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MEASUREMENT AND PERFORMANCE OF THE STRONG STABILITY METHOD

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ABSTRACT. The aim of this paper is to show how to use in practice the strong stability method and also to prove its efficiency. That's why we chose the $GI/M/1$ model for which there exists analytical results.

For this purpose, we first determine the approximation conditions of the characteristics of the $GI/M/1$ system. Under these conditions, we obtain the stability inequalities of the stationary distribution of the queue size.

We finally elaborate an algorithm for the approximation of the $GI/M/1$ system by the $M/M/1$ one, which calculate the approximation error with an exact computation. In order to give an idea about its application in practice, we give a numerical example.

The accuracy of the approach is evaluated by comparison with some known exact results.

1. Introduction

The Strong Stability method permits us to investigate the ergodicity and the stability of the stationary and non-stationary characteristics of Markov chains, after submitting their transition kernels to small perturbations. This method gives the possibility to clarify the conditions for which the characteristics of complex queues can be approximated by the correspondent ones of more simpler systems of queues. We should notice that, for the analysis of these described systems, we never know exactly the system's parameters (one estimate only the degree of proximity relatively to those given). That is why this kind of inequalities will allow us to numerically estimate the uncertainty shown during this analysis.

Our contribution in this work is to prove in concrete terms the applicability of this method by elaborating an algorithm which we apply to a numerical example. Furthermore, exact numerical results allow one to evaluate the performance and the robustness of this approach.

In section 3 we clarify the strong stability conditions of the $GI/M/1$ system (subsection 1), we next obtain the deviation of the transition kernel (subsection 2) and we finally estimate the error due to the approximation of the $GI/M/1$ system by an $M/M/1$ one with respect to the norm $\|\cdot\|_\beta$ (subsection 3).

In section 4, we elaborate an algorithm which verifies the approximation conditions of the $GI/M/1$ system and then determines if possible the minimal approximation error on the stationary distribution of the queue size.

In section 5, we successfully apply this algorithm to a numerical example. In particular, we are interested in the draw of the curve of the error. Indeed, this allows to give us an idea about the behavior of the approximation domain.

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2. Preliminary and position of the problem

2.1. Description of $M/M/1$ and $GI/M/1$ models. Let us consider a

$$GI/M/1(FIFO,\infty)$$

system where inter-arrival times are independently distributed with general distribution $H(t)$ and service times are distributed with $E_\gamma(t)$ (exponential with parameter γ).

Let X_n^* be the number of customers left behind in the system by the nth departure. It's easy to prove that X_n^* forms a Markov chain [4, 5] with a transition operator

$$P^* = \|P_{ij}^*\|_{i,j\geq 0}$$

where

$$P_{ij}^* = \begin{cases} d_{i+1-j}^* = \int_0^\infty \frac{1}{(i+1-j)!} e^{-\gamma t} (\gamma t)^{i+1-j}\, dH(t), & \text{if } 1 \leq j \leq i+1, \\ 1 - \sum_{k=0}^{i} d_k^*, & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

Consider also an $M/M/1(FIFO,\infty)$ system, which has Poisson arrivals with parameter λ and the same distribution of the service time as the precedent system. It is known that X_n (the number of customers left behind in the system by the nth departure), forms a Markov chain with a transition operator $P = \|P_{ij}\|_{i,j\geq 0}$ where

$$P_{ij} = \begin{cases} d_{i+1-j} = \frac{\lambda \gamma^{i+1-j}}{(\lambda+\gamma)^{i+2-j}}, & \text{if } 1 \leq j \leq i+1, \\ 1 - \sum_{k=0}^{i} d_k = \left(\frac{\gamma}{\gamma+\lambda}\right)^i, & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Suppose that the arrival flow of the $GI/M/1$ system is close to the Poisson one. This proximity is then characterized by the metric

$$w = w(H, E_\lambda) = \int_0^\infty |H - E_\lambda|\,(dt) \tag{2.3}$$

where $|a|$ is the variation of the measure a.

