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ON A CONJECTURE OF ERDÖS ABOUT ADDITIVE FUNCTIONS

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Abstract. For a real-valued additive function $f\colon \mathbb{N} \to \mathbb{R}$ and for each $n \in \mathbb{N}$ we define a distribution function

$$F_n(x) := \frac{1}{n}\#\{m \le n\colon f(m) \le x\}.$$

In this paper we prove a conjecture of Erdös, which asserts that in order for the sequence F_n to be (weakly) convergent, it is sufficient that there exist two numbers $a < b$ such that $\lim_{n\to\infty}(F_n(b)-F_n(a))$ exists and is positive.

The proof is based upon the use of the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to mimic the behaviour of an additive function as a sum of independent random variables.

Анотація. Для дійсної адитивної функції $f\colon \mathbb{N} \to \mathbb{R}$ при всіх $n \in \mathbb{N}$ ми визначаємо функцію розподілу

$$F_n(x) := \frac{1}{n}\#\{m \le n\colon f(m) \le x\}.$$

У статті ми доводимо гіпотезу Ердеша, яка стверджує, що для (слабкої) збіжності послідовності F_n достатньою умовою є існування двох чисел $a < b$ таких, що границя $\lim_{n\to\infty}(F_n(b) - F_n(a))$ існує і додатна.

Доведення базується на використанні компактифікації Стоуна–Чеха (Stone–Čech) $\beta\mathbb{N}$ для $\mathbb{N}$, що дає змогу дослідити поведінку адитивної функції, трактуючи її як суму незалежних випадкових величин.

Аннотация. Для вещественной адитивной функции $f\colon \mathbb{N} \to \mathbb{R}$ при всех $n \in \mathbb{N}$ мы определяем функцию распределения

$$F_n(x) := \frac{1}{n}\#\{m \le n\colon f(m) \le x\}.$$

В статье мы доказываем гипотезу Эрдеша, в которой утверждается, что для (слабой) сходимости последовательности F_n достаточным условием является существование двух чисел $a < b$ таких, что предел $\lim_{n\to\infty}(F_n(b) - F_n(a))$ существует и положителен.

Доказательство основано на использовании компактификации Стоуна–Чеха (Stone–Čech) $\beta\mathbb{N}$ для $\mathbb{N}$, что дает возможность исследовать поведение адитивной функции, трактуя ее как сумму независимых случайных величин.

1. Introduction

A function $f\colon \mathbb{N} \to \mathbb{R}$ is called *additive* if $f(mn) = f(m) + f(n)$ for any coprime integers m and n. Then f is defined by its values $f(p^k)$ on prime powers p^k (p prime, $k \in \mathbb{N}$) and $f(1) = 0$.

Given a real-valued additive function f, one can define, for each $n \in \mathbb{N}$, a distribution function

$$F_n(x) := \frac{1}{n}\#\{m \le n\colon f(m) \le x\}. \tag{1.1}$$

An old conjecture of Erdös in 1947 (see Erdös [4]) asserts that in order for the sequence F_n to be (weakly) convergent (in this case we say that the additive function f possesses a *limit distribution*), it is sufficient that there exist two numbers $a < b$ such

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that

$$\lim_{n\to\infty} (F_n(b) - F_n(a)) \quad \text{exists and is positive.} \tag{1.2}$$

In 1992 A. Hildebrand [6] could show that the conclusion of Erdös' conjecture is valid, provided (1.2) is strengthened to

$$L_a := \lim_{n\to\infty} F_n(a) \quad \text{and} \quad L_b := \lim_{n\to\infty} F_n(b) \tag{1.3}$$

both exists and $L_a \neq L_b$. Some further discussions are contained in [10] and [11].

In this paper we show that the above conjecture of Erdös holds.

Theorem. *Let $f : \mathbb{N} \to \mathbb{R}$ be an additive function. In order for the distributions* (1.1) *to converge, it is sufficient that* (1.2) *holds for some $a < b$.*

The proof is based upon a method, introduced in [7, 8] using the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$ to mimic the behaviour of an additive function as a sum of independent random variables.

