POISSON APPROXIMATION OF PROCESSES WITH LOCALLY INDEPENDENT INCREMENTS WITH MARKOV SWITCHING

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Abstract. In this paper, the weak convergence of additive functionals of processes with locally independent increments and with Markov switching in the scheme of Poisson approximation is investigated. Singular perturbation problem for the generator of Markov process is used to prove the relative compactness.

Анотація. В роботі досліджено слабку збіжність адитивних функціоналів від процесів з локально незалежними приростами та марковським перемиканням в схемі пуассонової апроксимації. Для доведення відносної компактності процесу використано задачу сингулярного збурення для генератора марковського процесу.

Аннотация. В работе исследована слабая сходимость аддитивных функционалов от процессов с локально независимыми приращениями и марковским переключением в схеме пуассоновской аппроксимации. При доказательстве относительной компактности процесса используется задача сингулярного возмущения для генератора марковского процесса.

1. Introduction

Let us consider the following stochastic additive functional

$$\xi(t) = \xi(0) + \int_0^t \eta(ds; x(s)), \qquad t \geq 0,$$

where $x(t)$, $t \geq 0$, is a jump Markov process with the state space $(E, \mathcal{E})$ and $\eta(t, x)$ is a family of processes with independent increments, $x \in E$, $t \geq 0$ with state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. This is an important process since we have as particular cases the following well-known stochastic systems:

• The integral functional

$$\alpha(t) = \int_0^t a(x(s))\, ds, \qquad t \geq 0$$

where a is a deterministic measurable function defined on $(E, \mathcal{E})$.

• The dynamical system

$$\dot{u}(t) = C(u(t), x(t)), \qquad t \geq 0,$$

where C is a deterministic $\mathbb{R}^d$-function defined on $\mathbb{R}^d \times E$.

• The compound Poisson process

$$\zeta(t) = \sum_{k=1}^{\nu(t)} a(x_k),$$

where x_k is the embedded Markov chain of the jump Markov process $x(t)$.

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In this paper we establish weak convergence results in a semimartingale framework. In fact, we prove that for time scaled switching Markov process $x(t/\varepsilon)$, the additive semimartingale $\xi^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$, weakly converges to a Poisson process with drift. The main difference from the results obtained in [8] is infinity of the measure of jumps corresponding to the processes with locally independent increments (see definition (2) below). The large deviations problem for the processes of this type was studied in [12].

The proof is given in two steps. In the first one we obtain relative compactness of the semimartingales representation of the family ξ^ε, $\varepsilon > 0$, by proving the following two facts [4]:

$$\lim_{c\to\infty} \sup_{\varepsilon\leq\varepsilon_0} \mathsf{P}\left\{\sup_{t\leq T} |\xi^\varepsilon(t)| > c\right\} = 0,$$

known as the compact containment condition (CCC), and

$$\mathsf{E}\,|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \leq k|t-s|,$$

for some positive constant $k > 0$. But due to infinity of the measure of jumps of the process we should check additional conditions (see Theorem A in Appendix).

In the second step we prove convergence of predictable characteristics of the semimartingales, which are integral functionals of the form:

$$\int_0^t a(\xi^\varepsilon(s), x^\varepsilon(s))ds,$$

by using singular perturbation technique as presented in [7].

Finally, we apply Theorem IX.3.27 from Jacod and Shiryayev [6] in order to prove the weak convergence of semimartingale.

The original part of this work is the use of relative compactness proof scheme given for averaging approximation (Bogolubov) to obtain a Poisson approximation result. Moreover, this kind of additive functionals are very useful in practice since they include the well-known stochastic systems.

The paper is organized as follows. In Section 2 we present the process with locally independent increments and the switching Markov process. In the same section we present the main results of Poisson approximation. In Section 3 we present the proof of the theorem. Two theorems we refer to are presented in the Appendix.

2. Main results

Let us consider the set of real numbers $\mathbb{R}$, and $(E, \mathcal{E})$, a *standard state space*, (i.e., E is a Polish space and $\mathcal{E}$ its Borel σ-algebra). Let $C_3(\mathbb{R})$ be a measure-determining class of real-valued bounded functions, such that $g(u)/u^2 \to 0$, as $|u| \to 0$ for $g \in C_3(\mathbb{R})$ and $C_2(\mathbb{R})$ be a measure-determining class of all continuous bounded functions which are 0 around 0 (see [6, 7]). We note that $C_2(\mathbb{R}) \subset C_3(\mathbb{R})$.

