ON COUPLING MOMENT INTEGRABILITY FOR TIME-INHOMOGENEOUS MARKOV CHAINS

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Abstract. In this paper, we find the conditions under which the expectation of the first coupling moment for two independent, discrete, time-inhomogeneous Markov chains will be finite. We consider discrete chains with a phase space $\{0, 1, \dots\}$ and as the coupling moment we understand the first moment of visiting zero state by the both chains at the same time.

Анотація. В даній роботі знаходяться умови за яких гарантовано існування математичного сподівання моменту склеювання для двох незалежних, дискретних, неоднорідних за часом Марківських ланцюгів. Розглядаються дискретні ланцюги з фазовим простором $\{0, 1, \dots\}$ та під моментом склеювання розуміється перший момент одночасного потрапляння в нуль обох ланцюгів.

Аннотация. В даной работа рассматриваются условия при которых гарантировано существует конечное математическое ожидание момента склеивания для двух независимых, дискретных, неоднородных по времени цепей Маркова. Рассматриваются дискретные цепи с пространством состояний $\{0, 1, \dots\}$ и под моментом склеивания мы понимаем первый момент одновременного попадания в нулевое состояние обеих цепей.

1. Introduction

The problem of finiteness for the moment of simultaneous hitting for two chains into certain set (or simultaneous renewal of two renewal processes) play a crucial role in evaluation of the stability estimates using coupling method. Similar estimates one can find in the authors' works [4, 5]. The problem of stability for a time-inhomogeneous Markov chain is investigated there using a coupling method as a key method of the research. Similar problems, but for the homogeneous Markov chains, are also considered in the work [7].

The key question for the stability estimate evaluation in these papers is how we can estimate the expectation for the moment of simultaneous hitting for two Markov chains. The coupling setup can be found in the following work [5].

The problem of integrability and finiteness for the coupling moment can be reduced to the problem of integrability and finiteness for the moment of simultaneous hitting into certain set or to the problem of finiteness for the moment of simultaneous renewal. Similar task is considered in the Lindvall's book [14]. It worth to mention, that this monograph is a classical book on the coupling method. There introduced different types of coupling: week coupling, maximal coupling, Ornshtein coupling, Mineka coupling and so on. Another famous book on the coupling method is a Torrison's work [15].

The coupling method is also used in many other works. The first works on coupling method are [1, 12, 13]. An example of how the coupling method is used to establish stability estimates for time-homogeneous chain with different initial distributions is proposed in [2].

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However, the problem of coupling only the same homogeneous Markov chain and related problems were considered in books mentioned above. In particular, the theorem about integrability of the coupling moment in the book by Lindvall [14] had been proved for two copies of the same time-homogeneous Markov chain whith different initial distributions. In the investigation of stability there arises the necessity to extend coupling moment for different, not necessarily homogeneous Markov chains. So, well-known classical Lindvall's and Thorrison's results do not work in this case. Meantime, it is important to note that the main theorem of this article uses the same proof schema as Lindvall's theorem 4.2 [14, p. 27].

The paper [9] is devoted to the investigation of such problem as integrability of the coupling moment for two different Markov chains. In this work the estimates for the expectation of a coupling moment for two different time-homogeneous Markov chains starting with a random delays are presented. The conditions under which these estimates were obtained are the strong aperiodicity ($g_1^1+g_1^2 > 0$) and the finiteness of second renewal moments.

The maximal coupling for two time-inhomogeneous chains is considered in other author's papers [10, 11].

In the current paper these results extended to the time-inhomogeneous case. It is important that in this case the fundamental principle of independence of the renewal times does not hold true anymore. Instead, the conditional independence should be considered given the fixed moments of the previous renewal process.

The main theorem of this paper gives the general conditions which guarantee the integrability of the coupling moment. They are the condition of the separation from a zero for renewal probabilities (in the time-homogeneous case this condition automatically holds true for the non-periodic renewal distribution with a finite mean) and the uniform integrability of the renewal distributions. It is interesting that similarity of the condition can be noted for homogeneous and inhomogeneous case. In particular, for the time-homogeneous case, an estimate similar to the one from the work [9] is derived in a principal different way.

2. Dependence of renewal moments for time-inhomogeneous Markov chain

The fundamental fact defining the proof schema in the time-inhomogeneous case is that elements of a renewal sequence are not independent and the distribution of the $k+1$-st renewal moment is completely defined by the k-th renewal value.

Let's examine an example that leads to the renewal sequence generated by the time-inhomogeneous Markov chain.

Consider some time-inhomogeneous discrete Markov chain $(X_t, t \geq 0)$ with a phase space $\{0, 1, 2, \ldots\}$. Its transition probabilities are defined in the following way:

$$\mathsf{P}\{X_{t+1} = j \mid X_t = i\} = P_t(i,j) = p_{ij}^{(t)}, \qquad t \geq 0. \tag{1}$$

In the zero moment of time the chain is in the zero state. Let's introduce the following notation:

$$\begin{aligned} \theta_1 &= \inf\{t > 0 \colon X_t = 0\} \\ \theta_2 &= \inf\{t > \theta_1 \colon X_t = 0\} \\ &\ldots \\ \theta_m &= \inf\{t > \theta_{m-1} \colon X_t = 0\}, \qquad m > 1, \end{aligned} \tag{2}$$

where θ_1 is time of the first returning to zero, θ_2 is time between first and second zero hitting, and so on. In this case $\tau_k = \sum_{j=1}^{k} \theta_k$ is the k-th hitting moment.