2. Finitely distributed additive functions

An additive function f is said to be *finitely distributed* if there are positive constants c_1 and c_2, and an unbounded sequence $n_1 < n_2 < \dots$ so that for every i there exists a sequence

$$a_1^{(i)} < a_2^{(i)} < \cdots < a_{t_i}^{(i)} < n_i$$

satisfying

$$\left| f\left(a_r^{(i)}\right) - f\left(a_s^{(i)}\right) \right| < c_1, \qquad t_i > c_2 n_i, \ 1 \le r, s \le t_i.$$

The necessary and sufficient condition that f should be finitely distributed is that there should exist a constant c and an additive function h so that

$$f(n) = c \log n + h(n) \tag{2.1}$$

where both the series

$$\sum_{|h(p)|>1} \frac{1}{p}, \qquad \sum_{\substack{p \\ |h(p)|\le 1}} \frac{h^2(p)}{p} \tag{2.2}$$

converge (Erdös [3], 1946). Further characterizations of finitely distributed additive functions can be found in Ch. 7 of Elliott's book [2]. For our purpose we shall apply the following ([2, p. 259]).

Proposition. *If the additive function has a representation* (2.1) *with convergent series* (2.2)*, then, if we define*

$$\alpha(n) = c \log n + \sum_{\substack{p \le n \\ |h(p)|\le 1}} \frac{h(p)}{p}, \tag{2.3}$$

the distribution functions

$$G_n(x) := \frac{1}{n} \#\{m \le n \colon f(m) - \alpha(n) \le x\} \tag{2.4}$$

weakly converge to some distribution function $G(x)$.

If (1.2) holds then f is finitely distributed. Now, assume that (2.1) holds and $\alpha(n)$ is unbounded. Then, if $\alpha(n_k') \to \infty$, $k \to \infty$, for some subsequence (n_k'), by (1.2),

$$\lim_{k\to\infty} \left\{ G_{n_k'}(b - \alpha(n_k')) - G_{n_k'}(a - \alpha(n_k')) \right\} = \lim_{k\to\infty} \left(F_{n_k'}(b) - F_{n_k'}(a) \right) > 0 \tag{2.5}$$

whereas the left side in (2.5) tends to zero since G_n converge weakly to some distribution function. Then, since $\alpha(n) = c\log n + O(\log\log n)$, we conclude $c = 0$, i.e. $f = h$, and

$$A(n) := \sum_{\substack{p\le n\\ |f(p)|\le 1}} \frac{f(p)}{p} = O(1) \quad \text{for all } n\in\mathbb{N}. \tag{2.6}$$

In the following we assume that

$$\sum_{\substack{p\\ |f(p)|\le 1}} \frac{f(p)}{p} \quad \text{diverges,} \tag{2.7}$$

which implies (see [3], Theorem II) that $G(x)$ is continuous and strictly increasing for all $x\in\mathbb{R}$.

For each $n\in\mathbb{N}$ define the additive function f_n by

$$f_n(p^k) = \begin{cases} f\left(p^k\right) & \text{if } p\le n,\\ 0, & \text{otherwise.}\end{cases}$$

and put, for $A\subset\mathbb{N}$,

$$\delta_n(A) := \frac{1}{n}\#\{m\le n\colon m\in A\}.$$

If the limit

$$\delta(A) := \lim_{n\to\infty}\delta_n(A) \tag{2.8}$$

exists we say that A *possesses the asymptotic density* $\delta(A)$.

If some sequence $\{n'_k\}$ is given we write

$$\delta'(A) := \lim_{k\to\infty}\delta_{n'_k}(A) \tag{2.9}$$

in the case the limit (2.9) exists.
With these notations we show

Lemma 1. *Assume that* (1.2) *holds. Then*

$$\lim_{n\to\infty}\delta(\{m\colon f_n(m)\in(a,b]\}) = \delta(\{m\colon f(m)\in(a,b]\}) =: c_0 > 0. \tag{2.10}$$

Proof. Observe that $\delta(\{m\colon f_n(m)\in(a,b]\})$ always exists. Assume that (2.10) does not hold. Then there exists a sequence $\{n_k\}$ of natural numbers such that

$$\lim_{k\to\infty}\delta(\{m:\ f_{n_k}(m)\in(a,b]\}) = c'\ne c_0.$$

Since $A(n_k) = O(1)$ there exists some subsequence $\{n'_k\}$ of $\{n_k\}$ so that

$$\lim_{k\to\infty}A(n'_k) =: A'$$

exists. Choose k_1 such that for every $k_0\ge k_1$

$$\begin{aligned} &\left|\delta\left(\left\{m\colon f_{n'_{k_0}}(m)\in(a,b]\right\}\right) - c_0\right|\\ &\quad = \left|\delta'\left(\left\{m\colon f_{n'_{k_0}}(m)\in(a,b]\}\right) - \delta'(\{m\colon f(m)\in(a,b]\right\}\right)\right|\\ &\quad \ge \frac{|c_0-c'|}{2}. \end{aligned} \tag{2.11}$$