The additive functional $\xi^\varepsilon(t)$, $t \geq 0$, $\varepsilon > 0$, on $\mathbb{R}$ in the series scheme with small series parameter $\varepsilon \to 0$, $\varepsilon > 0$, is defined by the stochastic additive functional ([7, Section 3.3.1])

$$\xi^\varepsilon(t) = \xi_0^\varepsilon + \int_0^t \eta^\varepsilon(ds; x(s/\varepsilon)). \tag{1}$$

The family of processes with *locally independent increments* $\eta^\varepsilon(t;x)$, $t \geq 0$, $x \in E$, on $\mathbb{R}$, is defined by the generators (see [1, Section I.2], [7, Section 1.2.4])

$$\mathbf{\Gamma}^\varepsilon(x)\varphi(u) = b_\varepsilon(u;x)\varphi'(u) + \int_{\mathbb{R}} \left[\varphi(u+v) - \varphi(u) - v\varphi'(u)\mathbb{1}_{(|v|\leq 1)}\right] \Gamma^\varepsilon(u, dv; x), \tag{2}$$

where $\varphi(u)$ is real-valued twice differentiable function on $\mathbb{R}$ vanishing at infinity, with the sup-norm $\|\varphi\| = \sup_{u\in\mathbb{R}} |\varphi(u)|$, $\varphi(u) \in C_0^2(\mathbb{R})$, $b_\varepsilon(u;x) = \int_{\mathbb{R}} v\,\Gamma^\varepsilon(u, dv; x)$, and

$\Gamma^\varepsilon(u, dv; x)$ is the intensity kernel that satisfies the condition

$$\Gamma^\varepsilon(u, \{0\}; x) = 0.$$

Let $\mathbf{B}$ be the Banach space, that is a complete linear normed space, of all bounded real-valued measurable functions on E, with the sup-norm $\|\varphi\| = \sup_{x\in E} |\varphi(x)|$, $\varphi(x) \in \mathbf{B}$. The switching Markov process $x(t)$, $t \geq 0$, on the standard phase space $(E, \mathcal{E})$, is defined by the generator

$$\mathbf{Q}\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \tag{3}$$

where $q(x)$, $x \in E$, is the intensity of jumps function of $x(t)$, $t \geq 0$, and $P(x, dy)$ is the transition kernel of the embedded Markov chain x_n, $n \geq 0$, defined by $x_n = x(\tau_n)$, $n \geq 0$, with $0 = \tau_0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \ldots$ the jump times of $x(t)$, $t \geq 0$.

It is worth noticing that the coupled process $\xi^\varepsilon(t)$, $x(t/\varepsilon)$, $t \geq 0$, is a Markov additive process (see, e.g., [7, Section 2.5]).

Let Π be a projector onto null-subspace of reducible-invertible operator Q (see in details [7, Section 1.2]), defined in (3):

$$\Pi\varphi(x) = \int_E \pi(dx)\varphi(x).$$

The following relation is true

$$Q\Pi = \Pi Q = 0.$$

The Poisson approximation of Markov additive process (2) is considered under the following conditions.

C1: The Markov process $x(t)$, $t \geq 0$, is uniformly ergodic with $\pi(B)$, $B \in \mathcal{E}$, its stationary distribution.

C2: *Poisson approximation.* The family of processes with locally independent increments $\eta^\varepsilon(t; x)$, $t \geq 0$, $x \in E$, satisfies the Poisson approximation conditions [7, Section 7.2.3]:

PA1: Approximation of the mean values:

$$b_\varepsilon(u; x) = \int_{\mathbb{R}} v\,\Gamma^\varepsilon(u, dv; x) = \varepsilon[b(u; x) + \theta_b^\varepsilon(u; x)],$$

and

$$c_\varepsilon(u; x) = \int_{\mathbb{R}} v^2\Gamma^\varepsilon(u, dv; x) = \varepsilon[c(u; x) + \theta_c^\varepsilon(u; x)].$$

PA2: Poisson approximation condition for intensity kernel

$$\Gamma_g^\varepsilon(u; x) = \int_{\mathbb{R}} g(v)\,\Gamma^\varepsilon(u, dv; x) = \varepsilon[\Gamma_g(u; x) + \theta_g^\varepsilon(u; x)]$$

for all $g \in C_3(\mathbb{R})$, and the function $\Gamma_g(u; x)$ is bounded for each $g \in C_3(\mathbb{R})$, that is,

$$|\Gamma_g(u; x)| \leq C_g \quad \text{(a constant depending on } g).$$

The kernel $\Gamma(u, dv; x)$ is defined on the class $C_3(\mathbb{R})$ by the relation

$$\Gamma_g(u; x) = \int_{\mathbb{R}} g(v)\,\Gamma(u, dv; x), \qquad g \in C_3(\mathbb{R}).$$

The above negligible terms θ_g^ε, θ_b^ε, θ_c^ε satisfy the condition

$$\sup_{x\in E} |\theta_\cdot^\varepsilon(u; x)| \to 0, \qquad \varepsilon \to 0.$$

In addition the following conditions are used:

C3: *Uniform square-integrability*:

$$\lim_{c\to\infty} \sup_{x\in E} \int_{|v|>c} v^2\,\Gamma(u, dv; x) = 0.$$

C4: *Growth condition*: there exists a positive constant L such that

$$|b(u;x)| \le L(1+|u|) \quad \text{and} \quad |c(u;x)| \le L(1+|u|^2),$$

C5: *Linear growth of kernel*: we assume that $\Gamma(u,B;x)$ is absolutely continuous with respect to Lebesgue measure dv in $\mathbb{R}$, that is,

$$\Gamma(u,dv;x) = \Lambda(u,v;x)\,dv,$$

thus $\Lambda(u,v;x)$ is the Radon–Nikodym derivative of $\Gamma(u,B;x)$ and the following inequality holds:

$$|\Lambda(u,v;x)| \le Lf(v)(1+|u|)$$

for any real-valued non-negative function $f(v)$, $v\in\mathbb{R}$, such that

$$\int_{\mathbb{R}\setminus\{0\}} (1+f(v))v^2\,dv < \infty.$$

The main result of our work is the following.

Theorem 1. *Under conditions* **C1**–**C5** *the weak convergence*

$$\xi^\varepsilon(t) \Rightarrow \xi^0(t), \qquad \varepsilon\to 0$$

takes place.

The limit process $\xi^0(t)$, $t\ge 0$, is defined by the generator

$$\overline{\mathbf{\Gamma}}\varphi(u) = \widehat{b}(u)\varphi'(u) + \int_{\mathbb{R}} \left[\varphi(u+v) - \varphi(u) - v\varphi'(u)\mathbb{1}_{(|v|\le 1)}\right] \widehat{\Gamma}(u,dv), \tag{4}$$

where the average deterministic drift is defined by

$$\widehat{b}(u) = \Pi b(u;x) = \int_E \pi(dx) b(u;x),$$

and the average intensity kernel is defined by

$$\widehat{\Gamma}(u,dv) = \Pi\,\Gamma(u,dv;x) = \int_E \pi(dx)\Gamma(u,dv;x).$$

REMARK 1. The limit generator in the Euclidean space $\mathbb{R}^d$, $d>1$, is represented in the following view:

$$\overline{\mathbf{\Gamma}}\varphi(u) = \sum_{k=1}^d \widehat{b}_k(u)\varphi'_k(u) + \int_{\mathbb{R}^d} \left[\varphi(u+v) - \varphi(u) - \sum_{k=1}^d v_k\varphi'_k(u)\mathbb{1}_{(|v|\le 1)}\right] \widehat{\Gamma}(u,dv),$$

$$\varphi'_k(u) := \partial\varphi(u)/\partial u_k, \qquad 1\le k\le d.$$

3. Proof of Theorem 1

The proof of Theorem 1 is based on the semimartingale representation of the additive functional process (1). The method used here to prove the weak convergence is quite different from the method proposed by other authors ([4]–[6], [9]–[18]): the main point is to prove convergence of predictable characteristics of semimartingales which are integral functionals of some switching Markov processes.

According to Theorems 6.27 and 7.16 [2] the predictable characteristics of the semimartingale (2) have the following representations:

•$B^\varepsilon(t) = \varepsilon^{-1}\int_0^t b_\varepsilon(\xi^\varepsilon(s);x_s^\varepsilon)\,ds = \int_0^t b(\xi^\varepsilon(s);x_s^\varepsilon)\,ds + t\theta_b^\varepsilon$ — the first predictable characteristic;

•$C^\varepsilon(t) = \varepsilon^{-1}\int_0^t c_\varepsilon(\xi^\varepsilon(s);x_s^\varepsilon)\,ds = \int_0^t c(\xi^\varepsilon(s);x_s^\varepsilon)\,ds + t\theta_c^\varepsilon$ — the second modified characteristic;

•$\Gamma^\varepsilon(t) = \varepsilon^{-1}\int_0^t\int_{\mathbb{R}} h(v)\,\Gamma^\varepsilon(\xi^\varepsilon(s),dv;x_s^\varepsilon)\,ds = \int_0^t\int_{\mathbb{R}} h(v)\,\Gamma(\xi^\varepsilon(s),dv;x_s^\varepsilon)\,ds + t\theta_h^\varepsilon$, where $x_t^\varepsilon := x(t/\varepsilon)$, $t\ge 0$, and $\sup_{x\in E}|\theta_\cdot^\varepsilon| \to 0$, $\varepsilon\to 0$, $h(v)$ is the truncated function.

