Теорія ймовір. та матем. статист.
Вип. 89, 2013, стор. 75–90

Teor. Ĭmovir. ta Matem. Statyst.
No. 89, 2013, pp. 75–90

LIFT ZONOID ORDER AND FUNCTIONAL INEQUALITIES

UDC 519.21

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Abstract. We introduce the notion of a weighted lift zonoid and show that the ordering condition on a measure μ, formulated in terms of the weighted lift zonoids of this measure, leads to certain functional inequalities for this measure, such as non-linear extensions of Bobkov's shift inequality and weighted inverse log-Sobolev inequality. The choice of the weight K, involved in our version of the inverse log-Sobolev inequality, differs substantially from those available in the literature, and requires the weight v, involved into the definition of the weighted lift zonoid, to equal the divergence of the weight K w.r.t. initial measure μ. We observe that such a choice may be useful for proving direct log-Sobolev inequality, as well.

Анотація. Введено поняття зваженого ліфт зоноїда та показано, що умова порядку на міру μ, накладена у термінах зважених ліфт зоноїдів цієї міри, приводить до таких функціональних нерівностей на цю міру, як нелінійне узагальнення нерівності зсуву Бобкова та зваженої оберненої логарифмічної нерівності Соболєва. Вибір ваги K у нашій версії оберненої логарифмічної нерівності Соболєва істотно відрізняється від наявних у літературі, та вимагає, щоб вага v з означення зваженого ліфт зоноїда дорівнювала дивергенції ваги K відносно вихідної міри μ. Ми показуємо, що такий вибір також може бути корисним при доведенні прямої логарифмічної нерівності Соболєва.

Аннотация. Введено понятие взвешенного лифт зоноида и показано, что условие порядка на меру μ, наложенное в терминах взвешенных лифт зоноидов этой меры, приводит к таким функциональным неравенствам для этой меры, как нелинейное обобщение неравенства сдвига Бобкова и взвешенного обратного логарифмического неравенства Соболева. Выбор веса K в нашей версии обратного логарифмического неравенства Соболева существенно отличается от имеющихся в литературе, и требует, чтобы вес v из определения взвешенного лифт зоноида был равен дивергенции веса K относительно исходной меры μ. Мы показываем, что такой выбор также может быть полезным при доказательстве прямого логарифмического неравенства Соболева.

1. Introduction

The notions of *zonoid* and *lift zonoid*, introduced in [9], have a diverse field of applications. Because the lift zonoid determines the underlying measure uniquely, this concept can be used in multivariate statistics for measuring the variability of laws of random vectors, and for ordering these laws, see [10]. The concept of *zonoid equivalence* appears to be both naturally motivated by financial applications, and useful for proving extensions of the ergodic theorem for zonoid stationary and zonoid swap-invariant random sequences, see [12, 13]. Lift zonoids lead naturally to definitions of associated α-trimming and data depth, see [9] and [7], and to barycentric representation of the points of a space with a given measure, see [9] and [11].

In this paper, we explore a new field, where the notion of lift zonoid can be applied naturally. As a straightforward extension of the definition of lift zonoid, we introduce a *weighted lift zonoid* $\hat{Z}^v(\mu)$ with a vector-valued weight function v. We show that, for properly chosen weights v, the ordering condition on a measure μ, formulated in terms

2010 *Mathematics Subject Classification.* Primary 26D10, 39B62, 47D07, 60E15, 60J60.

Key words and phrases. Lift zonoid, weight, shift inequality, log-Sobolev inequality.

The second author was partially supported by the Leonard Euler program, DAAD project # 55518603.

of the weighted lift zonoid of this measure, leads to certain functional inequalities for this measure, such as non-linear extensions of Bobkov's *shift inequality* [3] and weighted *inverse log-Sobolev inequality*. Weighted versions of the classical functional inequalities (Poincaré, log-Sobolev, etc) have been studied recently in various contexts. The choice of the weight K, involved in our version of inverse log-Sobolev inequality, is specific and differs substantially from those available in the literature. This choice is strongly motivated by (an extension of) the functional form of Bobkov's shift inequality, and requires the weight v, involved into the definition of the weighted lift zonoid, to equal the *divergence* of the weight K w.r.t. initial measure μ. We observe that such a choice may be useful for proving (weighted) direct log-Sobolev inequality, as well. In the case of a bounded weight, this may lead to new sufficient conditions for the log-Sobolev inequality. We illustrate the range of applications of these conditions in two examples in Section 4.

2. Weighted lift zonoids, non-linear shift inequalities, and weighted inverse log-Sobolev inequalities

Let μ be a probability measure on the Borel σ-algebra in $\mathbb{R}^d$, and $v\colon \mathbb{R}^d \to \mathbb{R}^d$ be a measurable function such that

$$\int_{\mathbb{R}^d} \|v(x)\|\, \mu(dx) < \infty;$$

here and below we denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^d$. We define the *weighted zonoid* $Z^v(\mu)$ with the weight v as the set of all the points in $\mathbb{R}^d$ of the form

$$\int_{\mathbb{R}^d} g(x)v(x)\, \mu(dx) \tag{1}$$

with arbitrary Borel measurable $g\colon \mathbb{R}^d \to [0,1]$. The *weighted lift zonoid* $\hat{Z}^v(\mu)$ is defined as the weighted zonoid of the measure $\delta_1 \times \mu$ in $\mathbb{R}^{d+1}$. Equivalently, the weighted zonoid $Z^v(\mu)$ and the weighted lift zonoid $\hat{Z}^v(\mu)$ are the sets of the points of the form

$$\mathsf{E}\, g(X)v(X) \in \mathbb{R}^d \quad \text{and} \quad (\mathsf{E}\, g(X), \mathsf{E}\, g(X)v(X)) \in \mathbb{R}^{d+1} \tag{2}$$

respectively, where X is a random vector with the distribution μ. This definition is a straightforward generalization of the definitions of the zonoid and the lift zonoid (see [10], Definition 2.1), where the function v has the form $v(x) = x$.

The lift zonoid $\hat{Z}(\mu)$ is a convex compact set in $\mathbb{R}^{d+1}$, symmetric w.r.t. the point $(\frac{1}{2}, \frac{1}{2}\,\mathsf{E}\, X)$, which identifies the underlying measure μ uniquely; see [10]. On the other hand, it can be seen easily that the definition of the weighted lift zonoid $\hat{Z}^v(\mu)$ would not change if one restricts the class of Borel measurable functions g within it to the class of the functions of the form

$$g(x) = G(v(x)), \qquad G\colon \mathbb{R}^d \to [0,1] \text{ is Borel measurable.}$$

This observation leads immediately to the identity $\hat{Z}^v(\mu) = \hat{Z}(\mu \circ v^{-1})$; that is, the weighted lift zonoid $\hat{Z}^v(\mu)$ equals the (usual) lift zonoid of the image of the measure μ under the mapping v. As a corollary, we get that the weighted lift zonoid $\hat{Z}^v(\mu)$ is a convex compact set in $\mathbb{R}^{d+1}$, symmetric w.r.t. the point $((1/2), (1/2)Ev(X))$, and identifies the image measure $\mu \circ v^{-1}$ uniquely.

