

LIMIT THEOREMS FOR SUMS OF INDEPENDENT RANDOM VARIABLES IN THE DOMAIN OF ATTRACTION OF A STABLE LAW: A SURVEY

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JOOP MIJNHEER

РЕЗЮМЕ. In this survey we consider the following type of limit theorems. Laws of the iterated logarithm, results for the heavy tails and the rate of escape of random walks, strong approximations for partial sums, the rate of convergence of the sum of the sample extremes and U -statistics.

1. INTRODUCTION

In samples of stable random variables or random variables in their domains of attraction the (sample) extremes play an important role. The influence of the extremes in sums of independent random variables is more dominant than in the case of random variables with a finite variance. The influence of the maximal term in the sum of independent random variables was first studied in DARLING (1952). In MIJNHEER (1968) we described the behaviour of the sample average for random variables with an infinite expectation. A more precise description is given in the next sections. There we will give upper and lower bounds.

For applications of stable random variables or random variables in the domain of attraction we refer to MANDELBROT (1982). In the chapters 31, 32 and 37 he gives many references. Here we give some recent references. This list is far from complete! We mention MONTROLL and BENDLER (1984), WERON (1986), NONNENMACHER and NONNENMACHER (1988) and TUCKER (1992). But the discussion about the applicability is still going on. See LAU, LAU and WINGERDER (1990) and LAU and LAU (1993).

MANDELBROT (1982, p. 367) asserts that stable random variables were first investigated by Paul Lévy. He calls them Lévy-stable. For the theory of stable random variables we refer to the second volume of FELLER (1971). A summary is given in MIJNHEER (1974). In this paper we use the same notation as in the mentioned monograph.

X is a stable random variable with distribution function $F(\cdot; \alpha, \beta, \gamma, c)$ and characteristic function f given by

$$\log f(t) = \begin{cases} i\gamma t - c|t|^\alpha \{1 - i\beta \operatorname{sign}(t) \tan(\pi\alpha/2)\}, & \text{if } \alpha \neq 1, \\ i\gamma t - c|t| - i\beta(2/\pi)ct \log |t|, & \text{if } \alpha = 1, \end{cases}$$

where α , β , γ and c are real constants with $c \geq 0$, $0 < \alpha < 2$, and $|\beta| \leq 1$. We write $F(\cdot; \alpha, \beta)$ for $F(\cdot; \alpha, \beta, 0, 1)$.

If X_1, \dots, X_n are independent identically distributed with common distribution function $F(\cdot; \alpha, \beta)$ then

$$X_1 + \dots + X_n \stackrel{d}{=} n^{1/\alpha} X_1 \quad \text{if } \alpha \neq 1$$

and

$$X_1 + \dots + X_n - \left(\frac{2}{\pi}\right) \beta n \log n \stackrel{d}{=} n X_1 \quad \text{if } \alpha = 1.$$

$\stackrel{d}{=}$ denotes that the random variables have the same distribution.

A random variable Y_1 with distribution function G belongs to the domain of attraction of the stable distribution $F(\cdot; \alpha, \beta)$ if there exist sequences of norming constants $\{a_n\}$ and $\{b_n\}$ such that, for Y_1, \dots, Y_n independent identically distributed with common distribution G , the distribution of $a_n^{-1}(Y_1 + \dots + Y_n - b_n)$ converges weakly to $F(\cdot; \alpha, \beta)$. (Notation $Y_1 \in \mathcal{D}(\alpha, \beta)$.) In the case that we can take $a_n = an^{1/\alpha}$ for some positive constant a , we say that Y_1 is in the domain of normal attraction. (Notation $Y_1 \in \mathcal{D}_N(\alpha, \beta)$.)

In the list of references given in section 7 one will find one rather unknown reference: the theses of FELDHEIM (1937).

2. LAWS OF THE ITERATED LOGARITHM

In this section we discuss analogues of HARTMAN and WINTNER's law of the iterated logarithm for stable random variables and random variables in the domain of attraction. For independent identically distributed random variables Y_1, Y_2, \dots with $EY_i = 0$, $EY_i^2 = 1$ and $S_n = Y_1 + \dots + Y_n$, HARTMAN and WINTNER (1941) showed

$$\limsup_{n \rightarrow \infty} \frac{S_n}{(2n \log \log n)^{1/2}} = 1 \quad \text{a.s.} \quad (2.1)$$

STRASSEN (1964) proves a beautiful generalization of the result (2.1), the so called functional law of the iterated logarithm.

