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BERNSTEIN–VON MISES TYPE THEOREM FOR A CLASS OF HILBERT SPACE VALUED STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. We prove a Bernstein–von Mises type theorem for a class of Hilbert space-valued stochastic differential equations and apply it to study the Bayes estimation problem for linear infinite dimensional stochastic differential equations. We obtain the strong consistency, asymptotic normality and the asymptotic efficiency of the Bayes estimator under some regularity conditions.

1. Introduction

Statistical inference for parameters occuring in finite dimensional stochastic differential equations has been discussed in Liptser and Shiryayev (1978), Basawa and Prakasa Rao (1980) and more recently in Prakasa Rao (1999 a,b).

Here we study the parameter estimation problem for an infinite dimensional stochastic differential equation. Loges (1984) proved a Girsanov type theorem in Hilbert space and applied it to the study of asymptotic properties of the maximum likelihood estimators of the parameters. We discuss the problem of Bayes estimation of the parameters. Similar results have been obtained earlier for diffusion processes and diffusion fields in Prakasa Rao (1981, 84). The theory of infinite dimensional stochastic differential equations was earlier developed in Curtain and Pritchard (1978) and more recently in Kallianpur and Xiong (1995).

2. Main Results

2.1. Bernstein–von Mises theorem.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space and H denote a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Consider the stochastic differential equation

$$dZ(t) = \theta A Z(t) + dW(t), \qquad Z(0) = z_0,\ t \geq 0, \tag{2.1}$$

where $\{Z(t), t \geq 0\}$ is a H-valued stochastic process defined on $(\Omega, \mathcal{F}, \mathsf{P})$, A is the infinitesimal generator of a strongly continuous semigroup acting on H and $\{W(t), t \geq 0\}$ is a H-valued Wiener process with covariance operator W which is nuclear (cf. Kallianpur and Xiong (1995)). We assume that $W(0) = 0$ and that all the eigen values λ_i, $i \geq 1$, of W are strictly positive.

Suppose the parameter $\theta \in \Theta$ is unknown and real-valued. The problem of interest is the estimation of the parameter θ based on observation of $Z(t)$, $0 \leq t \leq T$. It is known that the H-valued Wiener process $\{W(t), t \geq 0\}$ can be represented in the form

$$W(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i \quad \mathsf{P}\text{-a.s}, \tag{2.2}$$

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where $\{e_i, i \geq 1\}$ is a complete orthonormal basis for H consisting of eigen vectors (with respect to λ_i) of W. Note that

$$\beta_i(t) = \langle W(t), e_i \rangle, \qquad i \geq 1, \tag{2.3}$$

and $\{\beta_i, (t), t \geq 0\}$, $i \geq 1$, are independent real-valued standard Wiener processes such that $\mathsf{E}[\beta_i(t+1) - \beta_i(t)]^2 = \lambda_i$.

The following result is due to Loges (1984).

Theorem 2.1. *Let A be an infinitesimal generator of a strongly continuous semigroup in H and suppose the process $\{Z(t), t \geq 0\}$ satisfies the stochastic differential equation*

$$dZ(t) = AZ(t)\,dt + dW(t), \qquad Z(0) = z_0,\ t \geq 0. \tag{2.4}$$

Further suppose that there exist positive constants C and D such that

$$\mathsf{E}\left(\exp\left(D\left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \langle AZ(t), e_i \rangle^2\right)\right)\right) \leq C, \qquad 0 \leq t \leq T. \tag{2.5}$$