The following theorem motivates the above definition of the weighted lift zonoid. To formulate it, we need to introduce some notation. Denote by γ_c the centered Gaussian measure in $\mathbb{R}^d$ with the covariance matrix $c^2 I_{\mathbb{R}^d}$. Let

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy, \quad x \in \mathbb{R},$$

be the standard Gaussian distribution density function and the standard Gaussian cumulative distribution function, respectively, and let

$$I(p) = \varphi(\Phi^{-1}(p)), \qquad p \in (0,1), \quad I(0) = I(1) = 0, \tag{3}$$

be the *Gaussian isoperimetric function*.

For any measurable f on $\mathbb{R}^d$, we write $\mathbf{E}_\mu f$ for the integral of f w.r.t. μ; function f may be vector-valued, then the integral is understood in the component-wise sense. For a function f taking values in $\mathbb{R}^+$, its μ-entropy is defined by

$$\mathbf{Ent}_\mu f = \mathbf{E}_\mu(f \log f) - (\mathbf{E}_\mu f)\log(\mathbf{E}_\mu f),$$

with the convention $0 \log 0 = 0$.

In what follows, we assume that the measure μ has *the logarithmic gradient* v_μ; that is, a function $v_\mu \colon \mathbb{R}^d \to \mathbb{R}$, integrable w.r.t. μ and such that for every smooth $f \colon \mathbb{R}^d \to \mathbb{R}$ with a compact support

$$\mathbf{E}_\mu \nabla f = -\mathbf{E}_\mu(v_\mu f). \tag{4}$$

This assumption is equivalent to the following, see Proposition 3.4.3 in [6]: there exists the density p_μ of the measure μ w.r.t. the Lebesgue measure, which belongs to the Sobolev class $W_{1,1}(\mathbb{R}^d)$; in this case

$$[v_\mu]_i = \frac{\partial_{x_i} p_\mu}{p_\mu}, \qquad i = 1, \dots, d.$$

Theorem 1. *I. The following three statements are equivalent.*

A. $\widehat{Z}^{v_\mu}(\mu) \subset \widehat{Z}(\gamma_c)$.

B. *For any smooth function* $f \colon \mathbb{R}^d \to [0,1]$ *with a compact support, one has*

$$\| \mathbf{E}_\mu \nabla f \| \le c I(\mathbf{E}_\mu f). \tag{5}$$

C. *For any* $h \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d)$

$$\Phi\left(\Phi^{-1}(\mu(A)) - c\|h\|\right) \le \mu(A+h) \le \Phi\left(\Phi^{-1}(\mu(A)) + c\|h\|\right). \tag{6}$$

II. Under the conditions ***A****–**C** above, the following inverse log-Sobolev inequality holds true: for any smooth function* $f \colon \mathbb{R}^d \to [0,\infty)$ *with a compact support,*

$$\| \mathbf{E}_\mu \nabla f\|^2 \le 2c\, \mathbf{Ent}_\mu f \, \mathbf{E}_\mu f. \tag{7}$$

Remark 1. By the definition (see Definition 5.1 in [10]), two measures μ_1 and μ_2 are related by the *lift zonoid order* (notation: $\mu_1 \preccurlyeq_{LZ} \mu_2$), if

$$\hat{Z}(\mu_1) \subset \hat{Z}(\mu_2).$$

Recall that $\widehat{Z}^{v_\mu}(\mu)$ equals the lift zonoid of $\nu_\mu := \mu \circ v_\mu^{-1}$; that is, of the distribution of the logarithmic gradient of the measure μ. Hence statement **A** can be equivalently formulated as follows: the distribution ν_μ of the logarithmic gradient of the measure μ is dominated in the sense of the lift zonoid order by the canonical Gaussian measure in $\mathbb{R}^d$.

Theorem 1 is not a genuinely new one. The equivalence of the relations **B** and **C** is used by S. Bobkov in [3] as a key ingredient in the proof of the *shift inequality* (6) (in [3], the measure μ is supposed to be a product-measure, but the proof of the equivalence of (5) and (6) in fact does not rely on this assumption). The outline of the proof of (7) under (5) and (6) is given in [2]. What we would like to emphasize is that condition **B**, usually called the *functional version of the shift inequality*, is equivalent to the relation **A**, which according to Remark 1 can be written as the lift zonoid order relation

$$\nu_\mu \preccurlyeq_{LZ} \gamma_c. \tag{8}$$

It is instructive to compare (8) with the following necessary and sufficient condition for the functional version of the shift inequality to hold, given in [3] in the case where the measure μ is a product-measure with equal marginals μ_1. This condition states that there exists $c > 0$ such that (5) holds true, if and only if there exists $\varepsilon > 0$ such that

$$\int_{\mathbb{R}} e^{\varepsilon x^2}\,\nu_{\mu_1}(dx) \le 2; \tag{9}$$

in addition, the optimal constant c in (5) and ε in (9) are connected by the relation

$$\frac{1}{\sqrt{6\varepsilon}} \le c \le \frac{4}{\sqrt{\varepsilon}}. \tag{10}$$

For the product measure $\mu(dx) = \prod_{i=1}^d \mu_1(dx_i)$, respective distribution of the logarithmic gradient is again a product measure

$$\nu_\mu(dx) = \prod_{i=1}^d \nu_{\mu_1}(dx_i),$$

and in this case, due to Corollary 5.3 in [10], (8) is equivalent to

$$\nu_{\mu_1} \preccurlyeq_{LZ} \gamma_c^1, \tag{11}$$

where γ_c^1 is the $\mathcal{N}(0, c^2)$-Gaussian measure on $\mathbb{R}$. Both (9) and (11) are conditions on the tails of the distribution of the logarithmic gradient of μ_1, but (11) is more precise because it involves the same c with (5).

The main result of this section, Theorem 2 below, is a generalization of Theorem 1 and is motivated by an observation that in Theorem 1 the equivalence of the relations **A** and **B** follows in a very straightforward way from the integration-by-parts formula (4); see the proof of Theorem 2 below. With this observation in mind, we introduce a wide class of weights which admit an analogue of the integration-by-parts formula (4). To do that, we recall that the μ-divergence of a function $g\colon \mathbb{R}^d \to \mathbb{R}^d$, if exists, is defined as the function $\delta_\mu(g) \in L_1(\mathbb{R}^d, \mu)$ such that for every smooth $f\colon \mathbb{R}^d \to \mathbb{R}$ with a compact support

$$\mathbf{E}_\mu (\nabla f, g)_{\mathbb{R}^d} = \mathbf{E}_\mu\, f \delta_\mu(g).$$

The μ-divergence is well defined, for instance, for any $g \in C^1$ bounded together with its partial derivatives; in this case,

$$\delta_\mu(g) = -\sum_{i=1}^d [v_\mu]_i g_i - \sum_{i=1}^d \partial_{x_i} g_i.$$

This follows directly from (4); see [6], Chapter 6 for more information on this subject. Let function $v\colon \mathbb{R}^d \to \mathbb{R}$ be such that, for some function K taking values in $d\times d$-matrices,

$$v_i = \delta_\mu(K_i), \qquad i = 1, \dots, d, \tag{12}$$

where K_i denotes the i-th row of the matrix K. Then for every smooth f with a compact support

$$\mathbf{E}_\mu (K\nabla f) = \mathbf{E}_\mu\, f v; \tag{13}$$

here and below we treat elements of $\mathbb{R}^d$ as vectors-columns. Formula (13) is a straightforward extension of the integration-by-parts formula (4), where the gradient ∇ is replaced by the “weighted gradient” $K\nabla$ with the matrix-valued weight K, and the logarithmic gradient v_μ is replaced by the μ-divergence of K. Furthermore, if K satisfies some extra regularity condition, e.g.

$$K\colon \mathbb{R}^d \to \mathbb{R}^{d\times d} \quad \text{is Lipschitz}, \tag{14}$$

then for every $h \in \mathbb{R}^d$ there exists a flow of solutions $\{\Psi_t^{K,h}(x), t \in \mathbb{R}, x \in \mathbb{R}^d\}$ of the Cauchy problem

$$d\Psi_t(x) = (K^* h)(\Psi_t(x))\, dt, \qquad \Psi_0(x) = x. \tag{15}$$

Theorem 2. *I. Let $v = (v_i)_{i=1}^d$ satisfy* (12). *Then the following two statements are equivalent.*

A1. $\widehat{Z}^v(\mu) \subset \widehat{Z}(\gamma_c)$.