The sequence $\{\theta_m, m \geq 1\}$ is a renewal sequence generated by the time-inhomogeneous Markov chain X_t. In general case, for the chain starting from a non-zero state we may consider an initial delay θ_0. It is time that a chain takes till hitting zero for the first time.

Let's now investigate a problem of dependence for the θ_m variables. In the homogeneous case, these variables are independent. But if the chain is time-inhomogeneous there is dependence between θ_m's. Let's see an example below.

The random variable θ_1 has a following distribution:

$$\begin{aligned} pr\{\theta_1 = k\} &= \mathsf{P}\{X_k = 0, X_{k-1} \neq 0, \ldots, X_1 \neq 0, X_0 = 0\} \\ &= \sum_{i_0=0, i_1\neq 0, i_2\neq 0, \ldots, i_{k-1}\neq 0, i_k=0} \prod_{j=0}^{k-1} p^{(j)}_{i_j i_{j+1}}. \end{aligned} \tag{3}$$

So, we can see that a distribution potentially depends from all X_t, $t \le k$.

The distribution of the random variable θ_2 is as follows

$$\begin{aligned} \mathsf{P}\{\theta_2 = k\} &= \sum_{j=1}^{k-1} \mathsf{P}\{\theta_2 = k, \theta_1 = j\} \\ &= \sum_j \mathsf{P}\{X_k = 0, X_{k-1} \neq 0, X_{j+1} \neq 0, X_j = 0, \\ &\qquad\qquad X_{j-1} \neq 0, \ldots, X_1 \neq 0, X_0 = 0\}. \end{aligned} \tag{4}$$

Note, that for each term in the last sum, the following equality holds true:

$$\begin{aligned} &\sum \mathsf{P}\{X_k = 0, X_{k-1} \neq 0, X_{j+1} \neq 0 \mid X_j = 0\}\, \mathsf{P}\{\theta_1 = j\} \\ &\quad = \sum \mathsf{P}\{X_k = 0, X_{k-1} \neq 0, X_{j+1} \neq 0 \mid X_j = 0\}\, \mathsf{P}\{\tau_1 = j\}. \end{aligned}$$

So, the distribution of the random variable θ_2 depends on the variable τ_1 and all X_t, $t > \tau_1$. We'll show that this situation holds true for the other θ_m as well.

Let us now consider

$$\mathsf{P}\{\theta_m = k\} = \sum \mathsf{P}\{X_k = 0, X_{k-1} \neq 0, \ldots, X_{j+1} \neq 0 \mid X_j = 0\}\, \mathsf{P}\{\tau_{m-1} = j\}. \tag{5}$$

So the distribution of the θ_m depends on probabilities $p^{(t)}_{ij}$ where $t \ge \tau_{m-1}$. In other words, in order to write down a distribution for the θ_m, one should know the value of the variable τ_{m-1} but now necessarily the values of variables $\theta_1, \ldots, \theta_{m-1}$. Moreover, under fixed τ_{m-1} the distribution of θ_m does not depend on the values $\theta_1, \ldots, \theta_{m-1}$.

Now we have:

$$\begin{aligned} &\mathsf{P}\{\theta_m = i, \theta_{m-1} = j \mid \tau_{m-1} = t\} \\ &\quad = \mathsf{P}\{\theta_m = i, \theta_{m-1} = j \mid X_t = 0, X_l = 0, \text{exactly } m-2 \text{ times}, \ l < m-1\} \\ &\quad = \mathsf{P}\{X_k = 0, k \in \{i, t, t-j\}, X_k \neq 0 \text{ otherwise}, A\}\, \mathsf{P}^{-1}(A) \\ &\quad = \mathsf{P}\{X_i = 0, X_l \neq 0, \\ &\qquad\qquad l = t+1, \ldots, i-1 \mid X_t = 0, X_{t-1} \neq 0, \ldots, X_{t-j} = 0, X_{t-j-1} \neq 0, A\} \\ &\qquad \times \mathsf{P}\{X_t = 0, X_{t-1} \neq 0, \ldots, X_{t-j} = 0, X_{t-j-1} \neq 0 \mid A\} \\ &\quad = \mathsf{P}\{X_i = 0, X_l \neq 0, l = t+1, \ldots, i-1 \mid X_t = 0\}\, \mathsf{P}\{\theta_{m-1} = j \mid \tau_{m-1} = t\} \\ &\quad = \mathsf{P}\{X_i = 0, X_l \neq 0, l = i-1, \ldots, t+1 \mid X_t = 0, B\}\, \mathsf{P}\{\theta_{m-1} = j \mid \tau_{m-1} = t\} \\ &\quad = \mathsf{P}\{\theta_m = i \mid \tau_{m-1} = t\}\, \mathsf{P}\{\theta_{m-1} = j \mid \tau_{m-1} = t\}, \end{aligned}$$

where the set $A = \{X_t = 0, X_l = 0, \text{ exactly } m-2 \text{ times}, l < m-1\} = \{\tau_{m-1} = t\}$, $B = \{\text{exactly } m-1 \text{ zero hittings happened till time } t-1\}$.

