi_\beta(\alpha)\int_0^t g(u)\varphi(u,\alpha)H(t-u,\alpha)\,du. \tag{13}$$

Proof. It follows from Eq. (11) that

$$\begin{aligned}H(t,\alpha) &= \mathsf{E}\Big[\exp\Big\{i\Big(\overrightarrow{\alpha},\overrightarrow{x}^{(n)}(t)\Big)\Big\}\Big] = \\ &= \mathsf{E}\Big[\exp\Big\{i\Big(\overrightarrow{\alpha},\overrightarrow{x}^{(n)}(t)\Big)\Big\}\mathrm{I}\{\theta_1 > t\}\Big] + \mathsf{E}\Big[\exp\Big\{i\Big(\overrightarrow{\alpha},\overrightarrow{x}^{(n)}(t)\Big)\Big\}\mathrm{I}\{\theta_1 \le t\}\Big] = \\ &= \mathsf{E}\Big[\exp\Big\{i\Big(\overrightarrow{\alpha}, v\overrightarrow{\eta}_0^{(n)}t\Big)\Big\}\mathrm{I}\{\theta_1 > t\}\Big] + \\ &\quad + \mathsf{E}\Big[\exp\Big\{i\Big(\overrightarrow{\alpha}, v\overrightarrow{\eta}_0^{(n)}\theta_1 + \overrightarrow{\beta}_1 + S_\xi + v\overrightarrow{\eta}_{\xi(t)}^{(n)}\big(t-\tau_{\xi(t)}\big)\Big)\Big\}\mathrm{I}\{\theta_1 \le t\}\Big] = \\ &= (1-G(t))\mathsf{E}\Big[e^{itv\left(\overrightarrow{\alpha},\overrightarrow{\eta}_0^{(n)}\right)}\Big] + \int_0^t g(u)\mathsf{E}\Big[e^{iuv\left(\overrightarrow{\alpha},\overrightarrow{\eta}_0^{(n)}\right)}\Big]\varphi_\beta(\alpha)H(t-u,\alpha)du,\end{aligned}$$

where $S_\xi = \sum_{j=2}^{\xi(t)}\Big(v\overrightarrow{\eta}_j^{(n)}(\tau_{j+1}-\tau_j) + \overrightarrow{\beta}_j\Big)$, and $\varphi_\beta(\alpha) = \mathsf{E}\Big[e^{i\left(\overrightarrow{\alpha},\overrightarrow{\beta}_1\right)}\Big]$. □

Denote by $f_n(t,\overrightarrow{x})$ the pdf of the particle position at time t. By using the n-dimension inverse Fourier transform $\mathcal{F}^{-1}$ with respect to α, we get $f_n(t,\overrightarrow{x}) = \mathcal{F}^{-1}(H(t,\alpha))$.

5. The jump telegraph process in $\mathbb{R}^3$

Let us consider the three dimension case, that is $n=3$ and $\varphi(t,\alpha) = \frac{\sin(\alpha t v)}{\alpha t v}$. For Erlang-2 distribution $g(t) = \lambda^2 t e^{-\lambda t}\mathrm{I}\{t \ge 0\}$, we have

$$H(t,\alpha) = \left(e^{-\lambda t} + \lambda t e^{-\lambda t}\right)\frac{\sin(\alpha t v)}{\alpha t v} + \frac{\lambda^2\varphi_\beta(\alpha)}{\alpha v}\int_0^t e^{-\lambda u}\sin(\alpha u v)H(t-u,\alpha)\,du.$$

By denoting $h(t,\alpha) = H(t,\alpha)e^{\lambda t}$, we obtain

$$h(t,\alpha) = (1+\lambda t)\frac{\sin(\alpha t v)}{\alpha t v} + \frac{\lambda^2\varphi_\beta(\alpha)}{\alpha v}\int_0^t \sin(\alpha(t-u)v)h(u,\alpha)\,du. \tag{14}$$

By differentiating twice Eq. (14), we have

$$\frac{\partial^2}{\partial t^2}h(t,\alpha) + \left(\alpha^2 v^2 - \lambda^2\varphi_\beta(\alpha)\right)h(t,\alpha) = 2\frac{\sin(\alpha t v) - \alpha t v\cos(\alpha t v)}{\alpha t^3 v} \tag{15}$$

with initial conditions $h(0,\alpha) = 1$, $\frac{\partial}{\partial t}h(0,\alpha) = \lambda$, which can be obtained directly from Eq. (14).