Then the processes $\{Z(t), 0 \leq t \leq T\}$ and $\{W(t) + z_0, 0 \leq t \leq T\}$ induce equivalent measures $\nu^{(T)}$ and $\mu_{z_0}^{(T)}$ on $(C([0,T], H), \mathcal{B})$ and

$$\log \frac{d\nu^{(T)}}{d\mu_{z_0}^{(T)}}(\{Z(t,\omega), 0 \leq t \leq T\}) \tag{2.6}$$

$$= \frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \int_0^T \langle AZ(t,\omega), e_i \rangle^2\, dt + \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \int_0^T \langle AZ(t,\omega), e_i \rangle\, d(\langle W(t,\omega), e_i \rangle)$$

$$= -\frac{1}{2}\sum_{i=1}^{\infty} \frac{1}{\lambda_i} \int_0^T \langle AZ(t,\omega), e_i \rangle^2\, dt \tag{2.7}$$

$$+ \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \int_0^T \langle AZ(t,\omega), e_i \rangle\, d(\langle Z(t,\omega), e_i \rangle).$$

Remarks. The stochastic integrals on the right side of (2.6) and of (2.7) are defined on the L_2-mean. Here the space $C([0,T), H)$ is the Banach space of continuous functions $f\colon [0,T] \to H$ endowed with the supremum norm and $\mathcal{B}$ is the associated Borel σ-algebra of subsets.

Let us consider the SDE given by (2.1) and let ν_θ be the probability measure generated by the process $\{Z(t), 0 \leq t \leq T\}$ on $(C([0,T], H), \mathcal{B})$. Suppose the condition (2.5) holds. Then, it follows by Theorem 2.1 that

$$\log \frac{d\nu_\theta^T}{d\mu_{z_0}^T} = -\frac{1}{2}\theta^2 \beta_T + \theta\gamma_T,$$

where

$$\beta_T = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \int_0^T \langle AZ(t), e_i \rangle^2\, dt \tag{2.8}$$

and

$$\gamma_T = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \int_0^T \langle AZ(t), e_i \rangle\, d(\langle Z(t), e_i \rangle). \tag{2.9}$$

In particular, it follows that the MLE $\hat{\theta}_T$ of θ satisfies the relation

$$\hat{\theta}_T = \frac{\gamma_T}{\beta_T}. \tag{2.10}$$

Loges (1984) discussed sufficient conditions for the strong consistency and the asymptotic normality of the estimator $\hat{\theta}_T$.

Suppose that Λ is a prior probability measure on (Θ, ζ), where ζ is the σ-algebra of Borel subsets of $\Theta \subset \mathbf{R}$. Assume that Θ is open. Further suppose that Λ has the density

$\lambda(\cdot)$ with respect to the Lebesgue mean e and the density $\lambda(\cdot)$ is continuous and positive in an open neighbourhood of θ_0, the true parameter. The posterior density of θ given $Z^T = \{Z(t), 0 \leq t \leq T\}$ is

$$p(\theta(Z^T) = \frac{\frac{d\nu_\theta^{(T)}}{d\mu_{Z_0}^{(T)}}(Z^T)\lambda(\theta)}{\int_\Theta \frac{d\nu_\theta^{(T)}}{d\mu_{Z_0}^{(T)}}(Z^T)\lambda(\theta)\,d\theta}. \tag{2.11}$$

It can be checked that

$$\begin{aligned}\sqrt{T}(\hat{\theta}_T - \theta_0) &= \frac{T^{-1/2}\sum_{i=1}^{\infty}(1/\lambda_i)\int_0^T \langle AZ(t), e_i\rangle\, d(\langle W(t), e_i\rangle)}{T^{-1}\sum_{i=1}^{\infty}(1/\lambda_i)\int_0^T \langle AZ(t), e_i\rangle^2\, dt} \\ &= \frac{T^{-1/2}\sum_{i=1}^{\infty}(1/\lambda_i)\int_0^T \langle AZ(t), e_i\rangle\, d(\langle W(t), e_i\rangle)}{T^{-1}\beta_T} \\ &= \frac{T^{-1/2}\alpha_T}{T^{-1}\beta_T} \quad \text{(say)}.\end{aligned}$$

Let

$$\tau = T^{1/2}\left(\theta - \hat{\theta}_T\right)$$

and

$$p^*\left(\tau \mid Z^T\right) = T^{-1/2} p\left(\hat{\theta}_T + \tau T^{-1/2} \mid Z^T\right).$$