So we have proved that

$$\mathsf{P}\{\theta_m = i, \theta_{m-1} = j \mid \tau_{m-1} = t\} = \mathsf{P}\{\theta_m = i \mid \tau_{m-1} = t\}\, \mathsf{P}\{\theta_{m-1} = j \mid \tau_{m-1} = t\}, \tag{6}$$

which means that variables θ_m and θ_{m-1} are conditionally independent given τ_{m-1}.

Let us also note, that formula (5) implies that the distribution of the θ_m is parameterized by only one parameter j (values of a τ_{m-1}), and does not depend on index m. So we can write:

$$g_n^j = \mathsf{P}\{\theta_m = n \mid \tau_{m-1} = j\}.$$

This fact leads us to consideration of the random variables $\theta(t)$ which have the same distribution as $(g_n^t)_{n\geq 0}$. This variables can be handled as moments of the first after time t returning to zero, if we know that a chain is in the zero state at the moment t.

3. Key definitions

In this section and further on we'll consider two time-inhomogeneous Markov chains $(X_t^1, t \geq 0)$ and $(X_t^2, t \geq 0)$ defined on a phase space $E = \{0, 1, \ldots\}$. The chains are defined by their transition probabilities on the s-th step $P_s(x, A, 1)$, $P_s(x, A, 2)$ for chains X_t^1, X_t^2 respectively. Let's define transition probabilities for $n > 0$ steps:

$$P^{t,n}(x, A, l) = \left(\prod_{k=0}^{n-1} P_{t+k}\right)(x, A, l). \tag{7}$$

Having this set of transition probabilities and the initial conditions $\mu^l(\cdot)$ we can build a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where both chains (X_t^l), $l \in \{1, 2\}$, are defined and

$$\mathbb{P}\{X_s^l \in A\} = \int_E \mu^l(dx) P^{0,s}(x, A, l), \qquad \mathbb{P}\{X_{s+1}^l \in A \mid X_s^l = x\} = P_s(x, A, l).$$

Let's define renewal intervals θ_k^l, $l \in \{1, 2\}$:

$$\theta_0^l = \inf\{t \geq 0 \colon X_t = 0\}, \qquad \theta_m^l = \inf\{t > \theta_{m-1} \colon X_t = 0\}, \quad m > 1, \tag{8}$$

which are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The classes of variables $\{\theta_k^1\}_{k\geq 0}$ and $\{\theta_k^2\}_{k\geq 0}$ are independent. θ_k^l for each $l \in \{1, 2\}$ and $k > 0$ have only positive integer values while θ_0^l take non-negative integers. Let's define renewal sequences in the following way:

$$\tau_n^l = \sum_{k=0}^{n} \theta_k^l, \qquad l \in \{1, 2\}. \tag{9}$$

We will assume that neighboring variables inside each class are conditionally independent giving τ. In other words, for each k, t, l the following equality holds true:

$$\mathsf{E}\left[f\left(\theta_k^l\right) g\left(\theta_{k+1}^l\right) \mid \tau_k^l\right] = \mathsf{E}\left[f\left(\theta_k^l\right) \mid \tau_k^l\right] \mathsf{E}\left[g\left(\theta_{k+1}^l\right) \mid \tau_k^l\right], \tag{10}$$

for any bounded Borel functions f and g.

Let's introduce a definition for the conditional distribution of the θ_k^l variable (please, note that this distribution does not depend on k):

$$g_n^{t,l} = \mathsf{P}\left\{\theta_k^l = n \;\middle|\; \tau_{k-1} = t\right\}, \qquad l =\in \{1, 2\},\ n \geq 0, \tag{11}$$

and we assume that $g_0^{t,l} = \mathsf{P}\{\theta_k^l = 0 \mid \tau_{k-1} = t\} = 0$. The variables θ_k^l, $k \geq 1$ will be interpreted as renewal steps and θ_0^l as a delay.

We'll say that $T > 0$ is a coupling (or simultaneously hitting) moment if:

$$T = \min\left\{t > 0 \colon \exists m, n \colon t = \tau_m^1 = \tau_n^2\right\}. \tag{12}$$

Our goal is to find conditions which guarantee $T < \infty$ a.s. and $\mathsf{E}[T] < \infty$.

By $u_n^{(t,l)}$ we define a renewal sequence for the process τ^l. In other words, $u_n^{(t,l)}$ is a probability of a renewal at the moment $t+n$ having renewal at the moment t. Formally $u_n^{(t,l)}$ can defined in a following way:

$$u_0^{(t,l)} = 1, \qquad u_n^{(t,l)} = \sum_{k=0}^{n} u_k^{(t,l)} g_{n-k}^{t+k,l}. \tag{13}$$

4. Formal definition of the $\theta^l(t)$ variable

As we've seen before, the distribution of the $k+1$-st renewal interval is completely defined by the value of the τ_k variable, i.e. by the moment of the previous renewal and does not depend on the index k. That's why we have introduced the notations $g_n^{t,l}$ and $u_n^{(t,l)}$. Our goal is to define random variables $\theta^l(t)$ in such a way that $g_n^{t,l}$ be a distribution for $\theta^l(t)$.