The case $0 < \alpha < 1$.

In the case of completely asymmetric stable random variables X_i , $i = 1, \dots$, with distribution function $F(\cdot; \alpha, 1)$ and $S_n = X_1 + \dots + X_n$ we have

$$\liminf_{n \rightarrow \infty} n^{-1/\alpha} (2 \log \log n)^{(1-\alpha)/\alpha} S_n = c(\alpha) \quad \text{a.s.} \quad (2.2)$$

The value of $c(\alpha)$ is given in MIJNHEER (1974) section 6.2. There one can also find references for the proof and an analogue of the Kolmogorov-Erdős integral test for these stable random variables.

A result similar to (2.2) can also be proved for partial sums of random variables in the domain of attraction under some conditions on the right hand tail.

Let V_1, V_2, \dots be independent identically distributed random variables with

$$P(V_1 = 1) = \frac{1}{2} = P(V_1 = -1).$$

We set

$$W_0 = 0, \quad W_n = V_1 + \dots + V_n, \quad n \geq 1.$$

Let T_n be the subscript of the n^{th} zero in the sequence W_1, W_2, \dots then the results in CHUNG and HUNT (1949) imply (2.2) with S_n replaced by T_n .

An extension to the case where $(S_n)_{n=1}^{\infty}$ is replaced by a non-negative continued fraction mixing stationary process, is given in AARONSON and DENKER (1990).

A result like (2.2) gives an exact lower bound for the behaviour of the sample average when the number of observations increases.

In MIJNHEER (1982) we consider independent identically distributed random variables Y_i , $i = 1, 2, \dots$, with common distribution given by

$$P(Y_1 > y) = y^{-\alpha} L(y) \quad \text{for } y \geq y_0 > 0,$$

where L is a slowly varying function at infinity. Let $S_n = Y_1 + \dots + Y_n$. We study the set of limit points of $\{n^{-1/\alpha} S_n; n = 1, 2, \dots\}$. We distinguish two cases, i.e. L non-decreasing and L non-increasing. In both cases we give the conditions in terms of L for the possible limit sets.

The case $\alpha = 1$.

See MIJNHEER (1974) section 6.3.

The case $1 < \alpha < 2$.

See MIJNHEER (1974) section 6.4.

Functional laws of the iterated logarithm are proved in MIJNHEER (1974) chapter 9.

3. RESULTS FOR THE HEAVY TAILS AND THE RATE OF ESCAPE OF RANDOM WALKS

Let X_1, X_2, \dots be independent identically distributed random variables with common distribution function $F(\cdot; \alpha, \beta)$. We define the sequence $\{T(n; \alpha, \beta)\}$ by

$$T(n; \alpha, \beta) = \begin{cases} (X_1 + \dots + X_n)n^{-1/\alpha} & \text{for } \alpha \neq 1 \\ (X_1 + \dots + X_n - (2/\pi)\beta n \log n)n^{-1} & \text{for } \alpha = 1. \end{cases}$$

We also define $\{\tilde{T}(n; \alpha, \beta)\}$ by

$$\tilde{T}(n; \alpha, \beta) = \text{sign}(T(n; \alpha, \beta))|T(n; \alpha, \beta)|^{1/\log \log n}.$$

Denote by $L(\alpha, \beta)$ the set of limit points of $\{\tilde{T}(n; \alpha, \beta)\}$.