Then $p^*(\tau \mid Z^T)$ denotes the posterior density of $T^{1/2}(\theta - \hat{\theta}_T)$. Let

$$\begin{aligned}L_T(\tau) &= \frac{d\nu_{\hat{\theta}_T + \tau T^{-1/2}}^{(T)}}{d\mu_{z_0}^{(T)}} \Bigg/ \frac{d\nu_{\hat{\theta}_T}^{(T)}}{d\mu_{z_0}^{(T)}} \\ &= \frac{d\nu_{\hat{\theta}_T + \tau T^{-1/2}}^{(T)}}{d\nu_{\hat{\theta}_T}^{(T)}} \quad \text{a.s. } [P_{\theta_0}].\end{aligned} \tag{2.12}$$

In view of (2.6) and (2.10), it follows that

$$\log L_T(\tau) = -\frac{1}{2}\beta_T \tau^2 T^{-1}. \tag{2.13}$$

Let

$$C_T = \int_{-\infty}^{\infty} L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) d\tau. \tag{2.14}$$

It can be checked that

$$p^*\left(\tau \mid Z^T\right) = C_T^{-1} L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right). \tag{2.15}$$

Suppose that

(C1) $\beta_T/T \to \beta > 0$ a.s. $[P_{\theta_0}]$ as $T \to \infty$.

Then the following relations hold:

(i) $\lim_{T\to\infty} L_T(\tau) = \exp(-\beta\tau^2/2)$ a.s. $[P_{\theta_0}]$;

(ii) for any $0 < \gamma < \beta$,

$$\log L_T(\tau) \leq -\frac{1}{2}\tau^2(\beta - \gamma)$$

for every τ for sufficiently large T, and

(iii) for every $\delta > 0$, there exists $\gamma' > 0$ such that

$$\sup_{|\tau| > \delta T^{1/2}} L_T(\tau) \leq \exp\left\{-\frac{1}{4}\gamma' T^{-1}\right\}$$

as $T \to \infty$.

In addition to the condition (C1), suppose that the following condition holds:

(C2) the maximum likelihood estimator $\hat{\theta}_T$ is strongly consistent, that is,

$$\hat{\theta}_T \to \theta_0 \quad \text{a.s } [P_{\theta_0}] \text{ as } T \to \infty.$$

In view of (C1), a sufficient condition for (C2) to hold is

(C2)′ $I_T \equiv \mathsf{E}(\beta_T) < \infty$.

This can be seen from Theorem 3 of Loges (1984).

In addition to the conditions (C1) and (C2), suppose that

(C3) $K(\cdot)$ is a nonnegative function such that, for some $0 < \gamma < \beta$,

$$\int_{-\infty}^{\infty} K(\tau) \exp\left\{-\frac{1}{2}\tau^2(\beta-\gamma)\right\} d\tau < \infty.$$

Lemma 2.1. *Suppose the conditions* (C1)–(C3) *hold. Then there exists $\delta > 0$ such that*

$$\lim_{T\to\infty} \int_{|\tau|\le\delta T^{1/2}} K(\tau) \left| L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) - \lambda(\theta_0)\exp\left\{-\frac{1}{2}\beta\tau^2\right\}\right| d\tau = 0 \quad a.s.\ [P_{\theta_0}]. \tag{2.16}$$

Proof. For any $\delta > 0$,

$$\begin{aligned}
&\int_{|\tau|\le\delta T^{1/2}} K(\tau) \left| L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) - \lambda(\theta_0)\exp\left\{-\frac{1}{2}\beta\tau^2\right\}\right| d\tau \\
&\le \lambda(\theta_0) \int_{|\tau|\le\delta T^{1/2}} K(\tau) \left| L_T(\tau) - \exp\left\{-\frac{1}{2}\beta\tau^2\right\}\right| d\tau \\
&\quad + \int_{|\tau|\le\delta T^{1/2}} K(\tau) L_T(\tau) \left|\lambda(\theta_0) - \lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right)\right| d\tau \\
&= A_T + B_T \quad \text{(say)}.
\end{aligned}$$

It can be checked that, for any $\delta > 0$,

$$A_T \to 0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } T \to \infty \tag{2.17}$$

by the dominated convergence theorem in view of (C1) and (C3).