For simplicity we'll omit index l in this section.

Assume X_t is some time-inhomogeneous Markov chain with transition probabilities on the t-th step equal to $P_t(x, A)$. As before, let's define:

$$P^{t,n}(x, A) = \left(\prod_{k=0}^{n-1} P_{t+k} \right)(x, A),$$

transition probability for the time from t to $t+n$.

For each t we define probability space $(\Omega_t, \mathcal{F}_t, \mathbb{P}_t)$ as a canonical space for the Markov chain X_{t+n} which starts at zero. Let's note that

$$\theta(t) = \min\{j > 0 \colon X_{t+j} = 0\}, \tag{14}$$

and $g_n^t = \mathbb{P}_t\{\theta(t) = n\}$ is the distribution of the variable $\theta(t)$. Then,

$$g_n^t = \int_{(E\setminus\{0\})^{n-1}} P_t(0, dx_0) P_{t+1}(x_0, dx_1) \dots P_{t+n-1}(x_{n-1}, \{0\}). \tag{15}$$

As in the previous section let's define $\theta^l(t)$ as a moment of the first hitting zero state for the chain $(X_{t+k}^l, k \ge 0)$ which starts from zero. Then a variable $\theta^l(t)$ has the distribution $(g_n^{t,l})_{n\ge 0}$.

Let's define an overshoot:

$$D_n(t) = \min\{j \ge 0 \colon X_{t+n+j} = 0\}. \tag{16}$$

The variable $D_n(t)$ should be understood as a time that has left till hitting $\{0\}$ after moment $t+n$ having $X_t = 0$. Note that variables $D_n(t)$ and $\theta(t)$ are defined on the common probability space $(\Omega_t, \mathcal{F}_t, \mathbb{P}_t)$.

The following lemma is a key in proving the main theorem (the proof will be given later):

Lemma 4.1. *If a distribution family g_n^t (or, a family of random variables $\theta(t)$) is uniformly integrable then for each $\rho \in (0,1)$ there exists a constant $C = C(\rho) \ge 0$, such that for each t the following inequality holds true:*

$$\mathbb{E}_t[D_n(t)] \le \rho n + C.$$

5. Main theorem

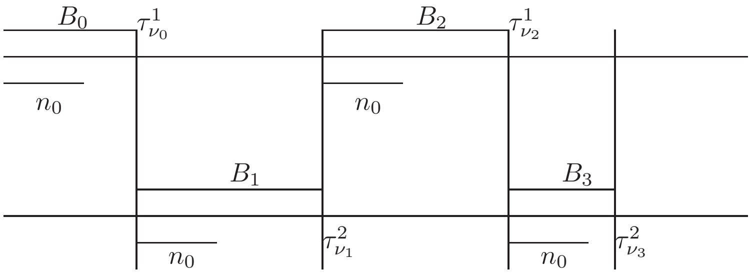
Theorem 5.1. *Assume that (in notations introduced before):*

1) *The set of random variables $\theta^l(t)$ is uniformly integrable (or, in other words, the family of distributions $g_n^{t,l}$ is uniformly integrable).*
2) *There exists a constat $\gamma > 0$ and a positive integer $n_0 > 0$ such that for all t, l and $n \ge n_0$: $u_n^{(t,l)} \ge \gamma$.*

Then the coupling moment is integrable: $\mathsf{E}[T] < \infty$.

6. Setup for the proof of the theorem 5.1

Following the Lindvall approach (see. [14, p. 27]) let's define the following random variables:



$$\nu_0 := \min\left\{j \geq 1 \colon \tau^1_j > n_0\right\},$$
$$B_0 := \tau^1_{\nu_0},$$
$$\nu_1 := \min\left\{j \geq \nu_0 \colon \tau^2_j - \tau^1_{\nu_0} > n_0, \text{ or } \tau^2_j - \tau^1_{\nu_0} = 0\right\},$$
$$B_1 := \tau^2_{\nu_1} - \tau^1_{\nu_0},$$

and further on

$$\nu_{2m} := \min\left\{j \geq \nu_{2m-1} \colon \tau^1_j - \tau^2_{\nu_{2m-1}} > n_0, \text{ or } \tau^1_j - \tau^2_{\nu_{2m-1}} = 0\right\},$$
$$B_{2m} := \tau^1_{\nu_{2m}} - \tau^2_{\nu_{2m-1}},$$
$$\nu_{2m+1} := \min\left\{j \geq \nu_{2m} \colon \tau^2_j - \tau^1_{\nu_{2m}} > n_0\,, \text{ or } \tau^2_j - \tau^1_{\nu_{2m}=0}\right\},$$
$$B_{2m+1} := \tau^2_{\nu_{2m+1}} - \tau^1_{\nu_{2m}}.$$

ν_k is called as coupling trials. Let's define $\tau = \min\{n \geq 1 \colon B_n = 0\}$ and a sequence of sigma-fields $\mathfrak{B}_n$, $n \geq 0$ in the following way:

$$\mathfrak{B}_n = \sigma\left[B_k, \nu_k, \tau^l_j, k \leq n, j \leq \nu_n\right].$$

Let's also define random variables: $D^{k,l}_n = \min\{j \colon \exists m, \tau^l_m = \tau^l_k + n + j\}$.