On the other hand we shall show that

$$\lim_{k_0\to\infty}\overline{\lim_{k\to\infty}}\,\delta_{n'_k}\left(\left\{m\colon \left|f(m)-f_{n'_{k_0}}(m)\right|>\varepsilon\right\}\right) = 0 \tag{2.12}$$

for every $\varepsilon>0$ which contradicts (2.11).

For the proof of (2.12) put

$$\mathcal{P}_0 := \{p\colon |f(p)| > 1\} \cup \left\{p^k\colon k \geq 2\right\}.$$

Define the functions

$$h'(m) := \sum_{\substack{p^k \| m \\ p^k \in \mathcal{P}_0 \\ p > n'_{k_0}}} f\left(p^k\right)$$

and

$$j(m) := \sum_{\substack{p \| m \\ |f(p)| \leq 1 \\ p > n'_{k_0}}} f(p).$$

From our definitions of these functions

$$f(m) - f_{n'_{k_0}}(m) = j(m) - \left(A(n'_k) - A\left(n'_{k_0}\right)\right) + h'(m) + \left(A(n'_k) - A\left(n'_{k_0}\right)\right).$$

We shall prove that for every $\varepsilon > 0$ each of the three expressions

$$L_1(k_0) = \overline{\lim_{k\to\infty}}\, \delta_{n'_k}\left(\left\{m\colon \left|j(m) - \left(A(n'_k) - A\left(n'_{k_0}\right)\right)\right| > \varepsilon\right\}\right),$$

$$L_2(k_0) = \overline{\lim_{k\to\infty}}\, \delta_{n'_k}(\{m\colon |h'(m)| > \varepsilon\})$$

and

$$L_3(k_0) = \overline{\lim_{k\to\infty}}\, \delta_{n'_k}\left(\left\{m\colon \left|A\left(n'_k\right) - A\left(n'_{k_0}\right)\right| > \varepsilon\right\}\right)$$

converge to zero as $k_0 \to \infty$. We may readily estimate the first of these three expressions by appealing to the Turan–Kubilius inequality. In our present circumstances it becomes

$$\frac{1}{n'_k} \sum_{m=1}^{n'_k} \left|j(m) - \left(A\left(n'_k\right) - A\left(n'_{k_0}\right)\right)\right|^2 \ll \sum_{\substack{n'_{k_0} < p \leq n'_k \\ |f(p)| \leq 1}} \frac{|f(p)|^2}{p}.$$

Appealing to the convergence of the second sum in (2.2) we see that

$$L_1(k_0) \ll \frac{1}{\varepsilon^2} \sum_{\substack{n'_{k_0} < p \\ |f(p)| \leq 1}} \frac{|f(p)|^2}{p} = o(1) \quad \text{as } k_0 \to \infty.$$

The estimate $L_3(k_0) = o(1)$ as $k_0 \to \infty$ is obvious.

If an integer m is counted in the expression $L_2(k_0)$ it must satisfy one of two divisibility criteria.

First, it may be divisible by the square of a prime $p > n'_{k_0}$. The frequency of these integers is at most

$$\delta_{n'_k}\left(\left\{m\colon p^2 | m, p > n'_{k_0}\right\}\right) \leq \sum_{n'_{k_0} < p} \frac{1}{p^2} = o(1) \quad \text{as } k_0 \to \infty.$$

Next, it may be exactly divisible by a prime in the range $n'_{k_0} < p$ for which $|f(p)| > 1$. From the hypothesis (2.2) we deduce that the frequencies of such integers is at most

$$\sum_{\substack{n'_{k_0} < p \\ |f(p)| > 1}} \frac{1}{p} = o(1) \quad \text{as } k_0 \to \infty$$

and thus $L_2(k_0) = o(1)$ as $k_0 \to \infty$. We have now shown that (2.12) holds and completed the proof of Lemma 1. □

In the next step we identify the additive function f with a sum $\sum_{p \text{ prime}} X_p$ of independent random variables.