On the other hand, for sufficiently large T,

$$0 \le B_T \le \sup_{|\theta-\theta_0|\le\delta} |\lambda(\theta) - \lambda(\theta_0)| \int_{|\tau|\le\delta T^{1/2}} K(\tau)\exp\left\{-\frac{1}{2}\tau^2(\beta-\gamma)\right\} d\tau \tag{2.18}$$

since $\hat{\theta}_T$ is strongly consistent as $T \to \infty$ by the condition (C2). The terms on the right hand side of the above inequality can be made smaller than any given $\eta > 0$ by choosing δ sufficiently small in view of the continuity of $\lambda(\cdot)$ at θ_0. Combining the relations (2.17) and (2.18), we obtain the lemma. □

Lemma 2.2. *In addition to the conditions* (C1) *to* (C3), *suppose the following condition* (C4) *holds*:

(C4) *for every $\eta > 0$ and $\delta > 0$,*

$$\exp\left\{-\eta T^{-1}\right\} \int_{|\tau|>\delta} K\left(\tau T^{-1/2}\right) \lambda(\hat{\theta}_T + \tau)\, d\tau \to 0 \quad a.s.\ [P_{\theta_0}]$$

as $T \to \infty$.

Then, for every $\delta > 0$,

$$\lim_{T\to\infty} \int_{|\tau|>\delta T^{1/2}} K(\tau) \left| L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) - \lambda(\theta_0)\exp\left\{-\beta\tau^2/2\right\}\right| d\tau = 0 \quad a.s.\ [P_{\theta_0}]. \tag{2.19}$$

Proof. For any $\delta > 0$, consider

$$\begin{aligned}
&\int_{|\tau|>\delta T^{1/2}} K(\tau)\left|L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) - \lambda(\theta_0)\exp\left\{-\beta\tau^2/2\right\}\right| d\tau \\
&\quad\le \int_{|\tau|>\delta T^{1/2}} K(\tau)L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) d\tau \\
&\quad+ \int_{|\tau|>\delta T^{1/2}} K(\tau)\lambda(\theta_0)\exp\left\{-\beta\tau^2/2\right\} d\tau \\
&\quad\le \exp\left\{-\gamma^1 T^{-1}/4\right\}\int_{|\tau|>\delta T^{1/2}} K(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) d\tau \\
&\quad+ \lambda(\theta_0)\int_{|\tau|>\delta T^{1/2}} K(\tau)\exp\left\{-\beta\tau^2/2\right\} d\tau \\
&\quad= F_T + G_T \quad \text{(say)}.
\end{aligned} \tag{2.20}$$

In view of the condition (C4), it follows that

$$F_T \to 0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } T \to \infty \tag{2.21}$$

for every $\delta > 0$. Condition (C3) implies that

$$G_T \to 0 \quad \text{as } T \to \infty. \tag{2.22}$$

Relations (2.21) and (2.22) prove the lemma. □

Lemma 2.1 and 2.2 together prove that

$$\begin{aligned}
&\lim_{T\to\infty}\int_{-\infty}^{\infty} K(\tau)\left|L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) - \lambda(\theta_0)\exp\left\{-\beta\tau^2/2\right\}\right| d\tau \\
&\quad= 0 \quad \text{a.s. } [P_{\theta_0}].
\end{aligned} \tag{2.23}$$

Let $K(\tau) \equiv 1$. It is easy to see that the conditions (C3) and (C4) hold and we have

$$\lim_{T\to\infty}\left|L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) - \lambda(\theta_0)\exp\left\{-\beta\tau^2/2\right\}\right| d\tau = 0 \quad \text{a.s. } [P_{\theta_0}].$$

Hence the term C_T, defined by (2.14), satisfies the property

$$\begin{aligned}
C_T &= \int_{-\infty}^{\infty} L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) d\tau \\
&\to \lambda(\theta_0)\int_{-\infty}^{\infty}\exp\left\{-\beta\tau^2/2\right\} d\tau \quad \text{a.s. } [P_{\theta_0}] \text{ as } T \to \infty.
\end{aligned} \tag{2.24}$$

We now have the following result giving an analogue of the Bernstein–von Mises theorem for a class of Hilbert-space valued stochastic differential equations (cf. Prakasa Rao (1981, 1984)).