Designate by $\{\pi_k^*\}$ and $\{\pi_k\}$ the stationary distributions of the states of X_n^* and X_n. We have then

$$\begin{aligned} \pi_k^* &= \lim_{n\to\infty} \mathsf{P}(X_n^* = k), \qquad k = 0,1,2,\ldots, \\ \pi_k &= \lim_{n\to\infty} \mathsf{P}(X_n = k), \qquad k = 0,1,2,\ldots. \end{aligned} \tag{2.4}$$

2.2. The strong stability criteria. Let $\mathcal{M} = \{\mu_j\}$ be the space of finite measures on $\mathbb{N}$, and $\mathcal{N} = \{f(j)\}$ the space of bounded measurable functions on $\mathbb{N}$. We associate with each transition kernel P the linear mapping:

$$(\mu P)_k = \sum_{j\geq 0} \mu_j P_{jk}, \tag{2.5}$$

$$(Pf)(k) = \sum_{i\geq 0} f(i) P_{ki}. \tag{2.6}$$

Introduce on $\mathcal{M}$ the class of norms of the form:

$$\|\mu\|_v = \sum_{j\geq 0} v(j)\,|\mu_j|\,. \tag{2.7}$$

Where v is an arbitrary measurable function (not necessary finite) bounded below away from a positive constant. This norm induce in the space $\mathcal{N}$ the norm:

$$\|f\|_v = \sup_{k\geq 0} \frac{|f(k)|}{v(k)}. \tag{2.8}$$

Let us consider B, the space of linear operators, with norm:

$$\|Q\|_v = \sup_{k\geq 0} \frac{1}{v(k)} \sum_{j\geq 0} v(j) Q_{kj}. \tag{2.9}$$

Definition. The Markov chain X with a transition kernel P and an invariant measure π is said to be strongly v-stable with respect to the norm $\|\cdot\|_v$, if $\|P\|_v < \infty$ and each stochastic kernel Q on the space $(\mathbb{N}, \mathcal{B}(\mathbb{N}))$ in some neighborhood

$$\{Q\colon \|Q - P\|_v < \epsilon\}$$

has a unique invariant measure $\mu = \mu(Q)$ and $\|\pi - \mu\|_v \to 0$ as $\|Q - P\|_v \to 0$.

Theorem 1 ([1]). *A Markov chain X, with transition kernel P, is strongly v-stable, if and only if there exists a measure α and a non-negative measurable function h on $\mathbb{N}$ such that:*

a) $\|P\|_v < \infty$;
b) $T = P - h\circ\alpha > 0$;
c) *there exists $m \geq 1$ and $\rho < 1$ such that $T^m v(x) \leq \rho v(x)$ for all $x \in E$.*

Theorem 2 ([5]). *Let X be a strongly v-stable Markov chain, with an invariant measure π and holding the conditions of the theorem 1. If μ is the invariant measure of a kernel Q, then for the norm $\|Q - P\|_v$ sufficiently small, we have:*

$$\mu = \pi\left[I - \Delta R_0 (I - \Pi)\right]^{-1} = \pi + \sum_{t\geq 1} \pi\left[\Delta R_0 (I - \Pi)\right]^t$$

where $\Delta = Q - P$, $R_0 = (I - T)^{-1}$ and $\Pi = 1\circ\pi$ is the stationary projector of the kernel P, 1 is the identity function, and I the identity kernel on $\mathcal{M}$.

Consequence 1. *Under the conditions of the theorem 1,*

$$\mu = \pi + \pi\Delta R_0 (I - \Pi) + o\left(\|\Delta\|_v^2\right)$$

for $\|\Delta\|_v \to 0$.

Consequence 2. *Under the conditions of the theorem 1, for*

$$\|\Delta\|_v < \frac{(1-\rho)}{c}$$

we have the estimation

$$\|\mu - \pi\|_v \leq \|\Delta\|_v c \|\pi\|_v \left(1 - \rho - c\|\Delta\|_v\right)^{-1}$$

where

$$c = m\|P\|_v^{m-1}\left(1 + \|\mathbb{1}\|_v \|\pi\|_v\right)$$

and

$$\|\pi\|_v \leq (\alpha v)(1-\rho)^{-1}(\pi h) m \|P\|_v^{m-1}$$

3. Strong stability in the $GI/M/1$ system

3.1. **Strong stability conditions.** The first step consists on the determination of the strong v-stability conditions of the $M/M/1$ system after a small perturbation of the arrival flow.