Theorem 3.1. *Let $\{\tilde{T}(n; \alpha, \beta)\}$ be defined as above for $0 < \alpha < 2$ and $|\beta| \leq 1$. Then, with probability 1, all limit points of $\{\tilde{T}(n; \alpha, \beta)\}$ are*

$$L(\alpha, \beta) = \begin{cases} [-e^{1/\alpha}, e^{1/\alpha}], & 1 \leq \alpha < 2, |\beta| \neq 1, \\ [-e^{1/\alpha}, 1], & 1 \leq \alpha < 2, \beta = -1, \\ [-1, e^{1/\alpha}], & 1 \leq \alpha < 2, \beta = 1, \\ [-e^{1/\alpha}, -e^{-1/(1-\alpha)}] \cup [e^{-1/(1-\alpha)}, e^{1/\alpha}], & 0 < \alpha < 1, |\beta| \neq 1, \\ [-e^{1/\alpha}, -1], & 0 < \alpha < 1, \beta = -1, \\ [1, e^{1/\alpha}], & 0 < \alpha < 1, \beta = 1. \end{cases}$$

Proof. In Theorem 8.12 of MIJNHEER (1974) we showed that all points of these intervals are limit points. It was mentioned in remark 8.12 that I did not know if I gave all the limit points. PAKSHIRAJAN and VASUDEVA (1977) showed that the intervals in this theorem contain all the limit points. \square

In chapter 8 of MIJNHEER (1974) we give more references for the law of the iterated logarithm type theorems for partial sums of stable random variables and random variables in their domain of attraction. For the sake of completeness we mention here that CHUNG and HUNT (1949) give an integral test for T_n as defined in section 2.

VASUDEVA (1978) considers $\{T_{j,n}\}$, $j = 1, \dots, k$, k independent copies of $\{T(n; \alpha, \beta)\}$ and gives the limit set of the sequence of vectors

$$\xi_n(k) = \{|T_{1,n}|^{1/\log n}, \dots, |T_{k,n}|^{1/\log n}\}.$$

From the law of the iterated logarithm type theorem for the heavy tail we may obtain an upper bound for the sample average for random variables in $\mathcal{D}_{\mathcal{N}}(\alpha, 1)$, $0 < \alpha < 1$.

Note the gap in the set of limit points in the case $0 < \alpha < 1$ and $|\beta| \neq 1$. In section 8.1 of MIJNHEER (1974) we showed, for these values for α and β ,

$$P(0 \leq T(n_k; \alpha, \beta) \leq a_k \text{ i.o.}) = 0 \quad \text{or} \quad 1 \quad (3.1)$$

according as $\sum a_k$ converges or diverges, where $n_k = \max(k, \lceil \gamma^{k^{\delta}} \rceil)$ for $\gamma > 1$ and $\delta \geq 1 - \alpha$ (i.o. means "infinitely often"). The zero-one law in (3.1) implies the rate of escape of $n_k^{-1/\alpha} |S_{n_k}| = T(n_k; \alpha, \beta)$. For different values for δ we obtain the limit points in $[e^{-1/(1-\alpha)}, 1]$.

Let Y_1, Y_2, \dots be independent identically distributed symmetric non-lattice random variables in the domain of (non-normal) attraction of the Cauchy distribution ($Y_i \in \mathcal{D}(1, 0)$) i.e.

$$P(|Y_1| > y) = y^{-1} L(y),$$

where L is slowly varying. Let $V_n = Y_1 + \dots + Y_n$. In MIJNHEER (1987) we studied the rate of escape of the transient random walk V_n . We proved an integral test for the almost sure $\liminf n^{-\gamma} |V_n|$ for some $\gamma \in (0, 1)$.

4. STRONG APPROXIMATION FOR PARTIAL SUMS

Throughout this section Y_1, Y_2, \dots will denote a sequence of independent identically distributed random variables in the domain of attraction of the stable law $F(\cdot; \alpha, \beta)$ and X_1, X_2, \dots is a sequence of independent identically distributed stable random variables with this stable law. In this section we give almost sure bounds for $|\sum_{i=1}^n (Y_i - \tau_i X_i)|$, where $\{\tau_i\}$ is a sequence of positive numbers. For $\alpha = 2$ we refer to the monograph of CSÖRGÓ and RÉVÉSZ (1981).

The case $Y_1 \in \mathcal{D}_{\mathcal{N}}(\alpha, 0)$, symmetric and $0 < \alpha < 2$, is considered in STOUT (1979). In that paper he takes $\tau_i = 1$ for $i = 1, 2, \dots$. In MIJNHEER (1983) we consider the case $Y_1 \in \mathcal{D}(\alpha, 0)$, symmetric and $\tau_i \neq 1$. The restriction to symmetrically distributed random variables is made for technical reasons.