B1. *For any smooth function $f\colon \mathbb{R}^d \to [0,1]$ with a compact support, one has*

$$\| \mathbf{E}_\mu K\nabla f\| \le cI(\mathbf{E}_\mu f). \tag{16}$$

If, in addition, the matrix-valued function K satisfies (14), *then* ***A1*** *and* ***B1*** *are equivalent to the following.*

C1. *For any $h \in \mathbb{R}^d$ and $A \in \mathcal{B}(\mathbb{R}^d)$*

$$\Phi\left(\Phi^{-1}(\mu(A)) - c\|h\|\right) \le \mu\left(\left[\Psi_1^{K,h}\right]^{-1}(A)\right) \le \Phi\left(\Phi^{-1}(\mu(A)) + c\|h\|\right). \tag{17}$$

II. Under the condition ***A1****, equivalently* ***B1****, the following weighted inverse log-Sobolev inequality holds true: for any smooth function $f\colon \mathbb{R}^d \to [0,\infty)$ with a compact support,*

$$\| \mathbf{E}_\mu K\nabla f\|^2 \le 2c^2\, \mathbf{Ent}_\mu f\, \mathbf{E}_\mu f. \tag{18}$$

Note that condition **A1** is just the lift zonoid order relation for the image measure of μ under v:

$$\mu \circ v^{-1} \preccurlyeq_{LZ} \gamma_c. \tag{19}$$

Before giving the proof of Theorem 2, let us summarize: a lift zonoid order condition (8) is a *criterion* for the shift inequality, written either in its explicit form (6), or in its functional form (5). This equivalence is rather flexible in the following sense: if the logarithmic gradient v_μ in (8) is replaced by another weight v of the form

$$v = \delta_\mu(K) \tag{20}$$

(see (12)), then respective lift zonoid order condition (19) is still equivalent to the weighted version (16) of the functional form of a (generalized) shift inequality. The explicit form of the (generalised) shift inequality in that case is available as well, and concerns, instead of linear shifts, the transformations of the initial measure μ by the flows of solutions to (15).

Proof of Theorem 2: statement I. The lift zonoid $\hat{Z}(\gamma)$ of a standard Gaussian measure γ in $\mathbb{R}^d$ can be identified in the following way: for a given $\alpha \in (0,1)$, the section of $\hat{Z}(\gamma)$ by the hyper-plane $\{\alpha\} \times \mathbb{R}^d$ has the projection on the last d coordinates equal to the ball centered at 0 and having the radius $I(\alpha)$; see [9], Section 6.3 or [11], Proposition 3.4. It is easy to see from the definition of the lift zonoid that

$$\hat{Z}(\gamma_c) = c\hat{Z}(\gamma).$$

Hence condition **A1** can be equivalently written as follows: for every Borel measurable $g\colon \mathbb{R}^d \to [0,1]$ such that $\mathbf{E}_\mu\, g = \alpha$,

$$\|\mathbf{E}_\mu(gv)\| \le cI(\alpha) = cI(\mathbf{E}_\mu\, g).$$

By the standard approximation argument, the above condition is equivalent to a similar one with Borel measurable g's replaced by smooth and compactly supported f's. Because for such f by (13)

$$\|\mathbf{E}_\mu(fv)\| = \|\mathbf{E}_\mu(K\nabla f)\|\,,$$

conditions **A1** and **B1** are equivalent.

The proof of the equivalence of **B1** and **C1** follows the same lines with the S.Bobkov's proof from [3] for the case of product measures and linear shifts; to make the exposition self-sufficient here we expose the key steps of this proof.

Denote $R_r(p) = \Phi(\Phi^{-1}(p) + r)$, $r \ge 0$, $p \in (0,1)$. Then the following properties hold true:

- for every $r \geq 0$ the function R_r is concave;
- the family $\{R_r, r \geq 0\}$ is a semigroup w.r.t. the composition of the functions, i.e.

$$R_{r_1} \circ R_{r_2} = R_{r_1+r_2};$$

- the function R_0 is an identity, and the "generator" of the semigroup $\{R_r, r \geq 0\}$ equals the Gaussian isoperimetric function I in the sense that

$$\frac{R_r(p) - p}{r} \to I(p), \qquad r \to 0+.$$

Similarly, the family of functions $S_r(p) = \Phi(\Phi^{-1}(p) - r)$, $r \geq 0$, $p \in (0,1)$ has the following properties:

- for every $r \geq 0$ the function S_r is convex;
- the family $\{S_r, r \geq 0\}$ is a semigroup w.r.t. the composition of the functions;
- the function S_0 is an identity, and the "generator" of the semigroup $\{S_r, r \geq 0\}$ equals $(-I)$.

Observe that **C1** is equivalent to the following.

C2. For any $h \in \mathbb{R}^d$ and Borel measurable $f\colon \mathbb{R}^d \to [0,1]$

$$S_{c\|h\|}(\mathbf{E}_\mu f) \leq \mathbf{E}_\mu \left(f \circ \Psi_1^{K,h} \right) \leq R_{c\|h\|}(\mathbf{E}_\mu f). \tag{21}$$

Indeed, taking $f = \mathbb{1}_A$ we get **C1** from **C2**. Inversely, under **C1** by the concavity of R_r and Jensen's inequality we have

$$\mathbf{E}_\mu \left(f \circ \Psi_1^{K,h} \right) = \int_0^\infty \mu\left(\left\{ x\colon f\left(\Psi_1^{K,h}(x) \right) \geq t \right\} \right) dt \leq \int_0^\infty R_{c\|h\|}\left(\mu(\{x : f(x) \geq t\}) \right) dt$$

$$\leq R_{c\|h\|}\left(\int_0^\infty \mu(\{x\colon f(x) \geq t\})\, dt \right) = R_{c\|h\|}(\mathbf{E}_\mu f).$$

The proof of the left hand side inequality in (21) is similar and omitted. Hence **C1** and **C2** are equivalent.