Theorem 3. *Suppose that the charge (λ/γ) of the $M/M/1$ system is smaller than 1. Therefore, for all β such that $1 < \beta < \gamma/\lambda$, the imbedded Markov chain X_n, is v-strongly stable, after a small perturbation of the inter-arrival time, for $v(k) = \beta^k$.*

Proof. Let us verify the conditions of the stability criteria (theorem1) for the function $v(k) = \beta^k$ where $\beta > 1$, and suppose that $\lambda/\gamma < 1$.

Let us choose $m = 1$ and:

$$h_i = \left(\frac{\gamma}{\lambda+\gamma}\right)^i \quad \text{for all } i \geq 0 \quad \text{and } \alpha_j = \begin{cases} 0, & \text{if } j \geq 1, \\ 1, & \text{if } j = 0. \end{cases} \tag{3.1}$$

⋆ *Verification of the condition* b). From (2.2), we have

$$T_{ij} = P_{ij} - h_i\alpha_j = \begin{cases} P_{i0} - 1.h_i = 0, & \text{if } j = 0, \\ P_{ij} - 1.0 = P_{ij}, & \text{if } j \geq 1, \end{cases} = \begin{cases} 0, & \text{if } j = 0, \\ P_{ij} \geq 0, & \text{otherwise.} \end{cases}$$

⋆ *Verification of the condition* c). From (2.2) and (2.6) we have

$$(Tv)(k) = \sum_{j\geq 0} \beta^j T_{kj} = \sum_{j\geq 1} \beta^j P_{kj} = \sum_{j=1}^{k+1} \beta^j d_{k+1-j},$$

$$(Tv)(k) = \sum_{j=1}^{k+1} \beta^j \frac{\lambda}{\gamma+\lambda} \left(\frac{\gamma}{\gamma+\lambda}\right)^{k+1-j}$$

$$= \frac{\lambda}{\lambda+\gamma}\beta^{k+1} \sum_{j=1}^{k+1} \left(\frac{\gamma}{\beta(\lambda+\gamma)}\right)^{k+1-j}$$

$$= \frac{\lambda}{\lambda+\gamma}\beta^{k+1} \frac{1-\left(\frac{\gamma}{\beta(\gamma+\lambda)}\right)^{k+1}}{1-\frac{\gamma}{\beta(\gamma+\lambda)}}$$

By assuming that $\gamma/\lambda < 1$, we obtain $\gamma/(\beta(\gamma+\lambda)) < 1$ for all $\beta > 1$. Hence

$$(Tv)(k) \leq \beta^k \frac{\beta\lambda}{\gamma - \gamma/\beta + \lambda}.$$

If we suppose, in addition, that $\beta < \gamma/\lambda$, then

$$\beta \frac{\lambda}{\gamma - \gamma/\beta + \lambda} < 1.$$

Therefore there exists $\rho < 1$ such that

$$(Tv)(k) \leq \rho v(k) \quad \text{where } \rho = \frac{\beta\lambda}{\gamma - \gamma/\beta + \lambda} \quad \text{for all } \beta \quad \text{such that } 1 < \beta < \gamma/\lambda.$$

⋆ *Verification of the condition* a). Using the precedent result, and according to the equations (2.7), (2.8), and (2.9), we obtain, for all β such that $1 < \beta < \gamma/\lambda$,

$$\|T\|_v = \sup_{k\geq 0} \frac{1}{\beta^k} \sum_{j\geq 0} \beta^j T_{kj} \leq \beta \frac{\lambda}{\gamma - \gamma/\beta + \lambda} = \rho < 1,$$

$$\|\alpha\|_v = 1,$$

and

$$\|h\|_v = \sum_{j\geq 0} \beta^j \alpha_j = 1.$$

Hence $\|P\|_v \leq \|T\|_v + \|h\|_v \|\alpha\|_v < \infty$. □

3.2. **Estimation of the strong stability.** In this section we obtain the inequalities of stability according to reference [2].

3.2.1. *Estimation of the transition kernel deviation.* To be able to estimate numerically the margin between the stationary distributions of the Markov chains X_n^* and X_n, we estimate the norm of the deviation of the transition kernel P^*.