3. Additive functions as a sum of independent random variables

For the sake of simplicity we restrict ourselves to strongly additive functions. Then f can be written in the form

$$f = \sum_p f(p)\varepsilon_p$$

where

$$\varepsilon_p(n) = \begin{cases} 1 & \text{if } p|n, \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathcal{A}$ denotes the algebra generated by the sets

$$A_p := \{n \in \mathbb{N} \colon p|n\}, \quad p \text{ prime},$$

then obviously each $A \in \mathcal{A}$ possesses an asymptotic density $\delta(A)$ and $\delta(A_p) = \frac{1}{p}$ (p prime). Thus δ defines a content on $\mathcal{A}$. Now the construction runs as follows. (For details see [7, 8].) We embed $\mathbb{N}$, endowed with the discrete topology, in the Stone–Čech compactification $\beta\mathbb{N}$,

$$\mathbb{N} \hookrightarrow \beta\mathbb{N}$$

and, if for any $A \subset \mathbb{N}$, the closure of A in $\beta\mathbb{N}$ is denoted by $\bar{A}$, then

$$\bar{\mathcal{A}} := \{\bar{A} \subset \beta\mathbb{N} \colon A \in \mathcal{A}\}$$

is an algebra, too. The extension $\bar{\delta}$ of δ

$$\bar{\delta}(\bar{A}) := \delta(A), \quad \bar{A} \in \bar{\mathcal{A}},$$

defines a premeasure on $\bar{\mathcal{A}}$ and leads to a measure P, induced by

$$\delta^*(A) := \overline{\lim_{n\to\infty}} \delta_n(A) \quad \text{for all } A \subset \mathbb{N},$$

and to a probability space $(\Omega, \sigma(\bar{\mathcal{A}}), \mathsf{P})$ with $\Omega = \beta\mathbb{N}$ and with $\mathsf{P}(\bar{A}_p) = 1/p$, p prime.

There is a unique extension of ε_p to a function $\bar{\varepsilon}_p$ on Ω, and putting $X_p = f(p)\bar{\varepsilon}_p$

$$f = \sum_p f(p)\varepsilon_p \to X = \sum_p f(p)\bar{\varepsilon}_p = \sum_p X_p$$

$$f_n \to S_n := \sum_{p \le n} X_p$$

with

$$\mathsf{P}(X_p = f(p)) = \frac{1}{p}$$

and

$$\mathsf{P}(X_p = 0) = 1 - \frac{1}{p}.$$

The $\bar{\varepsilon_p}$ are independent, i.e. $X = \sum_p X_p$ is a sum of independent random variables. If (1.2) holds then, by Lemma 1,

$$\lim_{n\to\infty} \mathsf{P}(S_n \in (a,b]) = c_0 > 0$$

and, by Proposition, $\sum_p X_p$ is essentially convergent (for the definition see [13, p. 262]).

Putting

$$a_p = \mathsf{E}(X_p^c), \qquad Y_p = X_p - a_p, \qquad T_n := \sum_{p \le n} Y_p$$

then $\lim_{n\to\infty} T_n$ holds a.s.. (Here X_p^c denotes the truncation of X_p at (a positive) c, i.e. we replace X_p by $X_p = X$ or 0 according as $|X_p| < c$ or $|X_p| \geq c$.) Denote $Y := \lim T_n$ a.s.

It is well-known that the a.s. convergence of $Y = \sum_p Y_p$ is equivalent to the weak convergence of the distributions of the partial sums of that series. Moreover, by Kolmogorov's three series theorem, $Y = \sum_p Y_p$ converges a.s. if and only if the series

$$\sum_p \mathsf{E}\left(Y_p^c\right), \qquad \sum_p \mathsf{P}(|Y_p| > c), \qquad \sum_p \mathrm{Var}\left(Y_p^c\right) \tag{3.1}$$

converge.

We choose $c = 1$, i.e. $a_p = \mathsf{E}(X_p^1)$ and put (see (2.6))

$$A(n) = \sum_{p\leq n} a_p.$$

Then $A(n) = O(1)$ and, the divergence of the sequence $A(n)$ implies (see [3, Theorem 2]).