The jump martingale part of the semimartingale (2) is represented as follows

$$\mu^\varepsilon(t)=\int_0^t\int_{|v|\le 1} v\left[\mu^\varepsilon\left(\xi^\varepsilon(s),ds,dv;x_s^\varepsilon\right)-\Gamma^\varepsilon\left(\xi^\varepsilon(s),dv;x_s^\varepsilon\right)ds\right].$$

Here $\mu^\varepsilon(u,ds,dv;x)$, $x\in E$, is the family of counting measures of jumps of the process, namely

$$\mathsf{E}\,\mu^\varepsilon(u,ds,dv;x)=\Gamma^\varepsilon(u,dv;x)\,ds.$$

We can see now that predictable characteristics depend on the process $\xi^\varepsilon(s)$. Thus, to prove convergence of $\xi^\varepsilon(s)$ we should prove convergence of predictable characteristics dependent on $\xi^\varepsilon(s)$. To avoid this difficulty, we combine two methods. The one based on semimartingales theory, is combined with a solution of singular perturbation problem instead of ergodic theorem.

We split the proof of Theorem 1 in the following two steps.

3.1. **Relative compactness.** At this step we establish the relative compactness of the family of processes $\xi^\varepsilon(t)$, $t\ge 0$, $\varepsilon>0$, by using the approach developed in [10]. Let us remind that the space of all probability measures defined on the standard space $(E,\mathcal{E})$ is also a Polish space; so the relative compactness and tightness are equivalent.

Proposition 1. *Under assumption* **C4,C5,PA1**, *the following compact containment condition (CCC) holds:*

$$\lim_{c\to\infty}\lim_{\varepsilon\to 0}\mathsf{P}\left\{\sup_{t\le T}|\xi^\varepsilon(t)|>c\right\}=0. \tag{5}$$

Proof. The proof of this corollary follows from Kolmogorov's inequality by using the estimation of Lemma 1. □

Lemma 1. *Under assumption* **C4, C5, PA1** *there exists a constant $k_T>0$, independent of ε and dependent on T, such that*

$$\mathsf{E}\sup_{t\le T}|\xi^\varepsilon(t)|^2\le k_T.$$

Proof. (Following [10]). For a process $y(t)$, $t\ge 0$, let us define the process

$$y_t^\dagger=\sup_{s\le t}|y(s)|.$$

It follows from **PA1** and **C4** that for any fixed $t>0$

$$\int_0^t\int_{\mathbb{R}} v^2\Gamma^\varepsilon(\xi^\varepsilon(s),dv;x_s^\varepsilon)\,ds=\varepsilon\int_0^t c(\xi^\varepsilon(s);x_s^\varepsilon)ds+\varepsilon t\theta_c^\varepsilon(u;x)$$
$$\le\varepsilon t\left[L\left(1+\left(\left(\xi_t^\varepsilon\right)^\dagger\right)^2\right)+\theta_c^\varepsilon(u;x)\right]<\infty\quad \mathsf{P}\text{-a.s.}$$

The increasing process $\int_0^t\int_{\mathbb{R}\setminus\{0\}}v^2\Gamma^\varepsilon(\xi^\varepsilon(s),dv;x_s^\varepsilon)ds$ is continuous in t so that by Theorem 28 from [3, Ch.5] (see also Theorem 1.6.3 [11]) it is the compensator of

$$\int_0^t\int_{\mathbb{R}\setminus\{0\}}v^2\mu^\varepsilon\left(\xi^\varepsilon(s),d(s/\varepsilon),dv;x_s^\varepsilon\right).$$

Therefore, (1) is the special semimartingale with the decomposition

$$\xi^\varepsilon(t)=u+A_t^\varepsilon+M_t^\varepsilon, \tag{6}$$

where $u=\xi^\varepsilon(0)$; A_t^ε is the predictable drift (see [4]):

$$A_t^\varepsilon=\int_0^t b\left(\xi^\varepsilon(s),x_s^\varepsilon\right)ds+\int_0^t\int_{|v|>1}v\Gamma\left(\xi^\varepsilon(s),dv;x_s^\varepsilon\right)ds+\theta_A^\varepsilon(t),$$

and M_t^ε is the locally square integrable martingale

$$M_t^\varepsilon = \int_0^t \int_{\mathbb{R}\setminus\{0\}} v[\mu(\xi^\varepsilon(s), ds, dv; x_s^\varepsilon) - \Gamma(\xi^\varepsilon(s), dv; x_s^\varepsilon)ds] + \theta_M^\varepsilon(t),$$

and for every finite $T > 0$

$$\sup_{0\le t\le T} |\theta_\cdot^\varepsilon(t)| \to 0, \qquad \varepsilon \to 0.$$

From (6) we have

$$\left((\xi_t^\varepsilon)^\dagger\right)^2 \le 3\left[u^2 + \left((A_t^\varepsilon)^\dagger\right)^2 + \left((M_t^\varepsilon)^\dagger\right)^2\right]. \tag{7}$$

Conditions **C4–C5** imply that

$$\begin{aligned}(A_t^\varepsilon)^\dagger &\le L\int_0^t \left(1 + (\xi_s^\varepsilon)^\dagger\right) ds + L\int_0^t \int_{|v|>1} |v| f(v)\left(1 + (\xi_s^\varepsilon)^\dagger\right) dv\, ds \\ &\le L(1 + r_1)\int_0^t \left(1 + (\xi_s^\varepsilon)^\dagger\right) ds,\end{aligned} \tag{8}$$

where $r_1 = \int_{\mathbb{R}\setminus\{0\}} v^2 f(v)\, dv$.