Theorem 2.2. *Suppose the conditions* (C1) *to* (C4) *hold where $\lambda(\cdot)$ is a prior density which is continuous and positive in an open neighbourhood of θ_0, the true parameter. Then*

$$\lim_{T\to\infty}\int_{-\infty}^{\infty} K(\tau)\left|p^*\left(\tau \mid Z^T\right) - \left(\frac{\beta}{2\pi}\right)^{1/2}\exp\left\{-\beta\tau^2/2\right\}\right| = 0 \quad a.s.\ [P_{\theta_0}]. \tag{2.25}$$

Proof. Note that

$$\begin{aligned}
&\int_{-\infty}^{\infty} K(\tau)\left|p^*\left(\tau \mid Z^T\right) - \left(\frac{\beta}{2\pi}\right)^{1/2}\exp\left\{-\beta\tau^2/2\right\}\right| d\tau \\
&\quad\le C_T^{-1}\int_{-\infty}^{\infty} K(\tau)\left|L_T(\tau)\lambda\left(\hat{\theta}_T + \tau T^{-1/2}\right) - \lambda(\theta_0)\exp\left\{-\beta\tau^2/2\right\}\right| d\tau \\
&\quad+ \int_{-\infty}^{\infty} K(\tau)\left|C_T^{-1}\lambda(\theta_0) - \left(\frac{\beta}{2\pi}\right)^{1/2}\right|\exp\left\{-\beta\tau^2/2\right\} d\tau
\end{aligned} \tag{2.26}$$

and the two terms on the right side of the inequality (2.26) tend to zero a.s. $[P_{\theta_0}]$ as $N \to \infty$ by the Lemma 2.2 and the relation (2.24). □

As a consequence of Theorem 2.2, the following theorem can be proved.

Theorem 2.3. *Suppose the following conditions hold:*

(D1) $\hat{\theta}_T \to \theta_0$ *a.s.* $[P_{\theta_0}]$ *as* $T \to \infty$;

(D2) $T^{-1}\beta_T \to \beta > 0$ *a.s.* $[P_{\theta_0}]$ *as* $T \to \infty$;

(D3) $\lambda(\cdot)$ *is a prior density which is continuous and positive in an open neighbourhood of* θ_0, *the true parameter; and*

(D4) $\int_{-\infty}^{\infty} |\theta|^m \lambda(\theta)\, d\theta < \infty$ *for some integer* $m \geq 0$.

Then

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} |\tau|^m \left| p^*(\tau \mid Z^T) - \left(\frac{\beta}{2\pi}\right)^{1/2} \exp\left\{-\beta\tau^2/2\right\} \right| d\tau = 0 \quad a.s.\ [P_{\theta_0}]. \tag{2.27}$$

Remarks. It is obvious that the condition (D4) holds for $m = 0$. Suppose the conditions (D1) to (D3) hold. Then it follows that

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \left| p^*\left(\tau \mid Z^T\right) - \left(\frac{\beta}{2\pi}\right)^{1/2} \exp\left\{-\beta\tau^2/2\right\} \right| d\tau = 0 \quad \text{a.s. } [P_{\theta_0}]. \tag{2.28}$$

This is the analogue of the Bernstein–von Mises theorem in the classical statistical inference. As a special case of Theorem 2.3, we obtain that

$$\mathsf{E}_{\theta_0}\left[T^{1/2}\left(\hat{\theta}_T - \theta_0\right)\right]^m \to \mathsf{E}[Z]^m \quad \text{as } T \to \infty, \tag{2.29}$$

where Z is $N(0,1)$.