By solving Eq. (15) with the initial conditions, we obtain

$$\begin{aligned}H(t,\alpha) = e^{-\lambda t}h(t,\alpha) &= \frac{\sin(\alpha t v)}{\alpha t v}e^{-\lambda t} + \frac{\sin(At)}{A}\lambda e^{-\lambda t} + \\ &+ \Big[\ln\frac{\alpha v - A}{\alpha v + A} + (\mathrm{Si}(t(A-\alpha v)) - \mathrm{Si}(t(A+\alpha v)))\sin(At) + \\ &+ (\mathrm{Ci}(t(A-\alpha v)) - \mathrm{Ci}(t(A+\alpha v)))\cos(At)\Big]\frac{\lambda^2\varphi_\beta(\alpha)}{2A\alpha v}e^{-\lambda t},\end{aligned}$$

where $A = \sqrt{\alpha^2 v^2 - \lambda^2\varphi_\beta(\alpha)}$, $\mathrm{Si}(x) = \int_0^x \frac{\sin(t)}{t}dt$, $\mathrm{Ci}(x) = \gamma + \ln(x) + \int_0^x \frac{\cos(t)-1}{t}dt$, and γ is the Euler–Mascheroni constant.

References

1. N. Ratanov, *A jump telegraph model for option pricing*, Quant. Finance, **7** (2007), no. 5, 575–583.
2. O. Lopez, N. Ratanov, *Option pricing driven by a telegraph process with random jumps*, J. Appl. Prob., **49** (2012), no. 3, 838–849.
3. N. Ratanov, *Option pricing model based on a Markov-modulated diffusion with jumps*, Braz. J. Probab. Stat., **24** (2010), no. 2, 413–431.
4. N. Ratanov, A. Melnikov, *On financial markets based on telegraph processes*, Stochastics, **80** (2008), no. 2-3, 247–268.

5. A. Di Crescenzo, B. Martinucci, *On the generalized telegraph process with deterministic jumps*, Methodology and Computing in Applied Probability, **15** (2013), no. 1, 215–235.
6. A. Di Crescenzo, A. Iuliano, B. Martinucci, S. Zacks, *Generalized telegraph process with random jumps*, Journal of Applied Probability, **50** (2013), no. 2, 450–463.
7. A. Di Crescenzo, *On random motions with velocities alternating at Erlang-distributed random times*, Adv. in Appl. Probab., **33** (2001), no. 3, 690–701.
8. A. Pogorui, R. M. Rodríguez-Dagnino, *One dimensional semi-Markov evolution with general Erlang sojourn times*, Random Operators and Stochastic Equations, **13** (2005), no. 4, 399–405.
9. A. Pogorui, R. M. Rodríguez-Dagnino, *Random motion with uniformly distributed directions and random velocity*, Journal of Statistical Physics, **147** (2012), no. 6, 1216–1225.
10. A. De Gregorio, *On random flights with non-uniformly distributed directions*, J. Stat. Phys., **147** (2012), no. 2, 382–411.
11. A. De Gregorio, *A family of random walks with generalized Dirichlet steps*, J. Math. Phys., **55** (2014), no. 2, 023302.
12. R. Garra, E. Orsingher, *Random flights governed by Klein–Gordon–type partial differential equations*, Stochastic Processes and their Applications, **124** (2014), no. 6, 2171–2187.
13. A. Di Crescenzo, A. Meoli, *On a jump-telegraph process driven by an alternating fractional Poisson process*, J. Appl. Probab., **55** (2018), no. 1, 94–111.
14. L. Angelani, *Run-and-tumble particles, telegrapher's equation and absorption problems with partially reflecting boundaries*, Journal of Physics A: Mathematical and Theoretical, **48** (2015), 495003.
15. V. S. Korolyuk, A. V. Swishchuk, *Semi-Markov Random Evolutions*, Kluwer Academic Publishers, Dordrecht, 1995.
16. V. S. Korolyuk, V. V. Korolyuk, *Stochastic Models of Systems*, Kluwer Academic Publishers, Dordrecht, 1999.
17. A. Kolesnik, N. Ratanov, *Telegraph processes and option pricing*, Springer Briefs in Statistics, Springer, Heidelberg, 2013.

Department of Algebra and Geometry, Zhytomyr State University, Valyka Berdychivska St., 40, Zhytomyr, Ukraine, 10008
E-mail address: pogor@zu.edu.ua

School of Engineering and Sciences, Tecnologico de Monterrey, Av. Eugenio Garza Sada 2501 Sur, C.P. 64849, Monterrey, N. L., México.
E-mail address: rmrodrig@tec.mx

Received 04.12.2018

ДИФЕРЕНЦІАЛЬНІ ТА ІНТЕГРАЛЬНІ РІВНЯННЯ ДЛЯ ВИПАДКОВИХ БЛУКАНЬ ІЗ СТРИБКАМИ

А. О. ПОГОРУЙ, Р. М. РОДРІГЕС-ДАГНІНО

Анотація. Отримано інтегро-диференціальне рівняння для характеристичної функції випадкового блукання із випадковими стрибками на прямій, де чергування напрямку та стрибки відбуваються відповідно до моментів відновлення, причому час між двома послідовними змінами швидкості має розподіл Ерланга. Ми також вивчаємо випадкові блукання частинки із стрибками у вищих розмірностях, для яких одержано рівняння типу відновлення для характеристичної функції положення частинки. У тривимірному випадку отримано диференціальне рівняння телеграфного типу для випадкового руху з випадковими стрибками, де зміни напрямків і стрибків відбуваються відповідно до моментів відновлення та час між двома послідовними моментами має розподіл Ерланга.