To get **B1** from **C2**, take th instead of h and differentiate the right hand side inequality in (21) w.r.t. t at the point $t = 0$. In more details, denote $f_t(x) = f\big(\Psi_1^{K,th}(x)\big)$, then

$$f_t(x) = f\left(\Psi_t^{K,h}(x) \right),$$

and therefore there exits a continuous derifative

$$\partial_t f_t(x) = \left((\nabla f)\left(\Psi_t^{K,h}(x) \right), (K^* h)\left(\Psi_t^{K,h}(x) \right) \right)_{\mathbb{R}^d}.$$

Because f is smooth and compactly supported and K satisfies (14), this derivative is bounded as a function of $(t,x) \in [0,T] \times \mathbb{R}^d$ for every fixed T. Therefore by the dominated convergence theorem

$$\frac{1}{t}\left(\mathbf{E}_\mu f_t - \mathbf{E}_\mu f \right) \to \mathbf{E}_\mu \left(\nabla f, K^* h \right)_{\mathbb{R}^d} = \left(\mathbf{E}_\mu K \nabla f, h \right)_{\mathbb{R}^d}, \qquad t \to 0+.$$

Because

$$\frac{1}{t}(R_{ct\|h\|}(\mathbf{E}_\mu f) - \mathbf{E}_\mu f) = c\|h\| I(\mathbf{E}_\mu f),$$

we get from (21)

$$(\mathbf{E}_\mu K \nabla f, h)_{\mathbb{R}^d} \leq c\|h\| I(\mathbf{E}_\mu f), \qquad h \in \mathbb{R}^d.$$

Taking sup over all h with $\|h\| = 1$, we get (16).

To get **C2** from **B1**, consider first the case where f is smooth and compactly supported and such that $0 < \mathbf{E}_\mu f < 1$. By (16), for a given $h \in \mathbb{R}^d$ we have that

$$\left. \frac{d}{dt} \right|_{t=0} \mathbf{E}_\mu f_t = (\mathbf{E}_\mu K \nabla f, h)_{\mathbb{R}^d} \leq c\|h\| I(\mathbf{E}_\mu f).$$

Recall that

$$\frac{d}{dt}\bigg|_{t=0} R_{c\|h\|t}(\mathbf{E}_\mu f) = c\|h\| I(\mathbf{E}_\mu f).$$

Therefore for every $\varrho > 1$ there exists $\delta = \delta(f) > 0$ such that for every $t \in (0, \delta)$:

$$\mathbf{E}_\mu f_t \le R_{\varrho c\|h\|t}(\mathbf{E}_\mu f). \tag{22}$$

Note that if $t_1 \in (0, \delta(f))$ and $t_2 \in (0, \delta(f_{t_1}))$, then

$$\mathbf{E}_\mu f_{t_1+t_2} = \mathbf{E}_\mu \left(f_{t_1} \circ \Psi^{K,h}_{t_2} \right) \le R_{\varrho c\|h\|t_1}(\mathbf{E}_\mu f_{t_1}) \le R_{\varrho c\|h\|(t_1+t_2)}(\mathbf{E}_\mu f); \tag{23}$$

here we have used the flow property of $\{\Psi^{K,h}_t, t \in \mathbb{R}\}$, the semigroup property of $\{R_r, r \ge 0\}$, and monotonicity of R_r. Because the derivative $\partial_t f_t$ is uniformly continuous w.r.t. $(t,x) \in [0,T] \times \mathbb{R}^d$ for every fixed T, it can be shown that

$$\delta_T = \inf_{t\in[0,T]} \delta(f_t) > 0.$$

Then, applying (23) at most T/δ_T times, we get that (22) holds true for every $t \in [0,T]$. Consequently, (22) holds true for every $t \in \mathbb{R}^+$ and ϱ therein can be replaced by 1. This gives the right hand side inequality in (21) for smooth and compactly supported f such that $0 < \mathbf{E}_\mu f < 1$. By an approximation argument, this can be extended to any measurable $f\colon \mathbb{R}^d \to [0,1]$. The proof of the left hand side inequality in (21) is completely analogous and omitted. □

Proof of Theorem 2: statement II. The following lemma is a straightforward extension of a part of Proposition 2 in [2] (the one which states the equivalence of $P_1(c)$ and $P_2(c)$ in the notation of [2]).

Lemma 1. *Statement **B1** is equivalent to the following.*

B2.: *For any smooth function $f\colon \mathbb{R}^d \to [0,1]$ with a compact support, one has*

$$\sqrt{(\mathbf{E}_\mu I(f))^2 + \frac{1}{c^2}\| \mathbf{E}_\mu K\nabla f\|^2} \le I(\mathbf{E}_\mu f). \tag{24}$$

The proof is completely analogous to the one from [2], therefore we just sketch it. The implication **B2** ⇒ **B1** is trivial. To get the inverse implication, recall first that the standard Gaussian measure γ^d on $\mathbb{R}^d$ satisfies **B2** with $c = 1$ and identity matrix K; see [2], Section 2. Consider a smooth function $f\colon \mathbb{R}^d \to [0,1]$ with a compact support, and let $F(r) = \mu(\{x\colon f(x) \le r\})$ be its distribution function w.r.t. μ. Assume that F is absolutely continuous w.r.t. Lebesque measure on $\mathbb{R}$, and take $r \in \mathbb{R}$, $\varepsilon > 0$. Define $\psi_\varepsilon(x) = \mathbb{I}_{[0,r]}(x) + (1 - \frac{x-r}{\varepsilon})\mathbb{I}_{[r,r+\varepsilon]}(x)$. Applying **B1** to the function $g = \psi_\varepsilon(f)$ and tending $\varepsilon \to 0$, we get

$$F'(r)\|\theta(r)\| \le cI(F(r)) \quad \text{for} \quad \mu \circ f^{-1}\text{-a.a. } r \in \mathbb{R}, \tag{25}$$

where $\theta(r) = E_\mu(K\nabla f | f = r)$. Denote $k = F^{-1} \circ \Phi$, then k transforms the standard Gaussian measure γ^1 on $\mathbb{R}$ to $\mu \circ f^{-1}$. Taking the derivative in the identity $F(k) = \Phi$, we get $k'F'(k) = \varphi$. Then from (25) with $r = k(x)$ we get inequality

$$\frac{1}{c}\|\theta(k(x))\| \le k'(x) \tag{26}$$

valid γ^1-a.s. We have already mentioned that a standard Gaussian measure satisfies **B2** with $c = 1$ and identity K; for the case $d = 1$ this can be written as

$$\sqrt{\left(\int_{\mathbb{R}} I(g)d\gamma^1\right)^2 + \left(\int_{\mathbb{R}} g'd\gamma^1\right)^2} \le I\left(\int_{\mathbb{R}} gd\gamma^1\right).$$

Applying this inequality to $g=k$ and using (26) we get

$$\sqrt{\left(\int_0^1 I(r)dF(r)\right)^2+\left(\int_0^1 \frac{1}{c}\|\theta(r)\|dF(r)\right)^2}\le I\left(\int_0^1 rdF(r)\right); \qquad (27)$$

here we have took into account that the image of γ^1 under k is $\mu\circ f^{-1}$, and $\mu\circ f^{-1}$ is supported in $[0,1]$. Using the inequality

$$\int_0^1 \|\theta(r)\|dF(r)\ge\left\|\int_0^1 \theta(r)dF(r)\right\|=\|\,\mathbf{E}_\mu K\nabla f\|,$$

we complete the proof of the required statement. The additional assumption of $\mu\circ f^{-1}$ to be absolutely continuous can be removed by an approximation argument. □