Now, by Doob's inequality (see, e.g., [11, Theorem 1.9.2]),

$$\mathsf{E}\left((M_t^\varepsilon)^\dagger\right)^2 \le 4\left|\mathsf{E}\langle M^\varepsilon\rangle_t\right|,$$

where by condition **C5** we obtain

$$|\langle M^\varepsilon\rangle_t| = \left|\int_0^t\int_{\mathbb{R}\setminus\{0\}} v^2\Gamma\left(\xi^\varepsilon(s), dv; x_s^\varepsilon\right) ds\right| \le Lr_1\int_0^t \left[1 + \left((\xi_s^\varepsilon)^\dagger\right)^2\right] ds. \tag{9}$$

Inequalities (7)–(9) and Cauchy–Bunyakovsky–Schwarz inequality,

$$\left[\int_0^t \varphi(s)\, ds\right]^2 \le t\int_0^t \varphi^2(s)\, ds$$

imply

$$\mathsf{E}\left((\xi_t^\varepsilon)^\dagger\right)^2 \le k_1 + k_2\int_0^t \mathsf{E}\left((\xi_s^\varepsilon)^\dagger\right)^2 ds,$$

where k_1 and k_2 are positive constants independent of ε.

By Gronwall inequality (see, e.g., [4, p. 498]), we obtain

$$\mathsf{E}\left((\xi_t^\varepsilon)^\dagger\right)^2 \le k_1\exp(k_2 t).$$

Hence the lemma is proved. □

Lemma 2. *Under assumption* **C4**, **C5**, **PA1** *there exists a constant $k > 0$, independent of ε such that*

$$\mathsf{E}\,|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le k|t - s|.$$

Proof. In the same manner with (7), we may write

$$|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le 2|A_t^\varepsilon - A_s^\varepsilon|^2 + 2|M_t^\varepsilon - M_s^\varepsilon|^2.$$

By using Doob's inequality, we obtain

$$\mathsf{E}\,|\xi^\varepsilon(t) - \xi^\varepsilon(s)|^2 \le 2\,\mathsf{E}\{|A_t^\varepsilon - A_s^\varepsilon|^2 + 8\,|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s|\}.$$

Now (8), (9), and assumption **C5** imply

$$|A_t^\varepsilon - A_s^\varepsilon|^2 + 8\,|\langle M^\varepsilon\rangle_t - \langle M^\varepsilon\rangle_s| \le k_3\left[1 + \left((\xi_T^\varepsilon)^\dagger\right)^2\right]|t-s|,$$

where k_3 is a positive constant independent of ε.

From the last inequality and Lemma 1 the desired conclusion is obtained. □

Finally, we have to use the Theorem 8.2.1 from [11] that states the relative compactness of semimartingales (see Appendix).

Lemma 3. *Under conditions **C1**–**C5** the family of processes $\xi^\varepsilon(t)$ is relatively compact.*

Proof. To verify the relative compactness we should check the conditions **LP1**–**LP5** of Theorem A (see Appendix).

We easily see that **LP1** follows from the conditions **C4**–**C5**.

Show, that under (5) **LP2** and second part of **LP5** hold. Really, by the condition **C5** and the definition of $\Gamma(u, v; x)$ on the set $\{\sup_{t\le T}|\xi^\varepsilon(t) \le c|\}$ we have for the function $g(v) = \begin{cases} 0, & |v| \le l \\ 1, & |v| > l \end{cases}$:

$$\begin{aligned}\int_0^T \int_{|v|>l} \Gamma^\varepsilon\left(\xi^\varepsilon(s), dv; x_s^\varepsilon\right) ds &= \int_0^T \int_{\mathbb{R}} g(v)\Gamma^\varepsilon\left(\xi^\varepsilon(s), dv; x_s^\varepsilon\right) ds \\ &= \varepsilon \int_0^T \int_{\mathbb{R}} g(v)\Gamma(\xi^\varepsilon(s), dv; x_s^\varepsilon) ds + \varepsilon T\theta_g^\varepsilon \\ &\le \varepsilon TL(1+c)\int_{|v|>l} f(v)dv + \varepsilon T\theta_g^\varepsilon \\ &\le \varepsilon \frac{TL(1+c)}{l}\int_{\mathbb{R}} v^2 f(v)dv + \varepsilon T\theta_g^\varepsilon \to 0, \\ l \to \infty, \qquad \varepsilon \to 0.\end{aligned}$$

Using of condition **PA2** here is stipulated by the fact that $g(v) \in C_2(\mathbb{R}^d) \subset C_3(\mathbb{R}^d)$.