Lemma 2. *Let $Y = \sum_p Y_p$ with $Y_p = X_p - a_p$ as above, where the partial sums $\sum_{p\leq N} a_p$ are bounded and divergent. Then the distribution function $G(x) = \mathsf{P}(Y \leq x)$ is continuous and strictly monotone for all $x \in \mathbb{R}$.*

Remark. The divergence of the sequence $A(n)$ implies

$$\begin{aligned} \sum_p a_p^- &= -\infty, \\ \sum_p a_p^+ &= +\infty \end{aligned} \tag{3.2}$$

where $a_p^+ = \max(a_p, 0)$ and $a_p^- = \max(-a_p, 0)$. Then the strict monotonicity of the distribution function $G(x)$ in Lemma 2 can be directly proved by a result of A. Hildebrand [6].

For this we define, following the notation of Hildebrand in [6], p. 1206, the range of a random variable X as the set

$$R(X) = \{x \in \mathbb{R} \colon \mathsf{P}(|X - x| \leq \varepsilon) > 0 \text{ for every } \varepsilon > 0\},$$

that is, it is equal to the set of points of increase of the distribution function $F(x) = \mathsf{P}(X \leq x)$. The form of this set was described by A. Hildebrand in Lemma 2 of [6] when X is given as an a.s. convergent series of independent random variables. A special version of this result is contained in the following lemma.

Lemma 3. *Let $\sum_{n=0}^{\infty} X_n$ be an a.s. convergent series of independent random variables and let X denote its sum. Suppose that for every $\varepsilon > 0$ and $n \geq n_0 = n_0(\varepsilon)$ there exist numbers $c_n^- = c_n^-(\varepsilon)$, $c_n^+ = c_n^+(\varepsilon) \in R(X_n)$ with $|c_n^-| \leq \varepsilon$ and $|c_n^+| \leq \varepsilon$ such that*

$$\lim_{N\to\infty} \sum_{n=n_0}^{N} c_n^- = -\infty$$

and

$$\lim_{N\to\infty} \sum_{n=n_0}^{N} c_n^+ = +\infty.$$

Then $R(X) = \mathbb{R}$.

Now it is easy to prove the assertions of Lemma 2. Put

$$c_p^- = \begin{cases} f(p) - a_p & \text{if } -\frac{\varepsilon}{2} \leq f(p) < 0, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$c_p^+ = \begin{cases} f(p) - a_p & \text{if } 0 < f(p) \le \frac{\varepsilon}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Then obviously, c_p^-, $c_p^+ \in R(Y_p)$, $|c_p^-| \le \frac{\varepsilon}{2} + |a_p| \le \varepsilon$ and $|c_p^+| \le \frac{\varepsilon}{2} + |a_p| \le \varepsilon$ for $p > n_0 = n_0(\varepsilon)$ since $|a_p| \le 1/p$. Further,

$$\begin{aligned} \sum_{n_0 \le p \le N} c_p^- &= \sum_{\substack{n_0 \le p \le N \\ -\frac{\varepsilon}{2} < f(p) < 0}} f(p) - \sum_{n_0 \le p \le N} a_p \\ &< \sum_{\substack{n_0 \le p \le N \\ -\frac{\varepsilon}{2} < f(p) < 0}} \frac{f(p)}{p} + O(1) \\ &< \sum_{\substack{n_0 \le p \le N \\ -1 < f(p) < 0}} \frac{f(p)}{p} + O(1) \\ &= \sum_{n_0 \le p \le N} a_p^- + O(1) \to -\infty \quad \text{as } N \to \infty. \end{aligned}$$

Here the last inequality holds because of the convergence of the second series in (3.1). Similarly,

$$\lim_{N \to \infty} \sum_{n_0 \le p \le N} c_p^+ = +\infty.$$

We use Lemma 3 and recall that the divergence of the series (2.7) implies, by Levy's theorem, the continuity of $G(x)$ to end the proof of Lemma 2.

This ends the remark.