Theorem 4. *Let P (resp. P^*) be the transition operator of the imbedded Markov chain in an $M/M/1$ (resp. $GI/M/1$) system. Then, for all β such that $1 < \beta < \gamma/\lambda$, we have:*

$$\|P^* - P\|_v \leq (1+\beta)\, w$$

where w is defined in 2.3.

Proof. From (2.9) and for all β such that $1 < \beta < \gamma/\lambda$, we have:

$$\begin{aligned}\|P^* - P\|_v &= \sup_{k\geq 0} \frac{1}{\beta^k} \sum_{j\geq 0} \beta^j \left|P^*_{kj} - P_{kj}\right| \\ &= \sup_{k\geq 0} \frac{1}{\beta^k} \left(|P^*_{k0} - P_{k0}| + \sum_{j\geq 1} \beta^j \left|P^*_{kj} - P_{kj}\right| \right).\end{aligned}$$

Denote by A and B the two terms of the precedent expression. From (2.1) and (2.2) we have

$$A = \sup_{k\geq 0} \frac{1}{\beta^k} |P^*_{k0} - P_{k0}| = \sup_{k\geq 0} \frac{1}{\beta^k} \left| \sum_{j=1}^{k} d - \sum_{j=1}^{k} d^* \right|,$$

$$A \leq \sup_{k\geq 0} \frac{1}{\beta^k} \sum_{j=0}^{k} \int \frac{1}{j!} e^{-\gamma t} (\gamma t)^j \, |H - E_\lambda| \,(dt).$$

Therefore

$$A \leq \sup_{k\geq 0} \frac{1}{\beta^k} \int_0^\infty \sum_{j=0}^{k} \frac{(\gamma t)^j}{j!} e^{-\gamma t} \, |H - E_\lambda| \,(dt)\,,$$

$$A \leq \sup_{k\geq 0} \frac{1}{\beta^k} \int_0^\infty e^{-\gamma t} e^{\gamma t} \, |H - E_\lambda| \,(dt).$$

In other words

$$A \leq w.$$

On the other hand

$$\begin{aligned}B &= \sup_{k\geq 0} \frac{1}{\beta^k} \sum_{j=1}^{k+1} \beta^j \left|P^*_{kj} - P_{kj}\right| \\ &\leq \sup_{k\geq 0} \frac{1}{\beta^k} \sum_{j=1}^{k+1} \beta^j \int_0^\infty \frac{1}{(k+1-j)!} e^{-\gamma t} (\gamma t)^{k+1-j} \, |H - E_\lambda| \,(dt),\end{aligned}$$

$$\begin{aligned}B &\leq \sup_{k\geq 0} \beta \int_0^\infty e^{-\gamma t} \sum_{j=1}^{k+1} \left(\frac{\gamma t}{\beta}\right)^{k+1-j} \frac{1}{(k+1-j)!} \, |H - E_\lambda| \,(dt) \\ &\leq \sup_{k\geq 0} \beta \int_0^\infty |H - E_\lambda| \,(dt).\end{aligned}$$

Therefore

$$B \leq \beta w.$$

Hence, we conclude that

$$\|P^* - P\|_v \leq (1+\beta) w. \quad \square$$

3.2.2. *Stability inequalities.* This subsection consists on the determination of the deviation of the stationary distribution with respect to the norm $\|\cdot\|_v$.

Theorem 5. *Suppose that in an $M/M/1$ system, the Markov chain X_n is strongly v-stable, and*

$$w < \frac{(1-\rho)(\gamma-\lambda\beta)}{(1+\beta)(2\gamma-\lambda(1+\beta))}.$$

Therefore

$$\|\pi^* - \pi\|_v \leq \frac{(1+\beta)(2\gamma-\lambda(1+\beta))(\gamma-\lambda)w}{\frac{(\beta-1)(\gamma-\lambda\beta)^2}{(\beta-1)\gamma+\lambda\beta} - (2\gamma-\lambda(1+\beta))(1+\beta)(\gamma-\lambda\beta)w}$$

for all β such that $1 < \beta < \gamma/\lambda$ where π^* and π are defined in (2.4) and

$$\rho = \beta\frac{\lambda}{\gamma - \frac{\gamma}{\beta} + \lambda}.$$

Proof. To be able to use the consequence 2 of the theorem 2, we first estimate $\|1\|_v$ and $\|\pi\|_v$.