By the same way we get

$$\begin{aligned}\int_0^T \int_{|v|\le\delta} v^2\widehat{\Gamma}\left(\xi^\varepsilon(s), dv\right) ds &= \int_0^T \int_{|v|\le\delta}\int_E v^2\Gamma\left(\xi^\varepsilon(s), dv; x\right)\pi(dx)\,ds \\ &\le TL(1+c)\int_{|v|\le\delta} v^2 f(v)\,dv \to 0, \qquad \delta \to 0,\end{aligned}$$

and by the conditions $\Gamma^\varepsilon(u, \{0\}; x) = 0$, **PA1** and **C5**

$$\begin{aligned}\int_0^T \int_{|v|\le\delta} v^2\Gamma^\varepsilon\left(\xi^\varepsilon(s), dv; x_s^\varepsilon\right) ds &\le \int_0^T \int_{\mathbb{R}} v^2\Gamma^\varepsilon\left(\xi^\varepsilon(s), dv; x_s^\varepsilon\right) ds \\ = \varepsilon\int_0^T c(\xi^\varepsilon(s); x_s^\varepsilon)ds + \varepsilon T\theta_c^\varepsilon(u;x) &\le \varepsilon TL\left(1+c^2\right) + \varepsilon T\theta_c^\varepsilon(u;x) \to 0, \\ \varepsilon \to 0, \qquad \delta \to 0.\end{aligned}$$

It is clear, that **LP3**, **LP4** and the first part of **LP5** follows from the weak convergence of predictable characteristics. Thus, the final step in proof of this Lemma will be made in the next subsection by the verifying of Lemma 4. □

3.2. **Convergence of predictable characteristics.** The next step of proof concerns the convergence of the predictable characteristics. To do that, we apply the results of Sections 3.2–3.3 in [7] and Theorem 6.3 from [7] (see Appendix).

Lemma 4. *Let's point $A^\varepsilon(t)$ any of three predictable characteristics of the process $\xi^\varepsilon(t)$. The following weak convergence takes place*

$$A^\varepsilon(t) \Rightarrow A^0(t),$$

where

$$A^0(t) := \int_0^t \widehat{a}\left(\xi^0(s)\right) ds,$$

here

$$\widehat{a}(u) := \int_E \pi(dx) a(u;x).$$

Proof. We consider the three component Markov process $A^\varepsilon(t)$, $\xi^\varepsilon(t)$, x_t^ε, $t \geq 0$, which can be characterized by the martingale

$$\mu_t^\varepsilon = \varphi(A^\varepsilon(t), \xi^\varepsilon(t), x_t^\varepsilon) - \int_0^t \mathbf{L}^\varepsilon \varphi(A^\varepsilon(s), \xi^\varepsilon(s), x_t^\varepsilon)\, ds.$$

The generator $\mathbf{L}^\varepsilon$ of the martingale has the following representation [7]

$$\mathbf{L}^\varepsilon = \varepsilon^{-1}\mathbf{Q} + \mathbf{\Gamma}^\varepsilon + \mathbf{A}^\varepsilon, \tag{10}$$

with $\mathbf{\Gamma}^\varepsilon$ given by (3), $\mathbf{Q}$ given by (4), and $\mathbf{A}^\varepsilon(u;x)\varphi(v) = \mathbf{A}\varphi(v) + \widetilde{\theta}_a^\varepsilon$, where $\mathbf{A}\varphi(v) := a(u;x)\varphi'(v)$, and $\widetilde{\theta}_a^\varepsilon \to 0$, $\varepsilon \to 0$.

In order to prove the convergence of predictable characteristics, it is sufficient to study the action of the generator $\mathbf{L}^\varepsilon$ on test functions of two variables $\varphi(v,x)$.