According to Lemma 1, to prove statement II of Theorem 2 it is enough to show that **B2** implies (18) for any non-negative smooth compactly supported f. Take ε small, then εf takes values in $[0,1]$ and one can apply **B2**. After trivial transformations, we get

$$\frac{1}{c^2}\|\,\mathbf{E}_\mu K\nabla f\|^2\le\frac{I^2(\varepsilon\,\mathbf{E}_\mu f)-(\mathbf{E}_\mu I(\varepsilon f))^2}{\varepsilon^2}.$$

Hence the required statement would follow from the relation

$$\lim_{\varepsilon\to0+}\frac{I^2(\varepsilon\,\mathbf{E}_\mu f)-(\mathbf{E}_\mu I(\varepsilon f))^2}{\varepsilon^2}=2\,\mathbf{Ent}_\mu f\,\mathbf{E}_\mu f. \qquad (28)$$

This relation can be proved straightforwardly using the following asymptotic expansion:

$$I(\varepsilon)=\varepsilon\sqrt{2\log\frac{1}{\varepsilon}}-\frac{\varepsilon\log(2\log\frac{1}{\varepsilon})}{2\sqrt{2\log\frac{1}{\varepsilon}}}+\frac{\varepsilon}{\sqrt{2\log\frac{1}{\varepsilon}}}+\frac{\varepsilon\kappa(\varepsilon)}{\sqrt{2\log\frac{1}{\varepsilon}}}, \qquad (29)$$

where $\kappa(\varepsilon)\to0$, $\varepsilon\to0+$; the detailed exposition is straightforward but cumbersome and therefore is omitted. The asymptotic expansion (29) follows from the standard expansion

$$\Phi(t)=-\frac{1}{t}\varphi(t)+\frac{1}{t^3}\varphi(t)+O\left(t^{-5}\varphi(t)\right),\qquad t\to-\infty,$$

which holds true e.g. by the integration-by-parts formula. □

Remark 2. The above proof of statement II follows, in main lines, the one sketched in [2] (the proof of the implication $P_3(c)\Rightarrow P_6(c\sqrt{2})$ in Proposition 2), where the authors referred to Beckner's lectures at the Institut Henri Poincaré. However, instead of using the equivalence

$$I(\varepsilon)\sim\varepsilon\sqrt{2\log\frac{1}{\varepsilon}},\qquad \varepsilon\to0,$$

which apparently is not sufficient to provide (28), we use stronger asymptotic expansion (29).

Let us mention that a more explicit condition, sufficient for the lift zonoid relation (19) tohold true, can be given in a way similar to (9).

Proposition 1. *There exists* $c>0$ *such that* (19) *holds true, if and only if, there exists* $\varepsilon>0$ *such that*

$$\mathbf{E}_\mu e^{\varepsilon(v,h)^2_{\mathbb{R}^d}}\le2,\qquad \|h\|\le1. \qquad (30)$$

The optimal constant c *in* (19) *and* ε *in* (30) *are connected by the relation* (10).

Because the lift zonoid order relation is equivalent to the same relation for all one-dimensional projections (see Section 5 in [10]), statement of Proposition 1 follow immediately from the one-dimensional statement given below.

Lemma 2. *For a measure ν on $\mathbb{R}$ there exists $c>0$ such that*

$$\nu \preccurlyeq_{LZ} \gamma_c$$

with $\gamma_c \sim \mathcal{N}(0,c)$ if, and only if, there exists $\varepsilon>0$ such that

$$\int_{\mathbb{R}} e^{\varepsilon x^2}\,dx \le 2;$$

in that case, the optimal constants c, ε are connected by the relation (10).

The proof of Lemma 2 is contained, in fact, in the proof of Lemma 4.1 in [3], hence we omit it here.

At the end of this section, let us indicate one further research possibility related to the above results. In [4], an approach is proposed, making it possible to give explicit bounds for ergodic rates of solutions to Lévy driven SDE's, which has a wide range of further applications e.g. to limit theorems for functionals of such processes, see [14]–[16]. The key ingredient of this approach is a stochastic control based on perturbations of time coordinates of jumps of the Lévy noise. A natural question is whether such an approach remains practical when perturbations of jump amplitudes are used instead, which is typical in the stochastic calculus of variations for processes with jumps. In this context, it would be helpful to bound from below the size of the absolutely continuous part of the image of the Lévy measure of the noise under a non-linear mapping which corresponds to the perturbation of the noise. The above results seemingly can be useful here, because shift inequalities yield upper bounds for the size of singular component of the image of a measure: respective result was obtained in [3] in the context of linear shift inequalities (6), and can be extended easily to non-linear shift inequalities (17).

3. Weighted log-Sobolev inequalities in $\mathbb{R}$

Theorem 2 above gives a sufficient condition for a weighted inverse log-Sobolev inequality, based on a pair of functions v and K related by (20). The main result of this section, Theorem 3 below, shows that the use of the same pair may lead to sufficient conditions for the (direct) log-Sobolev inequality, either in a weighted or in a classical form. What is surprising is that, even in the simplest one-dimensional case, Theorem 3 leads to new sufficient conditions for the log-Sobolev inequality, when compared with those available in a literature; see below Proposition 2, Proposition 3, and two examples in Section 4. We believe that the reason for that is a proper choice of the *pair* of the weight functions v and K, involved in (31) and connected by (20).

Theorem 3. *Let $d=1$ and functions v and K be related by* (20). *Assume that for some $\alpha>0$*

$$Kv' \ge \alpha. \tag{31}$$

Assume in addition that the functions K and

$$a := 2KK' + K^2 v_\mu \tag{32}$$

belong to C^∞, have at most linear growth at ∞, and all their derivatives have at most polynomial growth at ∞.

Then for every smooth f with a compact support

$$\mathbf{Ent}_\mu f^2 \le \frac{2}{\alpha}\,\mathbf{E}_\mu (Kf')^2. \tag{33}$$

As a corollary, if K is bounded then μ satisfies the (classical) log-Sobolev inequality: for every absolutely continuous f such that both f and f' are square integrable w.r.t. μ,

$$\mathbf{Ent}_\mu f^2 \le \frac{2}{\alpha}\left(\sup_x K^2(x)\right)\mathbf{E}_\mu (f')^2. \tag{34}$$

Remark 3. The proof of Theorem 3 is based on the classic Bakry–Emery criterion; see below. We strongly believe that similar technique is applicable in the multidimensional case as well, but because of possible non-commutativity of matrix-valued weights which appear therein, now we can not give a multidimensional version of Theorem 3; this is a subject for a further research.

Remark 4. The additional assumptions on the functions K and a to be smooth and to satisfy certain growth bounds, in particular cases, can be removed by an approximation procedure; see e.g. Propositions 2 and 3 below.