From (2.8), we have

$$\|1\|_v = \sup_{k\geq 0} \beta^{-k} = 1.$$

From (2.7), we have

$$\|\pi\|_v = \sum_{j\geq 0} \beta^j \pi_j = \left(1-\frac{\lambda}{\gamma}\right)\sum_{j\geq 0}\left(\frac{\beta\lambda}{\gamma}\right)^j = \frac{\gamma-\lambda}{\gamma-\beta\lambda}.$$

Then

$$c = 1 + \|1\|_v\,\|\pi\|_v = \frac{2\gamma-\lambda(1+\beta)}{\gamma-\lambda\beta}$$

for all β such that $1 < \beta < \gamma/\lambda$.

Therefore we obtain the result of theorem 5, for all $\|P^* - P\|_v \leq (1-\rho)/c$. In other words, for all $w < (1-\rho)/(c(1+\beta))$, such that $1 < \beta < \gamma/\lambda$. □

4. Approximation algorithm of the $GI/M/1$ system

In this section, we elaborate an algorithm STR-STAB-APP which allows to verify the approximation conditions of the $GI/M/1$ system and to determine the error on the stationary distribution due to the approximation.

4.1. Algorithm STR-STAB-APP.

(1) define the inter-arrival times density function $h(x)$ of the $GI/M/1$ system;
(2) introduce service mean rate γ of the $GI/M/1$ system;
(3) determine the arrival mean rate:

$$\lambda \longleftarrow \frac{1}{\int x h(x)\,dx}$$

of the $G/M/1$ system;
(4) verify the stability: if $\lambda/\gamma \geq 1$ then {∗ the system is unstable ∗} go to 8; else {∗ the system is strongly v-stable for $1 < \beta < \gamma/\lambda$∗} go to 5;
(5) determine the proximity of $h(x)$ and $e_\lambda(x) = \lambda e^{-\lambda x}$

$$w \longleftarrow \int |h(x) - e_\lambda(x)|\,dx;$$

6. determine the approximation domain ($\beta_{\min} < \beta < \beta_{\max}$) of the system:

$$\beta_{\min} = \min\left(\beta,\ 1 < \beta < \frac{\gamma}{\lambda} \text{ and } w < \frac{(\beta(\gamma+\lambda) - \gamma - \lambda\beta^2)(\gamma - \lambda\beta)}{(1+\beta)(2\gamma - \lambda(1+\beta))(\beta(\lambda+\gamma) - \gamma)}\right),$$

$$\beta_{\max} = \max\left(\beta,\ 1 < \beta < \frac{\gamma}{\lambda} \text{ and } w < \frac{(\beta(\gamma+\lambda) - \gamma - \lambda\beta^2)(\gamma - \lambda\beta)}{(1+\beta)(2\gamma - \lambda(1+\beta))(\beta(\lambda+\gamma) - \gamma)}\right).$$

If ($\beta_{\min} \geq \beta_{\max}$), then {* the proximity is insufficient *} go to 8,
else {* the approximation is validated *} go to 7.

7. determine the minimal error E_β on the stationary distribution

$$E_\beta = \min\left(\frac{(1+\beta)(2\gamma - \lambda(1+\beta))(\gamma - \lambda)w}{\frac{(\beta-1)(\gamma-\lambda\beta)^2}{(\beta-1)\gamma+\lambda\beta} - (2\gamma - \lambda(1+\beta))(1+\beta)(\gamma - \lambda\beta)w},\ \beta_{\min} \leq \beta \leq \beta_{\max}\right).$$

8. end.

4.2. Interpretation. From theorem 3, we notice that the stability domain $[1, \gamma/\lambda]$ depends strongly on the system charge (λ/γ) and hence on its arrival rate λ. Indeed, the more λ increases and attains progressively γ, the more the β domain is restricted and becomes empty. In practice, this may be explained by the fact that the queue will increase and will become unstable.

From the theorem 5, we notice also that the approximation domain depends strongly on the w quantity, so as w becomes important, the more restricted is the approximation domain.