Thus, it has the representation

$$\mathbf{L}^\varepsilon \varphi(v,x) = [\varepsilon^{-1}\mathbf{Q} + \mathbf{A}]\varphi(v,x) + \widetilde{\theta}_a^\varepsilon \varphi(v,x). \tag{11}$$

The solution of the singular perturbation problem at the test functions $\varphi^\varepsilon(v,x) = \varphi(v) + \varepsilon\varphi_1(v,x)$ in the form $\mathbf{L}^\varepsilon\varphi^\varepsilon = \widehat{\mathbf{L}}\varphi + \theta^\varepsilon\varphi$ can be found in the following manner. We have:

$$\begin{aligned}\mathbf{L}^\varepsilon\varphi^\varepsilon(v,x) &= [\varepsilon^{-1}\mathbf{Q} + \mathbf{A}][\varphi(v) + \varepsilon\varphi_1(v,x)] \\ &= \varepsilon^{-1}\mathbf{Q}\varphi(v) + [\mathbf{Q}\varphi_1(v,x) + \mathbf{A}\varphi(v)] + \varepsilon\mathbf{A}\varphi_1(u,x) + \widetilde{\theta}_a^\varepsilon\varphi(v,x).\end{aligned}$$

We may write down the following equalities:

$$\begin{gathered}\mathbf{Q}\varphi(v) = 0, \\ \mathbf{Q}\varphi_1(v,x) + \mathbf{A}\varphi(v) = \widehat{\mathbf{L}}\varphi.\end{gathered}$$

From the first equality we see that the function $\varphi(v)$ belongs to the null-space of operator Q and thus does not depend on x. So, using the solvability condition, we have from the second equality

$$\widehat{\mathbf{L}}\varphi(v) = \Pi\mathbf{A}\Pi\varphi(v) + \Pi\mathbf{Q}\Pi\varphi_1(v,x) = \Pi\mathbf{A}\Pi\varphi(v).$$

That is

$$\widehat{\mathbf{L}} = \widehat{\mathbf{A}}, \tag{12}$$

where $\widehat{\mathbf{A}}\varphi(v) = \int_E \pi(dx)a(u;x)\varphi'(v)$.

Now Theorem B may be applied (see Appendix).

We see from (11) and (12) that the solution of singular perturbation problem for $\mathbf{L}^\varepsilon\varphi^\varepsilon(u,v;x)$ satisfies the conditions **CD1, CD2**. Condition **CD3** of this theorem implies that the quadratic characteristics of the martingale, corresponding to a coupled Markov process, is relatively compact. The same result follows from the CCC (see Corollary 1 and Lemma 2) by [6]. Thus, the condition **CD3** follows from the Corollary 1 and Lemma 2.

As soon as $A^\varepsilon(0) = A^0(0)$, $\xi^\varepsilon(0) = \xi^0(0)$ we see that the condition **CD4** is also satisfied. Thus, all the conditions of above Theorem 2 are satisfied, so the weak convergence $A^\varepsilon(t) \Rightarrow A^0(t)$ takes place.

Lemma is proved. □

Thus, by the weak convergence of predictable characteristics, we obtain **LP3**, **LP4** and the first part of **LP5**. As a result, by the Theorem 8.2.1 from [11] the process $\xi^\varepsilon(t)$ is relatively compact and Lemma 3 is proved.

The final step of the proof of Theorem 1 is achieved now by using Theorem IX.3.27 in [6]. Indeed all the conditions of this theorem are fulfilled.

As we have mentioned, the square integrability condition 3.24 follows from CCC (see [6]). The strong dominating hypothesis is true with the majoration functions are presented in the Conditions **C4**–**C5**. Condition **C5** implies the condition of big jumps for the last predictable measure of Theorem IX.3.27 in [6]. Conditions iv and v of Theorem IX.3.27 [6] are obviously fulfilled.

The weak convergence of predictable characteristics is proved by solving the singularly perturbation problem for the generator (10).

The last condition (3.29) of Theorem IX.3.27 is also fulfilled due to CCC proved in Proposition 1 and Lemma 2. Thus, the weak convergence is true.

We can see now that the limit Markov process is characterized by the following predictable characteristics

$$B^0(t) = \int_0^t \widehat{b}\left(\xi^0(s)\right)\, ds, \qquad C^0(t) = \int_0^t \widehat{c}\left(\xi^0(s)\right)\, ds, \qquad \Gamma^0_g(t) = \int_0^t \widehat{\Gamma}_g\left(\xi^0(s)\right)\, ds.$$

Here $C^0(t)$ is the second modified characteristic of the limit process. So, according to [2] the limit Markov process $\xi^0(t)$ can be expressed by the generator (4).

Theorem 1 is proved.

4. Appendix

Theorem A ([11, Theorem 8.2.1]). *Let Q^ε be the distribution of probabilities for $\mathbb{P}^\varepsilon$-semimartingale $\xi^\varepsilon = (\xi^\varepsilon(t), \mathcal{F}^\varepsilon_t)$ with the triplet of predictable characteristics $\mathcal{T}^\varepsilon = (B^\varepsilon, C^\varepsilon, \Gamma^\varepsilon)$ and Q is the distribution of semimartingale $\xi^0 = (\xi^0(t), \mathcal{D}^Q_t)$ with triplet $\mathcal{T}^0 = (B^0, C^0, \Gamma^0)$.*

If for the triplet $\mathcal{T}$ the following condition is true:

LP1:

$$\left|\int_E b(\xi(t), x)\, \pi(dx)\right| \le L(1 + \xi^\dagger(t)),$$

$$\left|\int_E c(\xi(t), x)\, \pi(dx)\right| \le L\left(1 + \left(\xi^\dagger(t)\right)^2\right),$$

and for any nonnegative measurable function $f(v) \le v^2 \wedge 1$

$$\int_E \int_{\mathbb{R}} f(v)\Gamma(\xi(t), dv; x)\, \pi(dx) \le L\left(1 + \xi^\dagger(t)\right).$$

And for the triplets $\mathcal{T}^\varepsilon$ for any fixed $T > 0$:

LP2:

$$\lim_{l\to\infty} \lim_{\varepsilon\to 0} \sup \int_0^T \int_{|v|>l} \Gamma^\varepsilon(\xi^\varepsilon(s), dv; x^\varepsilon_s)\, ds = 0,$$

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \sup \int_0^T \int_{|v|\le\delta} v^2 \widehat{\Gamma}(\xi^\varepsilon(s), dv)\, ds = 0.$$

LP3: *For every bounded measurable function $f(v)$*

$$\lim_{\varepsilon\to 0}\sup_{t\le T}\left|\int_0^t\int_{|v|>\delta} f(v)\left[\Gamma^\varepsilon\left(\xi^\varepsilon(s),dv;x_s^\varepsilon\right)-\widehat{\Gamma}\left(\xi^\varepsilon(s),dv\right)\right]\,ds\right|=0.$$

LP4:

$$\lim_{\varepsilon\to 0}\sup_{t\le T}\left|\int_0^t\left[b^\varepsilon\left(\xi^\varepsilon(s),x_s^\varepsilon\right)-\widehat{b}\left(\xi^\varepsilon(s)\right)\right]\,ds\right|=0.$$

LP5:

$$\lim_{\varepsilon\to 0}\sup_{t\le T}\left|\int_0^t\left[c^\varepsilon\left(\xi^\varepsilon(s),x_s^\varepsilon\right)-\widehat{c}\left(\xi^\varepsilon(s)\right)\right]ds\right|=0,$$

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\sup\int_0^T\int_{|v|\le\delta} v^2\Gamma^\varepsilon\left(\xi^\varepsilon(s),dv;x_s^\varepsilon\right)\,ds=0,$$

then under compact containment condition

$$\lim_{c\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{|\xi^\varepsilon(0)|\ge c\}=0$$

the family Q^ε, $\varepsilon>0$, is relatively compact.

Theorem B ([7, Theorem 6.3])**.** *Put $C_0^2(\mathbb{R}\times E)$ be the space of real-valued twice continuously differentiable by the first argument functions, defined on $\mathbb{R}\times E$ and vanishing at infinity, and $C(\mathbb{R}\times E)$ is the space of real-valued continuous bounded functions defined on $\mathbb{R}\times E$.*

Let the following conditions hold for a family of coupled Markov processes $\xi^\varepsilon(t)$, $x^\varepsilon(t)$, $t\ge 0$, $\varepsilon>0$:

CD1: *There exists a family of test functions $\varphi^\varepsilon(u,x)$ in $C_0^2(\mathbb{R}\times E)$, such that*

$$\lim_{\varepsilon\to 0}\varphi^\varepsilon(u,x)=\varphi(u),$$

uniformly on u, x.

CD2: *The following convergence holds*

$$\lim_{\varepsilon\to 0}\mathbf{L}^\varepsilon\varphi^\varepsilon(u,x)=\mathbf{L}\varphi(u),$$

uniformly on u,x. The family of functions $\mathbf{L}^\varepsilon\varphi^\varepsilon$, $\varepsilon>0$, is uniformly bounded, and $\mathbf{L}\varphi(u)$ and $\mathbf{L}^\varepsilon\varphi^\varepsilon$ belong to $C(\mathbb{R}\times E)$.

CD3: *The quadratic characteristics of the martingales that characterize a coupled Markov process $\xi^\varepsilon(t)$, $x^\varepsilon(t)$, $t\ge 0$, $\varepsilon>0$, have the representation*

$$\langle\mu^\varepsilon\rangle_t=\int_0^t\zeta^\varepsilon(s)\,ds,$$

where the random functions ζ^ε, $\varepsilon>0$, satisfy the condition

$$\sup_{0\le s\le T}\mathsf{E}\,|\zeta^\varepsilon(s)|\le c<+\infty.$$

CD4: *The convergence of the initial values holds and*

$$\sup_{\varepsilon>0}\mathsf{E}\,|\zeta^\varepsilon(0)|\le C<+\infty.$$

Then the weak convergence

$$\xi^\varepsilon(t)\Rightarrow\xi(t),\qquad\varepsilon\to 0,$$

takes place.

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