FISHER (1984) obtains a different solution to the same problem. The main difference between the solutions of MIJNHEER (1983) and FISHER (1984) is that in the case that $Y_i \in \mathcal{D}_{\mathcal{N}}(\alpha, 0)$ Fisher takes $\tau_i \equiv 1$. Both papers give several examples.

ZINCHENKO (1985) considers $Y_1 \in \mathcal{D}_{\mathcal{N}}(\alpha, \beta)$ and some restriction on the characteristic function of Y_1 . For $\tau_i \equiv 1$ she obtains the "natural" upper bound $o(n^{1/\alpha})$.

BERKES, DABROWSKI, DEHLING, and PHILIPP (1986) take $Y_1 \in \mathcal{D}_{\mathcal{N}}(\alpha, 0)$ and symmetric. They approximate $\sum_{i=1}^n Y_i$ not by $\sum_{i=1}^n X_i$, a partial sum of independent identically distributed stable random variables, but by $\sum_{i=1}^n X_i + \sum_{i=1}^n Z_i$, where X_i are independent identically distributed stable random variables and Z_i are independent identically distributed correcting random variables independent of the X_i 's.

BERKES and DEHLING (1989) give a new method to prove upper bounds and also obtain lower bounds.

5. RATE OF CONVERGENCE OF THE SAMPLE EXTREMES

Rates of convergence of the distribution function of normalized sums of random variables $Y_i \in \mathcal{D}(\alpha, \beta)$ to a stable law are investigated in the monograph of CHRISTOPH and WOLF (1992). This book contains many interesting examples and counterexamples. In this section we give the rate of convergence of the distribution function of the normalized sum of the sample extremes to a stable law.

As mentioned in the introduction DARLING (1952) investigated the role of the maximal term $M_n = \max(Y_1, \dots, Y_n)$ in the sum $S_n = Y_1 + \dots + Y_n$ in the case $Y_1 \geq 0$ and $Y_1 \in \mathcal{D}(\alpha, 1)$ with $0 < \alpha < 1$. From now on we restrict ourselves to this case.

Let $Y_{n:n} \geq Y_{n:n-1} \geq \dots \geq Y_{n:1}$ be the ordered sample. AROV and BOBROV (1960) showed, for $k_n \rightarrow \infty$ and $k_n n^{-1} \log n$ as $n \rightarrow \infty$,

$$S_n = Y_{n:n} + \dots + Y_{n:n-k_n+1} + [\alpha(1-\alpha)^{-1} + o(1)]k_n Y_{n:n-k_n}.$$

They obtain similar results in the case $Y_1 \in \mathcal{D}(\alpha, \beta)$ for $\alpha \in (0, 1) \cup (1, 2)$. It follows from Theorem 3 in CSÖRGÖ a.o. (1986) that $n^{-1/\alpha}(Y_{n:n} + \dots + Y_{n:n-k_n+1})$ converges in distribution to the completely asymmetric stable law $F(\cdot; \alpha, 1)$. They also give results for the general case. The main tool in their proof is a Brownian bridge approximation to the uniform empirical process. In MIJNHEER (1986) we derive the rate of convergence. If k_n tends slowly to infinity the rate of convergence is rather poor as we see in the following theorem.

Theorem 5.1. *Let Y_1, Y_2, \dots be independent identically distributed random variables with*

$$P(Y_1 > y) = y^{-\alpha} + r(y) \quad \text{for } y \geq y_0 > 0$$

and $r(y) = O(y^{-\gamma})$ for $y \rightarrow \infty$ and $\alpha < \gamma$. Let $n, k_n \rightarrow \infty$ and $k_n n^{-1} \rightarrow 0$. Then

$$\sup_y |P(n^{-1/\alpha}(Y_{n:n} + \dots + Y_{n:n-k_n+1}) \leq y) - F(y; \alpha, 1)| = O(\max(k_n^{1-1/\alpha}, n^{-1})).$$

This is part a) of the result in MIJNHEER (1986). If we take $k_n = (\log n)^p$, $p > 0$, we notice how poor this rate is. CRAMÉR (1963) showed under the same conditions

$$\sup_y |P(n^{-1/\alpha}(Y_1 + \dots + Y_n) \leq y) - F(y; \alpha, 1)| = O(n^{-\lambda/\alpha}),$$

where $\lambda = \min(1, \gamma - \alpha)$. Unfortunately the reference to the paper of Cramér is missing in the section on this subject in the monograph of Christoph and Wolf.