Proof of Theorem 3. Consider a Markov process X defined as the strong solution to the SDE

$$dX_t = a(X_t)\,dt + \sqrt{2}K(X_t)\,dW_t;$$

see (32) for the formula for the coefficient a. Then on the Schwartz space $\mathcal{S}(\mathbb{R})$ of C^∞ functions s.t. all their derivatives decay at ∞ faster than any polynomial, the generator L of the process X has the form

$$Lf = af' + bf'' = v_\mu f' + (bf')', \qquad b := K^2.$$

By the construction, the measure μ is a symmetric measure for the semigroup $\{T_t\}$ generated by the process X:

$$\mathbf{E}_\mu\, fT_tg = \mathbf{E}_\mu\, gT_tf, \qquad t \geq 0;$$

in particular,

$$\mathbf{E}_\mu\, T_tf = \mathbf{E}_\mu\, f, \qquad t \geq 0,$$

i.e. μ is an invariant measure for X. The class $\mathcal{G} = \mathcal{S}(\mathbb{R})$ is an algebra, invariant w.r.t. superpositions with C^∞-functions and dense in every $L_p(\mu)$, $p \geq 1$. In addition, thanks to the smoothness conditions and growth bounds imposed on coefficients a and K, the class $\mathcal{G}$ is invariant w.r.t. the semigroup T_t and the generator L. Define for $f, g \in \mathcal{G}$

$$\Gamma(f,g) = \frac{1}{2}(L(fg) - fLg - gLf), \qquad \Gamma_2(f,g) = \frac{1}{2}(L\Gamma(f,g) - \Gamma(Lf,g) - \Gamma(f,Lg)).$$

We will prove that

$$\Gamma_2(f,f) \geq \alpha\Gamma(f,f), \qquad f \in \mathcal{G}, \tag{35}$$

then the required statement would follow from the Bakry–Emery criterion [1].

Straightforward calculations give

$$\Gamma(f,g) = bf'g',$$

$$\begin{aligned} 2\Gamma_2(f,f) &= (ab' + bb'' - 2a'b)(f')^2 - 2bb'f'f'' + 2b^2(f'')^2 \\ &= \left(ab' + bb'' - 2a'b - \frac{(b')^2}{2}\right)(f')^2 + \left(\frac{b'f'}{\sqrt{2}} - bf''\sqrt{2}\right)^2 \\ &\geq \left(ab' + bb'' - 2a'b - \frac{(b')^2}{2}\right)(f')^2. \end{aligned}$$

Hence to prove (35) it is enough to show that

$$2ab' + 2bb'' - 4a'b - (b')^2 \geq 4\alpha b. \tag{36}$$

Recall that

$$v = \delta_\mu(K) = -Kv_\mu - K',$$

hence we can express the coefficients a and b through the functions K and v:

$$a = KK' - Kv, \qquad b = K^2.$$

Substituting these expressions into (36), after some transformations, which are straightforward but cumbersome and therefore omitted, we re-write (36) to the following form:

$$K^3 v' \geq \alpha K^2.$$

The last inequality clearly holds true under (31). Hence, applying the Bakry–Emery criterion, we get (33) for every $f \in \mathcal{S}(\mathbb{R})$.

If K is bounded, then for every $f \in \mathcal{S}(\mathbb{R})$ (34) holds true as a corollary of (33). It is a standard procedure to approximate a given absolutely continuous f such that $f, f' \in L_2(\mu)$ by a sequence of smooth compactly supported f_n in such a way that $f_n \to f$ and $f_n' \to f'$ in $L_2(\mu)$; see e.g. the proof of Corollary 2.6.10 in [6]. Passing to the limit in (34) for f_n, $n \geq 1$, we complete the proof. □

There is a wide choice for the pair of functions v and K related by (20). Below we give two versions of Theorem 3 which correspond to particular choices of this pair. The first one arise when one just takes $v(x) = x - \langle \mu \rangle$,

$$\langle \mu \rangle = \int_{\mathbb{R}} y \, \mu(dy).$$

Proposition 2. *Let measure μ on $\mathbb{R}$ have the first absolute moment and have a positive continuous distribution density p_μ. Denote*

$$\bar{K}_\mu(x) = \frac{1}{p_\mu(x)} \int_x^\infty (y - \langle \mu \rangle) p_\mu(y) \, dy, \qquad x \in \mathbb{R}.$$

The following statements hold true.

I. *If $\inf_x \bar{K}_\mu(x) = \alpha > 0$, then for every smooth f with a compact support*

$$\mathbf{Ent}_\mu f^2 \leq \frac{2}{\alpha} \mathbf{E}_\mu \left(\bar{K}_\mu f' \right)^2.$$

II. *If, in addition, $\sup_x \bar{K}_\mu(x) = \beta < \infty$, then for every absolutely continuous f such that both f and f' are square integrable w.r.t. μ,*

$$\mathbf{Ent}_\mu f^2 \leq 2\bar{c}_\mu \, \mathbf{E}_\mu (f')^2$$

with

$$\bar{c}_\mu = \frac{\beta^2}{\alpha}.$$

In the second version of Theorem 3, we choose K in a more intrinsic way, namely, we take K such that $\delta_\mu(K) = v$ with

$$v = \Phi^{-1}(F_\mu), \qquad F_\mu(x) = \mu((-\infty, x]), \tag{37}$$

then $\mu \circ v^{-1} = \gamma$, $\gamma \sim \mathcal{N}(0,1)$. Such a choice of the weight v is motivated by our intent to have

$$\hat{Z}^v(\mu) = \hat{Z}(\gamma);$$

that is, to make the order condition (19) with $c = 1$ as precise as it is possible, i.e. to replace an inequality by an identity. Because $\hat{Z}^v(\mu) = Z(\mu \circ v^{-1})$ identifies the law of $\mu \circ v^{-1}$ uniquely, such an intent naturally leads to the formula (37).

Proposition 3. *Let measure μ on $\mathbb{R}$ have a positive continuous distribution density p_μ. Denote*

$$\hat{K}_\mu(x) = \frac{I(F_\mu(x))}{p_\mu(x)}.$$

The following statements hold true.

I. *For every smooth f with a compact support,*

$$\mathbf{Ent}_\mu f^2 \leq 2 \, \mathbf{E}_\mu (\hat{K}_\mu f')^2. \tag{38}$$

II. *If, in addition, $\hat{K}_\mu$ is bounded, then for every absolutely continuous f such that both f and f' are square integrable w.r.t. μ,*

$$\mathbf{Ent}_\mu f^2 \le 2\hat{c}_\mu \,\mathbf{E}_\mu (f')^2$$

with

$$\hat{c}_\mu = \sup_x (\hat{K}_\mu(x))^2.$$

Remark 5. Define the *isoperimetric function* of the measure μ by

$$I_\mu(p) = p_\mu\left(F_\mu^{-1}(p)\right), \qquad p \in (0,1), \qquad I_\mu(0) = I_\mu(1) = 1.$$

Then, clearly, the function I defined by (3) equals I_γ, $\gamma \sim \mathcal{N}(0,1)$. The function $\hat{K}_\mu(x)$ above can be expressed as the ratio

$$\left.\frac{I_\gamma(p)}{I_\mu(p)}\right|_{p=F_\mu(x)},$$

and under the conditions of Proposition 3 the function F_μ gives a one-to-one correspondence between $(-\infty,\infty)$ and $(0,1)$. Hence the constant $\hat{c}_\mu$ above can be alternatively expressed as

$$\hat{c}_\mu = \left(\sup_{p\in(0,1)} \frac{I_\gamma(p)}{I_\mu(p)}\right)^2.$$

Proofs of Proposition 2 and Proposition 3. If $v(x) = x - \langle \mu \rangle$, we have $\bar{K}_\mu v' = \bar{K}_\mu$, and therefore the assumption $\inf \bar{K}_\mu = \alpha > 0$ made in Proposition 2 implies the principal condition (31). For the function v defined by (37) and the function $\hat{K}_\mu$, this condition takes even a more simple form because straightforward calculation shows that

$$\hat{K}_\mu v' = 1.$$

Hence one can expect that statements of Proposition 2 and Proposition 3 would follow from the version of the Bakry–Emery criterion given in Theorem 3. However, we can not apply this theorem here directly, because of extra smoothness and growth conditions on functions K and a, imposed therein. The strategy of the proof will be the following: first, we consider a family of measures, which approximate μ properly and satisfy both (31) for the respective pair of K and v, and extra smoothness and growth conditions on respective functions K and a. Then, by passing to a limit, we get respective weighted log-Sobolev inequality, i.e. prove statements I in Propositions 2, 3. Finally, using the same approximation procedure as in the proof of Theorem 3 above, we extend the class of f in the case where the weight K is bounded.