7. The proof of the theorem 5.1

At the beginning we assume that $\theta^2_0 = 0$.

The following inequality is true:

$$T \leq \theta^1_0 + \sum_{n=0}^{\tau} B_n = \theta^1_0 + \sum_{n \geq 0} B_n \nVdash_{\tau \geq n}. \tag{17}$$

According to the lemma 8.4 for each $n \geq 0$, $\rho \in (0,1)$ the following inequality holds true:

$$\mathsf{E}[B_n \mid \mathfrak{B}_{n-1}] \leq \rho B_{n-1} + C, \tag{18}$$

which implies that

$$\begin{aligned}\mathsf{E}[B_k \nVdash_{\tau \geq k} \mid \mathfrak{B}_{k-1}] &= \mathsf{E}\left[B_k \prod_{n=0}^{k-1} \nVdash_{B_k \neq 0} \,\middle|\, \mathfrak{B}_{k-1}\right] = \nVdash_{\tau \geq n} \mathsf{E}[B_n \mid \mathfrak{B}_{n-1}] \\ &\leq \nVdash_{\tau \geq n}(\rho B_{n-1} + C) = \rho B_{n-1} \nVdash_{\tau \geq n} + C \nVdash_{\tau \geq n} \\ &\leq \rho B_{n-1} \nVdash_{\tau \geq n-1} + C \nVdash_{\tau \geq n},\end{aligned}$$

where the latest equality follows from the relation $\{\tau \geq n\} \subset \{\tau \geq n-1\}$ and so $\nVdash_{\tau \geq n} \leq \nVdash_{\tau \geq n-1}$.

So we've proved the following inequality:

$$\mathsf{E}[B_n \mathbb{1}_{\tau \geq n}] \leq \rho\, \mathsf{E}\left[B_{n-1} \mathbb{1}_{\tau \geq n-1}\right] + C\, \mathsf{P}\{\tau \geq n\}. \tag{19}$$

It follows from lemma 8.5 that

$$\mathsf{P}\{\tau \geq n\} \leq (1-\gamma)^n.$$

Let's define $a_n = \mathsf{E}[B_n \mathbb{1}_{\tau \geq n}]$. Then (19) implies:

$$a_n \leq \rho a_{n-1} + C(1-\gamma)^n \leq C \sum_{k=0}^{n} \rho^k (1-\gamma)^{n-k} \leq Cn \max(\rho, (1-\gamma))^n.$$

Note, since ρ is arbitrary, we can choose it be equal to $(1-\gamma)$. In this case

$$a_n \leq Cn(1-\gamma)^n.$$

So

$$\mathsf{E}[T] \leq \mathsf{E}\left[\theta_0^1\right] + \sum_{n\geq 0} a_n \leq \mathsf{E}\left[\theta_0^1\right] + \frac{C}{\gamma^2} < \infty. \tag{20}$$

Recall our assumption $\theta_0^2 = 0$. Now we will get rid of it. Let's define as T' a coupling moment for the processes with the following delays:

$$\theta_0'^1 = \max\left(\theta_0^1, \theta_0^2\right) - \min\left(\theta_0^1, \theta_0^2\right),$$
$$\theta_0'^2 = 0.$$

Note that $T = T' + \min(\theta_0^1, \theta_0^2)$. So

$$\mathsf{E}[T] \leq \mathsf{E}\left[\min\left(\theta_0^1, \theta_0^2\right)\right] + \mathsf{E}[T'] < \infty.$$

Note that

$$\mathsf{E}[T'] \leq \mathsf{E}\left[\theta_0'^1\right] + \frac{C}{\gamma^2},$$

or

$$\mathsf{E}[T] \leq \mathsf{E}\left[\max\left(\theta_0^1, \theta_0^2\right)\right] + \frac{C}{\gamma^2}.$$

8. Auxiliary lemmas

Lemma 8.1. *Let $x_n^{(t)}$, $y_n^{(t)}$ be some inhomogeneous sequences of real numbers, $u_n^{(t)}$ be some inhomogeneous renewal sequence defined by the formula* (13)*: $g_0^{(t)} = 0$, for all t. Assume the following conditions are true*

$$x_n^{(t)} = \sum_{k=0}^{n} g_k^{(t)} x_{n-k}^{(t+k)} + y_n^{(t)}, \tag{21}$$

$$x_n^0 \geq \sum_{k=0}^{n} g_k^{(t)} x_{n-k}^0 + y_n^{(t)}. \tag{22}$$

Then for any t, n:

$$x_n^{(t)} \leq x_n^0.$$

Proof. Let's show that

$$x_n^{(t)} = \sum_{k=0}^{n} u_k^{(t)} y_{n-k}^{(t+k)}. \tag{23}$$

We'll do this by induction:

For the $n = 0$: $x_0^{(t)} = g_0^{(t)} x_0^{(t)} + y_0^{(t)} = y_0^{(t)} = u_0^{(t)} y_0^{(t)}$.