In 1986 I knew one other result about the rate of convergence. In HALL (1978) he considers Y_1 symmetric and $Y_1 \in \mathcal{D}_{\mathcal{N}}(\alpha, 0)$. In MIJNHEER (1986) we discussed the difference between the two approaches.

More results about the influence of extreme negative and positive terms one can find in CSÖRGÖ (1989).

6. U-STATISTICS

U -statistics are introduced by Hoeffding (1948). He gave conditions under which the standardized statistics have a normal limit law.

MALEVIC and ABDALINOV (1977) is the first paper where we have a stable limit law. They restrict themselves to the case $1 < \alpha < 2$. The same restriction is made in Chapter 6 of the monograph of CHRISTOPH and WOLF (1992). Under this restriction the U -statistic possesses a finite expectation and has the U -statistic the Hoeffding decomposition. See lemma 6.1 in CHRISTOPH and WOLF (1992).

In the case $0 < \alpha \leq 1$ the expectation does not exist. In this case less is known. In MIJNHEER (1991) we consider the (simple) U -statistic

$$U_n = \sum_{i=1}^n \sum_{j=1, i \neq j}^n X_i X_j,$$

where $X_i \in \mathcal{D}_{\mathcal{N}}(\alpha, 1)$ with $0 < \alpha < 1$ and obtain the tail behaviour for large values of the limit distribution of $n^{-2/\alpha}U_n$ if $n \rightarrow \infty$.

In MIJNHEER (1993) we give the influence of the maximal term in U_n . I.e. we compute

$$\lim_{n \rightarrow \infty} E \left(\frac{U_n}{X_{(n-1)} X_{(n)}} \right).$$

Obviously there exists a relation with double stable integrals. They have been studied in several papers. For reference see MIJNHEER (1991).

ЛИТЕРАТУРА

1. J. Aaronson and M. Denker, *Upper bounds for ergodic sums of infinite measure preserving transformations*, Trans. Amer. Math. Soc. **319** (1990), 101–138.
2. D. Z. Arov and A. A. Bobrov, *The extreme terms of a sample and their role in the sums of independent variables*, Th. Probab. Appl. **5** (1960), 377–395.
3. I. Berkes, A. Dabrowski, H. Dehling, and W. Philipp, *A strong approximation theorem for sums of random vectors in the domain of attraction to a stable law*, Acta Math. Hung. **48** (1986), 161–172.
4. I. Berkes and H. Dehling, *Almost Sure and Weak Invariance Principles for Random Variables Attracted by a Stable Law*, Probab. Th. Rel. Fields **83** (1989), 331–353.
5. J. Chover, *A law of the iterated logarithm for stable summands*, Proc. Amer. Math. Soc. **17** (1966), 441–443.
6. G. Christoph and W. Wolf, *Convergence theorems with a stable limit law*, Mathematical Research, vol. 70, Akademie Verlag, Berlin, 1992.
7. K. L. Chung and G. A. Hunt, *On the zeros of $\sum_1^n \pm 1$* , Ann. Math. **50** (1949), 385–400.
8. H. Cramer, *On asymptotic expansions for sums of independent random variables with a limiting stable distribution*, Sankhya **A25** (1963), 13–24.
9. M. Csörgő and P. Révész, *Strong approximations in probability and statistics*, Academic Press, New York, 1981.
10. S. Csörgő, L. Horváth, and D. M. Mason, *What portion of the sample makes a partial sum asymptotically stable or normal?*, Probab. Th. Rel. Fields **72** (1986), 1–16.
11. S. Csörgő, *Notes on extreme and self-normalized sums from the domain of attraction of a stable law*, J. London Math. Soc. **39** (1989), 369–384.
12. D. A. Darling, *The influence of the maximum term in the addition of independent random variables*, Trans. Amer. Math. Soc. **73** (1952), 95–107.
13. M. E. Feldheim, *Etude de la stabilité des lois de probabilité: Thèses.*, Faculté des sciences, Université de Paris, Paris, 1937.
14. W. Feller, *An introduction to probability theory and its applications*, vol. II, Wiley, New York, 1971.
15. E. Fisher, *An almost sure invariance principle for random variables in the domain of attraction of a stable law*, Z. Wahrscheinlichkeitstheorie verw. Gebiete **67** (1984), 461–471.
16. P. Hall, *On the extreme terms of a sample from the domain of attraction of a stable law*, J. London Math. Soc. **18** (1978), 181–191.
17. P. Hartman and A. Wintner, *On the law of the iterated logarithm*, Amer. J. Math. **63** (1941), 169–176.
18. A. H.-L. Lau, H.-S. Lau, and J. R. Wingender, *The distribution of stock returns: new evidence against the stable model*, Journ. Bus. & Econ. Stat. **8** (1990), 217–224.
19. H.-S. Lau and A. H.-L. Lau, *The reliability of the stability-under-addition test for the stable-Paretian hypothesis*, J. Statist. Comput. Simul. **48** (1993), 67–80.
20. T. L. Malevich and Abdalimov, *Stable limit distributions for U-statistics*, Theory Probab. Appl. **22** (1977), 370–377.
21. B. B. Mandelbrot, *The fractal geometry of nature*, W. H. Freeman and Company, New York, 1982.
22. J. L. Mijneer, *The conduct of the sample average when the first moment is infinite*, Statistica Neerlandica **22** (1968), 37–41.
23. ———, *Sample path properties of stable processes*, Math. Centre Tracts, vol. 59, Mathematisch Centrum, Amsterdam, 1974.
24. ———, *Limit points of $\{n^{-1/\alpha} S_n\}$* , Ann. of Prob. **10** (1982), 382–395.
25. ———, *Strong approximations of partial sums of independent identically distributed random variables in the domain of attraction of a symmetric distribution*, 9th Prague Conference on