It is evident that the stability domain includes the approximation domain. Then, for a small value of w, when the value of λ is small, the approximation domain is large.

5. Application of the algorithm

In order to appreciate the performance of the approach, we simplified some hypotheses. In particular we represented the inter-arrival time density by the hyperexponential one.

5.1. Position of the problem. Let us consider a $GI/M/1$ system (see subsection 2.1), where the density function of the inter-arrival times is:

$$h(x) = \begin{cases} \frac{1}{2}e^{-x} + e^{-2x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

and $\gamma = 10$.

We would like to test if the characteristics of the $GI/M/1$ system (real model) are sufficiently close to the characteristics of the $M/M/1$ system (ideal model) and then to determine the approximation error.

5.2. Solution of the problem.

- determination of the arrival mean rate
$$\lambda = \frac{1}{\int x h(x)\, dx} = 4/3;$$
- verification of the geometric ergodicity condition: $\lambda/\gamma = \frac{2}{15} < 1$ (the strong stability domain is then the set of values of β such that $1 < \beta < 7.5$);
- determination of the proximity of $h(x)$ and $e_\lambda(x) = \lambda e^{-\lambda x}$
$$w = \int |h(x) - e_\lambda(x)|\, dx = 0.07114;$$

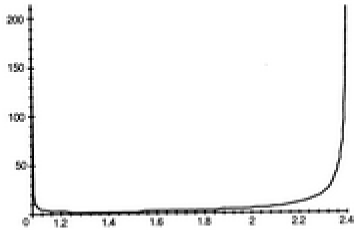

FIGURE 1. Approximation error as a function of β

- determination of the approximation domain: the exact values of $\beta_{\min}$ and $\beta_{\max}$ have been determined such as they are described in the step 6. $\beta_{\min} = 1.07235$ and $\beta_{\max} = 2.41114$;
- determination of the minimal error due to the approximation.

In order better to illustrate the behavior of the error $\|\pi^* - \pi\|_v$ as a function of β, we are going to draw its curve.

Notice that the error, being important at the start, decreases speedily for the values of β in the neighborhood of the lower bound ($1.07235 < \beta < 1.3$). This may be explained by the fact that one are at the boundary of the stability domain (critical region).

We notice also that it increases speedily in the neighborhood of the upper bound ($2.2 < \beta < 2.4$) (critical region).

In contrast everywhere else, the error increases reasonably with the values of β (favorable region). Nevertheless, it would be necessary to consider the minimal error E_β which correspond in our case to $\beta = 1.3$

$$E_\beta = 0.21475311.$$

Therefore, the error on the stationary distribution obtained by the strong stability method is such that

$$\sum_{i=0}^{\infty} (1.3)^i \, |\pi_i^* - \pi_i| \leq 0.21475311.$$

5.3. **Comparison with exact error.** It is known that the exact solution of π is such that:

$$\pi_i = \left(\frac{2}{15}\right)^i \left(\frac{13}{15}\right)$$

and the exact value of π^* is such that: $\pi_i^* = (1-x)x^i$ where x is the solution of the system

$$x = \int_0^\infty e^{-\gamma t(1-x)}\, dH(t).$$

We may then determine the exact error E_{exact} with respect to the norm $\|\cdot\|_v$:

$$E_{\text{exact}} = \|\pi^* - \pi\|_v = \sum_{i\geq 0} \beta^i \, |\pi_i^* - \pi_i|.$$

Moreover, we may estimate the proximity of π and $\pi^*(E_{\max})$ by using a metric defined and used in [7], also with respect to the norm$\|\cdot\|_v$:

$$\sum_{i\geq 0} \beta^i \, |\pi_i - \pi_i^*| \leq \sum_{i\geq 0} \beta^i \max|\pi_i - \pi_i^*| = \max|\pi_i - \pi_i^*| \sum_{i\geq 0} \beta^i = E_{\max}.$$

It is easy to see that, the more β is small, the more E_{exact} (as $E_{\max}$) is small (this is not the case of E_β).

We finally obtain for $\beta = 1.3$

$$E_{\text{exact}} = 0.035, \qquad E_{\max} = 1.054,$$

and

$$E_\beta = 0.21.$$

It is evident that E_β is smaller than $E_{\max}$.

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