Assuming the statement holds true for all $k \le n$, lets prove it for the $n+1$.

$$x_{n+1}^{(t)} = \sum_{k=0}^{n+1} g_k^{(t)} x_{n+1-k}^{(t+k)} + y_{n+1}^{(t)} = \sum_{k=1}^{n+1} g_k^{(t)} \sum_{j=0}^{n+1-k} u_j^{(t+k)} y_{n+1-k-j}^{(t+k+j)} + g_0^{(t)} x_{n+1}^{(t)} + y_{n+1}^{(t)}$$
$$= \sum_{k=1}^{n+1} u_k^{(t)} y_{n-k}^{(t+k)} + y_n^{(t)} = \sum_{k=0}^{n+1} u_k^{(t)} y_{n-k}^{(t+k)}.$$

Then for any t, n:

$$y_n^{(t)} \le x_n^{(0)} - \sum_{k=0}^{n} g_k^{(t)} x_{n-k}^{0},$$
$$x_n^{(t)} \le \sum_{k=0}^{n} u_k^{(t)} x_{n-k}^{(0)} - \sum_{k=0}^{n} u_k^{(t)} \sum_{j=0}^{n-k} g_j^{(t+k)} x_{n-k-j}^{(0)}. \tag{24}$$

Let us consider the second term

$$\sum_{k=0}^{n} u_k^{(t)} \sum_{j=0}^{n-k} g_j^{(t+k)} x_{n-k-j}^{(0)} = x_0^0 \sum_{k=0}^{n} u_k^{(t)} g_{n-k}^{(t+k)} + x_1^0 \sum_{k=0}^{n-1} u_k^{(t)} g_{n-1-k}^{(t+k)} + \ldots + x_n^0 u_0^{(t)} g_0^{(t)}$$
$$= \sum_{k=0}^{n-1} x_k^0 u_{n-k}^{(t)} = \sum_{k=1}^{n} u_k^{(t)} x_{n-k}^0.$$

Applying the last relation to the (24) we derive:

$$x_n^{(t)} \le \sum_{k=0}^{n} u_k^{(t)} x_{n-k}^{(0)} - \sum_{k=1}^{n} u_k^{(t)} x_{n-k}^0 = u_0^{(t)} x_n^0 = x_n^0. \qquad \square$$

Lemma 8.2. *Assume A is a some set defined by the variables $\tau_{\nu_k}^l$, ν_k, $k < n$. Then:*

$$\mathsf{E}\left[D_{k+n_0}^{m,l} \mid B_{n+1} = k, \tau_{\nu_n}^l = t, \nu_n = m, A\right] = \mathbb{E}_t\left[D_{k+n_0}^l(t)\right].$$

Proof. Let's denote $t + k + n_0 = q$. Then:

$$\mathsf{P}\left\{D_{k+n_0}^{m,l} = r, B_{n+1} = k, \tau_{\nu_n}^l = t, \nu_n = m, A\right\}$$
$$= \mathbb{P}\left\{X_{q+r}^l = 0, X_{q+s}^l \ne 0, s = 0, \ldots, r-1, X_t^l = 0, \tau_{\nu_n}^l = t, \nu_n = m, B_{n+1} = k, A\right\}$$
$$= \left(\int_{(E\setminus 0)^r} P^{t,k+n_0}(0, dx_0, l) P_q(x_0, dx_1, l) \ldots P_{q+r-1}(x_{r-1}, dx_r, l) P_{q+r}(x_r, 0, l)\right)$$
$$\times \mathbb{P}\left\{X_t = 0, \tau_{\nu_n}^l = t, \nu_n = m, B_{n+1} = k, A\right\}$$
$$= \mathbb{P}_t\left\{D_{k+n_0}^l(t) = r\right\} \mathbb{P}\left\{\tau_{\nu_n}^l = t, \nu_n = m, B_{n+1} = k, A\right\}. \qquad \square$$

Lemma 8.3.

$$\mathsf{E}[B_{2n} \mid \mathfrak{B}_{2n-1}] = \sum_{t,k} \mathbb{E}_t\left[D_{k+n_0}^1(t)\right] \mathbb{1}_{\tau_{\nu_{2n-2}}^1 = t} \mathbb{1}_{B_{2n-1} = k},$$
$$\mathsf{E}[B_{2n+1} \mid \mathfrak{B}_{2n}] = \sum_{t,k} \mathbb{E}_t\left[D_{k+n_0}^2(t)\right] \mathbb{1}_{\tau_{\nu_{2n-1}}^2 = t} \mathbb{1}_{B_{2n} = k}.$$

Proof. At the beginning we should note that the sigma-field $\mathfrak{B}_m$ is generated by the finite amount of random variables, and each of them takes only no more than countable number of values. So, for each m, $\mathfrak{B}_m$ is generated by the finite number of events.