To shorten the exposition, we explain in details the way this strategy is implemented for the proof of Proposition 3, only. The detailed proof of Proposition 2 is similar and omitted. We also does not repeat the approximation arguments from the proof of Theorem 3 above, and concentrate on the proof of (38) for smooth compactly supported f.

Consider first the following auxiliary case: $p_\mu \in C^\infty$, and for some $R > 0$

$$p_\mu(x) = \varphi(x), \qquad |x| \ge R. \tag{39}$$

Then v_μ (which, let us recall, equals p_μ'/p_μ) and $\hat{K}_\mu$ belong to C^∞ and

$$v_\mu(x) = -x, \qquad \hat{K}_\mu(x) = 1, \qquad |x| \ge R.$$

Then the functions $K = \hat{K}_\mu$ and a defined by (32) satisfy the assumptions of Theorem 3. Hence, applying Theorem 3, we get (38).

Next, consider the general case. Fix some function $\chi \in C^\infty$ taking values in $[0,1]$, such that $\chi(0) = 0$, $\chi(x) = 1$, $x \ge 1$, and define

$$\varphi_{r,\delta}(x) = \varphi(x)(\delta + (1-\delta)\chi(|x| + r)), \qquad x \in \mathbb{R};$$

then every $\varphi_{r,\delta}$, $r > 0$, $\delta \geq 0$ belongs to C^∞. Denote

$$M(r) = \int_{\mathbb{R}} \varphi_{r,0}(x)\, dx,$$

then M is a strictly decreasing function on $[0, \infty)$ and $M(0) < 1$. For a given $Q > 0$, consider the restriction p_μ^Q of p_μ to the segment $[-Q, Q]$, and assume that Q is large enough for

$$\int_{|x|>Q} p_\mu(x)\, dx < M(0).$$

Then for every δ small enough there exists unique $r = r(Q, \delta) > 0$ such that

$$\int_{\mathbb{R}} \left(p_\mu^Q(x) + \varphi_{r,\delta}(x)\right)\, dx = 1.$$

Take some non-negative $\psi \in C^\infty$, supported in $[-1, 1]$ and such that $\int_{\mathbb{R}} \psi(x)\, dx = 1$, and consider the probability measure $\mu_{Q,\varepsilon}$ with the density

$$p_{\mu_{Q,\delta}}(x) = \frac{1}{\delta} \int_{[-\delta,\delta]} p_\mu^Q(y) \psi\left(\frac{x-y}{\delta}\right)\, dy + \varphi_{r,\delta}(x).$$

By the construction, every $\mu_{Q,\delta}$ has positive C^∞ density and satisfy (39) for some large R. Therefore, (38) holds true with $\mu_{Q,\delta}$ instead of μ. It can be seen easily that

$$p_{\mu_{Q,\delta}} \to p_\mu, \qquad K_{\mu_{Q,\delta}} \to K_\mu, \qquad \delta \to 0,\ Q \to \infty,$$

uniformly on every finite segment. Passing to the limit, we obtain (38) for the initial measure μ and arbitrary smooth and compactly supported f. □

4. Examples

Example 1. Let μ on $\mathbb{R}$ have a positive C^1-density p_μ, such that for some $a, R > 0$

$$v_\mu(x)x \geq -ax^2, \qquad |x| > R \tag{40}$$

Let us show that then condition $\inf \bar{K}_\mu > 0$ from Proposition 2 holds true. Changing the variables $x \mapsto x - \langle\mu\rangle$, we can restrict ourselves to the case of $\langle\mu\rangle = 0$. Then we have for $x > R$

$$\begin{aligned} \bar{K}_\mu(x) &= \int_x^\infty y \exp(\log p_\mu(y) - \log p_\mu(x))\, dy \\ &= \int_x^\infty y \exp\left(\int_x^y v_\mu(z)\, dz\right)\, dy \geq \int_x^\infty y \exp\left(-a \int_x^y z\, dz\right)\, dy \\ &= e^{ax^2/2} \int_x^\infty y e^{-ay^2/2}\, dy = 1/a. \end{aligned}$$

Similar relation holds true for $x < -R$; to see this, one should note that

$$\bar{K}_\mu = -\frac{1}{p_\mu(x)} \int_{-\infty}^x y p_\mu(y)\, dy$$

because μ is centered. Finally, because $p_\mu \in C^1$ is positive, $\bar{K}_\mu$ has positive infimum over $[-R, R]$, which completes the proof.

Similarly, if in addition for some $b > 0$

$$v_\mu(x)x \leq -bx^2, \qquad |x| > R, \tag{41}$$

then $\sup \bar{K}_\mu < \infty$. Hence, by statement II of Proposition 2, for a measure μ satisfying (40) and (41) the log-Sobolev inequality holds true.

Note that (41) is just the well known drift condition, sufficient for the Poincaré inequality, e.g. Theorem 3.1 and Remark 3.2 in [8]. However, various sufficient conditions for the log-Sobolev inequality, available in the literature, typically require additional assumptions on the *curvature*, which in the current context equals $-v_\mu'$. Namely, the famous Bakry–Emery condition ([1])) requires $-v_\mu' \geq \delta > 0$; conditions by Wang ([17]) and Cattiaux–Guillin ([8], Theorem 5.1) are more flexible, but still contain a requirement that the curvature is bounded from below, i.e. in our case

$$-v_\mu' \geq \delta \tag{42}$$

with some $\delta \in \mathbb{R}$. The above condition (40) can be understood as an "integral" version of (42), and it is easy to give an example of measure μ satisfying (40) and (41) such that (42) fails.

Example 2. Let γ^3 be a standard Gaussian measure on $\mathbb{R}^3$, and B_R be a ball of radius R, touching the origing and with the center located at the first basis vector e_1; that is, $B_R = B(Re_1, R)$. Denote by $\gamma^{3,R}$ the measure γ^3 conditioned outside the ball B_R:

$$\gamma^{3,R}(A) = \frac{\gamma^3(A \setminus B_R)}{\gamma^3(\mathbb{R}^3 \setminus B_R)}.$$

Consider a measure μ_R on $\mathbb{R}$ which is a projection of $\gamma^{3,R}$ on the first coordinate. We will show that there exists some constant $\hat{c}$ such that uniformly by $R \geq 0$ the constants $\hat{c}_\mu$ for the measures $\mu = \mu_R$ from Proposition 3 are dominated by $\hat{c}$. This would yield that for the family μ_R, $R \geq 0$ the log-Sobolev inequality holds true with uniformly bounded constants.