2.2. Bayes estimation.

We define an estimator $\tilde{\theta}_T$ for θ to be a Bayes estimator based on the path $Z^T = \{Z(t), 0 \leq t \leq T\}$ and the prior density $\lambda(\theta)$, corresponding to the loss function $\tilde{L}(\theta, \varphi)$, if $\tilde{\theta}_T$ minimizes the function

$$B_T(\varphi) = \int \tilde{L}(\theta, \varphi) p\left(\theta \mid Z^T\right) d\theta, \qquad \varphi \in \Theta, \tag{2.30}$$

where $\tilde{L}(\theta, \varphi)$ is defined on $\Theta \times \Theta$.

Suppose there exists a Bayes estimator $\tilde{\theta}_T$ corresponding to the loss function $\tilde{L}(\theta, \varphi)$ satisfying the following properties:

(E1) $\tilde{L}(\theta, \varphi) = L(|\theta - \varphi|) \geq 0$;

(E2) $L(t)$ is nondecreasing for $t \geq 0$;

(E3) there exist nonnegative functions $R(T)$, $K(\tau)$, and $G(\tau)$ such that

(1) (a) $R(T)L(\tau T^{-1/2}) \leq G(\tau)$ for all $T \geq 0$;

(2) (b) $R(T)L(\tau T^{-1/2}) \to K(\tau)$ as $T \to \infty$ uniformly on bounded intervals of τ.

(3) (c) the function

$$\int_{-\infty}^{\infty} K(\tau + m) \exp\left\{-\beta\tau^2/2\right\} d\tau$$

achieves its minimum at $m = 0$, and

(4) (d) $G(\tau)$ satisfies the conditions similar to the conditions (C3) and (C4).

The following result can be proved by arguments similar to those given in Borwanker et al. (1971). We omit the proof.

Theorem 2.4. *Suppose the conditions* (D1)–(D3) *of Theorem* 2.3 *hold. In addition, suppose that the loss function* $\tilde{L}(\theta, \varphi)$ *satisfy the conditions* (E1)–(E3) *stated above. Then*

$$T^{1/2}\left(\hat{\theta}_T - \tilde{\theta}_T\right) \to 0 \quad a.s.\ [P_{\theta_0}] \text{ as } T \to \infty \tag{2.31}$$

and

$$\lim_{T\to\infty} R(T)B_T\left(\tilde{\theta}_T\right) = \lim_{T\to\infty} R(T)B_T\left(\hat{\theta}_T\right) = \left(\frac{\beta}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} K(\tau)\exp\left\{-\beta\tau^2/2\right\} d\tau. \tag{2.32}$$

Loges (1984) proves that

$$\hat{\theta}_T \to \theta_0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } T\to\infty \tag{2.33}$$

and

$$T^{1/2}\left(\hat{\theta}_T - \theta_0\right) \xrightarrow{\mathcal{L}} N(0,\beta) \quad \text{as } T\to\infty \tag{2.34}$$

under some conditions (cf. Theorem 3 and Theorem 4 of Loges (1984)). As a consequence of Theorem 2.4, it follows that

$$\tilde{\theta}_T \to \theta_0 \quad \text{a.s. } [P_{\theta_0}] \text{ as } T\to\infty \tag{2.35}$$

and

$$T^{1/2}\left(\hat{\theta}_T - \theta_0\right) \xrightarrow{\mathcal{L}} N(0,\beta) \quad \text{as } T\to\infty \tag{2.36}$$

under some conditions stated in Theorem 2.4 and the conditions in Theorem 3 and Theorem 4 of Loges (1984).

In other words the Bayes estimator $\tilde{\theta}_T$ of the parameter θ in the Hilbert-space valued stochastic differential equation given by (2.1) is strongly consistent, asymptotically normal and asymptotically efficient under the conditions stated above.

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