Let us define a set of events $\{A_n(i), i \in \mathfrak{I}_n\}$ as $A_n(i) = \{\tau^l_{\nu_k} = t_{lk}, \nu_k = n_k, k \le n\}$ and note that $\mathfrak{I}_n$ is a countable set. Let's add the following notation

$$C_n(s,t,m,k) = \left\{\tau^2_k = t, \tau^1_m = s, \nu_{2n-1} = k, \nu_{2n-2} = m, A_{2n-3}(i)\right\}.$$

Note that it follows from the definition of B_{2n} that

$$B_{2n} = D^{\nu_{2n-2},1}_{B_{2n-1}+n_0} + n_0, \tag{25}$$

which implies

$$\begin{aligned}&\mathsf{E}[B_{2n} - n_0 \mid \mathfrak{B}_{2n-1}]\\ &\quad= \sum_{s<t,m,k,i\in\mathfrak{I}_{2n-3}} \mathsf{E}\left[D^{m,1}_{t-s+n_0} \,\middle|\, C_n(s,t,m,k), A_{2n-3}(i)\right] \mathbb{1}_{C_n(s,t,m,k)} \mathbb{1}_{A_{2n-3}(i)}.\end{aligned}$$

Using lemma 8.2 we derive that the last term is equal

$$\begin{aligned}&\sum_{s<t,m,k,i\in\mathfrak{I}_{2n-3}} \mathbb{E}_s\left[D^1_{t-s+n_0}(s)\right] \mathbb{1}_{C_n(s,t,m,k)} \mathbb{1}_{A_{2n-3}(i)}\\ &\quad= \sum_{s<t,m,k} \mathbb{E}_s\left[D^1_{t-s+n_0}(s)\right] \mathbb{1}_{C_n(s,t,m,k)} = \sum_{s<t} \mathbb{E}_s\left[D^1_{t-s+n_0}(s)\right] \mathbb{1}_{\tau^2_{\nu_{2n-1}}=t} \mathbb{1}_{\tau^1_{\nu_{2n-2}}=s}\\ &\quad= \sum_{s,k} \mathbb{E}_s\left[D^1_{k+n_0}\right] \mathbb{1}_{\tau^1_{\nu_{2n-2}}=s} \mathbb{1}_{B_{2n-1}=k},\end{aligned}$$

where we used the following equality $B_{2n-1} = \tau^2_{\nu_{2n-1}} - \tau^1_{\nu_{2n-2}}$ in the last relation.

The corresponding statement for $\mathsf{E}[B_{2n+1} \mid \mathfrak{B}_{2n}]$ can be derived in a similar way. □

Lemma 8.4. *Assuming the conditions of the theorem 5.1 holds true for each $\rho \in (0,1)$ there exists a constant $C \in (0,\infty)$, that for every $n \ge 0$ a following inequality is true*

$$\mathsf{E}[B_n \mid \mathfrak{B}_{n-1}] \le \rho B_{n-1} + C.$$

Proof. Using lemmas 8.3 and 4.1 we will get

$$\begin{aligned}\mathsf{E}[B_{2n} \mid \mathfrak{B}_{2n-1}] &= \sum_{t,k} \mathbb{E}_t\left[D^1_{k+n_0}(t)\right] \mathbb{1}_{\tau^1_{\nu_{2n-2}}=t} \mathbb{1}_{B_{2n-1}=k}\\ &\le \sum_{t,k} (\rho(k+n_0)+C) \mathbb{1}_{\tau^1_{\nu_{2n-2}}=t} \mathbb{1}_{B_{2n-1}=k} = \rho B_{2n-1} + C'.\end{aligned}$$

The same statement holds true for the $\mathsf{E}[B_{2n+1} \mid \mathfrak{B}_{2n}]$. □

Lemma 8.5. *The following inequality is true*

$$\mathsf{P}\{\tau > n\} \le (1-\gamma)^n.$$

Proof. Recall that $\tau = \min(n\colon B_n = 0)$. An event $\{\tau > n\} = \{\prod_{k=0}^n B_k \ne 0\}$.

$$\begin{aligned}\mathsf{E}\left[\mathbb{1}_{\prod_{k=0}^n B_k \ne 0}\right] &= \mathsf{E}\left[\mathbb{1}_{\prod_{k=0}^{n-1} B_k \ne 0} \mathsf{E}\left[\mathbb{1}_{B_n \ne 0} \mid \mathfrak{B}_{n-1}\right]\right]\\ &= \mathsf{E}\left[\mathbb{1}_{\prod_{k=0}^{n-1} B_k \ne 0}\right] \mathsf{P}\left\{\theta^l_\eta > B_n + B_{n-1}\right\}\\ &\le \mathsf{E}\left[\mathbb{1}_{\prod_{k=0}^{n-1} B_k \ne 0}\right] \mathsf{P}\left\{\theta^l_\eta > n_0\right\} \le \mathsf{E}\left[\mathbb{1}_{\prod_{k=0}^{n-1} B_k \ne 0}\right](1-\gamma) \le (1-\gamma)^n,\end{aligned}$$

where η is a number of the next after B_{n-1} renewal in the l-th series. □

9. The proof of the lemma 4.1

Let's consider the random variable $D_n(t)\nVdash_{\theta(t)=j}$, $j \le n$. By the direct calculation it is easy to verify that

$$\mathbb{P}_t\{D_n(t) = k, \theta(t) = j\} = \mathbb{P}_t\{\theta(t) = j\}\mathbb{P}_{t+j}\{D_{n-j}(t+j) = k\}. \tag{26}$$