For a given $x \in [0, 2R]$, the section of the ball B_R by the hyperplane

$$\{y = (y_1, y_2, y_3) \colon y_1 = x\},$$

projected on the last two coordinates, is the ball in $\mathbb{R}^2$, centered at the origin and having the radius

$$r_R(x) = \sqrt{2Rx - x^2}.$$

Define $r(x) = 0$ for $x \notin [0, 2R]$. Then we have for $\mu = \mu_R$

$$p_\mu(x) = C_R \varphi(x) \psi_2(r_R(x)),$$

where

$$C_R = \left(\gamma^3(\mathbb{R}^3 \setminus B_R)\right)^{-1},$$

$$\psi_2(r) = \int_{\|y\| \geq r} \frac{1}{2\pi} e^{-(y_1^2 + y_2^2)/2} \, dy_1 \, dy_2 = \frac{1}{2\pi} \int_0^{2\pi} \int_r^\infty e^{-\rho^2/2} \rho \, d\rho \, d\theta = e^{-r^2/2}.$$

Consequently,

$$p_\mu(x) = \frac{C_R}{\sqrt{2\pi}} \begin{cases} e^{-Rx}, & x \in [0, 2R], \\ e^{-x^2/2}, & \text{otherwise}. \end{cases} \tag{43}$$

To bound $\hat{K}_\mu(x)$ consider separately three cases.

I. $x < 0$. Recall that $I'(p) = -\Phi^{-1}(p)$. Then for any $c > 1$ we have

$$[I(c\Phi(x))]' = -\Phi^{-1}(c\Phi(x)) c\varphi(x) \leq (-x) c\varphi(x) = c\varphi'(x)$$

because Φ^{-1} is an increasing function. Clearly, both $I(c\Phi(x))$ and $\varphi(x)$ vanish as $x \to -\infty$, hence

$$I(c\Phi(x)) = \int_{-\infty}^x [I(c\Phi(y))]' \, dy \leq c \int_{-\infty}^x \varphi'(y) \, dy = c\varphi(x), \qquad x \leq \Phi^{-1}(1/c). \tag{44}$$

Note that for $x < 0$

$$F_\mu(x) = C_R \Phi(x), \qquad p_\mu(x) = C_R \varphi(x),$$

and $C_R > 1$. In addition, the half-space $\{y = (y_1, y_2, y_3) \colon y_1 \leq x\}$ is contained in $\mathbb{R}^3 \setminus B_R$, hence

$$\Phi(x) = \gamma^3(\{y = (y_1, y_2, y_3) : y_1 \leq x\}) \leq \frac{1}{C_R} \Leftrightarrow x \leq \Phi^{-1}\left(\frac{1}{C_R}\right),$$

and we can apply (44) to get

$$\hat{K}_\mu(x) = \frac{I(C_R\Phi(x))}{C_R\varphi(x)} \le 1, \qquad x < 0.$$

II. $x > 2R$. In this case $1 - F_\mu(x) = C_R(1 - \Phi(x))$. Recall that $I(p) = I(1-p)$ and $\Phi^{-1}(1 - \Phi(x)) = -x$, hence we can use the same argument as in the case **I** to show that $\hat{K}_\mu(x) \le 1$, because for any $c > 1$

$$I(c(1 - \Phi(x)) = -c\int_x^\infty \Phi^{-1}\big(c(1 - \Phi(y))\big)\, dy \le c\int_x^\infty y\varphi'(y)\, dy = c\varphi(x).$$

III. $x \in [0, 2R]$. Recall that there exists a constant c_* such that

$$I(p) \le c_* p\sqrt{\log\frac{1}{p}}, \qquad p \in \left(0, \frac{1}{2}\right).$$

One has

$$C_R\gamma^3(\{y = (y_1, y_2, y_3)\colon y_1 > R\}) \le 1 - F_\mu(x) \le C_R\gamma^3(\{y = (y_1, y_2, y_3)\colon y_1 > 0\}) < \frac{1}{2},$$

hence we can write, using the identity $I(p) = I(1-p)$,

$$\hat{K}_\mu(x) = \frac{I(1 - F_\mu(x))}{p_\mu(x)} \le c_*\frac{1 - F_\mu(x)}{p_\mu(x)}\sqrt{\log\frac{1}{1 - F_\mu(x)}}.$$

Because $C_R > 1$, we have

$$\log\frac{1}{1 - F_\mu(x)} \le \log\frac{1}{1 - F_\mu(2R)} = \log\frac{1}{C_R(1 - \Phi(2R))} \le \log\frac{1}{1 - \Phi(2R)} \le c^*(1+R)^2$$

with some $c^* > 2$. By (43), we have

$$\frac{1 - F_\mu(x)}{p_\mu(x)} = e^{Rx}\left(\int_x^{2R} e^{-Ry}\, dy + \int_{2R} e^{-y^2/2}\, dy\right),$$

and the right hand side term can be estimated either by

$$e^{Rx}\int_x^\infty e^{-Ry}\, dy = \frac{1}{R},$$

(when R is large), or by

$$e^{2R^2}\int_0 e^{-y^2/2}\, dy = \sqrt{\frac{\pi}{2}}e^{2R^2}$$

(when R is small). Then for any $R > 0$ for $\mu = \mu_R$

$$\hat{c}_\mu = \sup_x \hat{K}_\mu \le \hat{c} := c_* c^* \sup_{Q>0} \min\left(\frac{1+Q}{Q}, \sqrt{\frac{\pi}{2}}(1+Q)e^{2Q^2}\right);$$

for $R = 0$ the measure μ just equals γ and therefore $\hat{c}_\mu = 1$.

This example is motivated by the manuscript [5], where the problem of estimating of the Poincaré constant for a Gaussian measure conditioned outside a ball is considered. One approach proposed therein is based on the decomposition of variance, and requires an estimate for the Poincaré constant of one-dimensional projection of the "punctured" Gaussian measure on the line which contains the center of the ball. Such an estimate depend on the position and the size of the ball, see Lemma 4.7 in [5], and the case of a large ball touching the origin relates the case (4) of that lemma. Our estimate for the log-Sobolev constant implies that the Poincaré constant for μ is uniformly bounded by $\hat{c}$, which drastically improves the bound ce^{R^2} from Lemma 4.7 [5], statement (4). Heuristically, the reason for this is the following. The measure μ contain "cavities", which appear due to the "puncturing" procedure, and if the ball is "large" and is located not so "far from the origin", then these "cavities" make the bounds for the Poincaré inequality obtained via classic sufficient conditions to be very inaccurate. On the other hand, the form of the weight $\hat{K}_\mu$ in Proposition 3 is highly adjusted to these "cavities", which makes respective bounds more precise. We believe that similar calculations can be made in a general setting, i.e. for arbitrary $d \geq 2$ and arbitrary position and size of the ball; this is a subject of a further research.

Acknowledgements. The paper was prepared partially during a visit of authors to the University of Potsdam; the authors express their deep gratitude to the University of Potsdam and personally to Sylvie Roelly for the hospitality. The authors are also grateful to Emmanuel Boissard for fruitful discussions concerning the manuscript [5].

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