Information Theory, Statistical Decision Functions, Random Processes, Czechoslovak Acad. Sci., Prague, 1983, pp. 83–89.

26. ———, *On the rate of escape of a transient random walk*, *Serdica* **13** (1987), 98–105.
27. ———, *On the rate of convergence of the sum of the sample extremes*, *Probab. Th. Rel. Fields* **79** (1988), 317–325.
28. ———, *U-statistics and double stable integrals*, *I.M.S. Lecture Notes* **18** (1991), 256–268.
29. ———, *On the influence of the maximal term in a (simple) U-statistic*, *Probability Theory and Mathematical Statistics* **50** (1994), 101–104.
30. E. W. Montroll and J. T. Bendler, *On Lévy (or stable) Distributions and the Williams Watts Model of Dielectric Relaxation*, *Journ. Stat. Physics* **34** (1984), 129–163.
31. T. F. Nonnenmacher and D. J. F. Nonnenmacher, *On Lévy distributions and some applications to biophysical systems*, *Stoch. Proc. Phys. & Geometry Proc.*, Ascona/Locarno (1988), World Scientific, Singapore, 627–638.
32. R. P. Pakshirajan and R. Vasudeva, *A law of the iterated logarithm for stable summands*, *Trans. Amer. Math. Soc.* **232** (1977), 33–42.
33. W. Stout, *Almost sure invariance principles when $EX_1^2 = \infty$* , *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **49** (1979), 23–32.
34. V. Strassen, *An invariance principle for the law of the iterated logarithm*, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **3** (1964), 211–226.
35. A. L. Tucker, *A reexamination of finite — and infinite — variance distributions as models of daily stock returns*, *Journ. Bus. & Econ. Stat.* **10** (1992), 73–81.
36. R. Vasudeva, *A logarithmic law for sums of stable random variables*, *Sankhya* **A40** (1978), 174–184.
37. K. Weron, *Relaxation in glassy materials from Lévy stable distribution*, *Acta Physica Polonica* **A 70** (1986), 529–540.
38. N. M. Zinchenko, *A strong invariance principle for sums of random variables from the domain of attraction of a stable law*, *Theor. Prob. Appl.* **30** (1985), 148–152.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF LEIDEN, LEIDEN, THE NETHERLANDS

Электронна адреса: mijnher@rulwinv.leidenuniv.nl

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