The following inequality holds true:

$$D_n(t)\nVdash_{\theta(t)>n} = (\theta(t) - n)\nVdash_{\theta(t)>n}. \tag{27}$$

Then, having in mind inequalities (26) and (27) we'll get

$$\begin{aligned}
\mathbb{E}_t[D_n(t)] &= \sum_{j=1}^{n} \mathbb{E}_t\left[D_n(t)\nVdash_{\theta(t)=j}\right] + \mathbb{E}_t\left[D_n(t)\nVdash_{\theta(t)>n}\right] \\
&= \sum_{j=1}^{n}\left(\sum_{k=0}^{\infty} k\mathbb{P}_t\{D_n(t) = k, \theta(t) = j\}\right) + \mathbb{E}_t\left[(\theta(t) - n)\nVdash_{\theta(t)>n}\right] \\
&= \sum_{j=1}^{n} \mathbb{P}_t\{\theta(t) = j\}\left(\sum_{k=0}^{\infty} k\mathbb{P}_{t+j}\{D_{n-j}(t+j) = k\}\right) + \mathbb{E}_t[(\theta(t) - n)\nVdash_{\theta(t)>n}] \\
&= \sum_{j=1}^{n} g_j^t \mathbb{E}_{t+j}[D_{n-j}(t+j)] + \mathbb{E}_t\left[(\theta(t) - n)\nVdash_{\theta(t)>n}\right].
\end{aligned}$$

So we have the following equality

$$\mathbb{E}_t[D_n(t)] = \sum_{j=1}^{n} g_j^t \mathbb{E}_{t+j}[D_{n-j}(t+j)] + \mathbb{E}_t\left[(\theta(t) - n)\nVdash_{\theta(t)>n}\right]. \tag{28}$$

After that we'll use the lemma 8.1. Let's define:

$$x_n^{(t)} = \mathbb{E}_t[D_n(t)],$$

$$y_n^{(t)} = \mathbb{E}_t\left[\nVdash_{\theta(t)>n}(\theta(t) - n)\right],$$

then (28) implies the condition (21).

We define as

$$x_n^0 = \rho n + C.$$

Let's proof that the condition (22) of the lemma 8.1 holds true. For doing that we should show, that for any $\rho \in (0, 1)$ there exists such $C = C(\rho)$, that

$$\rho n + C \ge \sum_{j=0}^{n} g_j^t(\rho(n - j) + C) + \sum_{j>n}(j - n)g_j^t, \tag{29}$$

We'll derive the following from the statement (29)

$$\begin{aligned}
(29) \Leftrightarrow \rho n + C &\ge n\rho \sum_{j=0}^{n} g_j^t + C\sum_{j=0}^{n} g_j^t - \rho\sum_{j=0}^{n} jg_j^t + \sum_{j>n} jg_j^t - nG_n^t \\
&\Leftrightarrow n\rho G_n^t + CG_n^t \ge \mathbb{E}_t\left[\theta(t)\nVdash_{\theta(t)>n}\right] - \rho\mathbb{E}_t\left[\theta(t)\nVdash_{\theta(t)\le n}\right] - nG_n^t \\
&\Leftrightarrow n(\rho + 1)G_n^t + CG_n^t + \rho\mathbb{E}_t\left[\theta(t)\nVdash_{\theta(t)\le n}\right] \ge \mathbb{E}_t\left[\theta(t)\nVdash_{\theta(t)>n}\right] \\
&\Leftrightarrow n(\rho + 1)G_n^t + CG_n^t + \rho\mathbb{E}_t\left[\theta(t)\right] \ge (1 + \rho)\mathbb{E}_t\left[\theta(t)\nVdash_{\theta(t)>n}\right],
\end{aligned} \tag{30}$$

So, the inequalities (29) are equivalent to (30). Note that, in the case of $G_n^t = 0$ the equality (30) holds true automatically. Assume than $G_n^t > 0$. But $\mathbb{E}_t[\theta(t)] \ge 1$ and the uniform integrability implies that there is a number n_0, such that for all $t > 0, n \ge n_0$: $\mathbb{E}_t[\theta(t)\nVdash_{\theta(t)>n}] \le \rho/(1 + \rho)$. The constant C we'll choose in the way to satisfy (30) for $n \le n_0$.

Let's show now, that C could be chosen disregarding of t. For $\varepsilon = \rho/(1+\rho)$ we'll find such $\delta > 0$, that for each set A, such that $\mathbb{P}(A) < \delta$ it follows that $\mathbb{E}_t[\theta(t)\mathbb{1}_A] < \varepsilon$. It is possible, since $\theta(t)$ are uniformly integrable. Let's define then

$$C := \frac{(1+\rho)sup_t\mathbb{E}_t[\theta(t)] - \rho}{\delta}.$$

Now having $G_n^t < \delta$ inequality (30) holds true automatically. In the case of $G_n^t \geq \delta$, we'll get:

$$\begin{aligned} n(\rho+1)G_n^t + CG_n^t + \rho\mathbb{E}_t[\theta(t)] > (1+\rho)\sup_t \mathbb{E}_t[\theta(t)] &\geq (1+\rho)\mathbb{E}_t[\theta(t)] \\ &\geq (1+\rho)\mathbb{E}_t\left[\theta(t)\mathbb{1}_{\theta(t)>n}\right]. \end{aligned}$$

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