For every subsequence $n' = (n'_k)$ of the natural numbers we defined

$$\delta'(A) = \lim_{k \to \infty} \delta_{n'_k}(A)$$

if the limit exists. This leads to a content δ' on $\mathcal{A}$ and a measure P' on $\beta\mathbb{N}$ induced by

$$\delta'^*(A) = \overline{\lim_{k \to \infty}}\, \delta_{n'_k}(A) \quad \text{for all } A \subset \mathbb{N}.$$

Obviously, if $\Omega_0 \subset \beta\mathbb{N}$ is P-measurable it is P'-measurable and $\mathsf{P}(\Omega_0) = P'(\Omega_0)$.

Since every bounded real-valued function g on $\mathbb{N}$ extends uniquely to a (continuous) function $\bar{g}$ on $\beta\mathbb{N}$ (for details see R. Walker [14, p. 8 et seq.]), we conclude

$$\Omega_0 := \overline{\{m \colon f(m) \in (a,b]\}} = \{\omega \colon \bar{f}(\omega) \in [a,b]\},$$

where $\bar{f}$ is the unique extension of the (bounded) function $f_{(a,b]}$, defined by

$$f_{(a,b]}(m) = \begin{cases} f(m), & \text{if } f(m) \in (a,b], \\ |a| + |b| + 1, & \text{if } f(m) \notin (a,b]. \end{cases}$$

If (1.2) holds then

$$\mathsf{P}(\Omega_0) = c_0 > 0.$$

4. Proof of the conjecture of Erdös

We suppose that $A(n)$ is not convergent so that

$$\underline{A} := \liminf_{n \to \infty} A(n) < \limsup_{n \to \infty} A(n) =: \bar{A}, \tag{4.1}$$

and we shall show that this leads to a contradiction.

We fix two increasing sequences $n' = \{n'_k\}$ and $n'' = \{n''_k\}$ of positive integers so that

$$\underline{A} = \lim_{k\to\infty} A(n''_k) \quad \text{and} \quad \bar{A} = \lim_{k\to\infty} A(n'_k).$$

We put

$$g_n = \sum_{p\le n} (f(p)\varepsilon_p - a_p)$$

and define

$$g' = g_{n'_1} + \sum_{k=1}^{\infty} \left(g_{n'_{k+1}} - g_{n'_k} \right). \tag{4.2}$$

Then

$$\left\{m\colon g'(m) \in (a - \overline{A}, b - \overline{A}]\right\} = \{m\colon f(m) \in (a,b]\}$$

since $g'(m) = f(m) - \overline{A}$ for every $m \in \mathbb{N}$. Further

$$\delta'\left(\left\{m\colon g'(m) \in (a - \overline{A}, b - \overline{A}]\right\}\right) = \lim_{k\to\infty} \delta' \left(\left\{ m\colon g_{n'_k}(m) \in (a - \overline{A}, b - \overline{A}] \right\} \right) = c_0.$$

In the same way we define

$$g'' = g_{n''_1} + \sum_{k=1}^{\infty} \left(g_{n''_{k+1}} - g_{n''_k} \right)$$

with $g''(m) = f(m) - \underline{A}$, $m \in \mathbb{N}$, and obtain

$$\delta''\left(\{m\colon g''(m) \in (a - \underline{A}, b - \underline{A}]\}\right) = \lim_{k\to\infty} \delta'' \left(\left\{ m\colon g_{n''_k}(m) \in (a - \underline{A}, b - \underline{A}] \right\} \right) = c_0.$$

Defining the corresponding extensions $\overline{g'}$ and $\overline{g''}$ and P' and P'', respectively, we arrive at

$$\Omega_0 = \left\{\omega\colon \overline{g'}(\omega) \in [a - \overline{A}, b - \overline{A}]\right\} = \left\{\omega\colon \overline{g''}(\omega) \in [a - \underline{A}, b - \underline{A}]\right\}$$

together with

$$P'\left(\left\{\omega\colon \overline{g'}(\omega) \in [a - \overline{A}, b - \overline{A}]\right\}\right) = P''\left(\left\{\omega\colon \overline{g''}(\omega) \in [a - \underline{A}, b - \underline{A}]\right\}\right) = c_0.$$

Since

$$g' \quad \text{corresponds to} \quad Y' = \lim_{k\to\infty} T_{n'_k}$$

$$g'' \quad \text{corresponds to} \quad Y'' = \lim_{k\to\infty} T_{n''_k}$$

and since

$$Y = \sum_p Y_p = \lim_{n\to\infty} T_n$$

converges a.s. with respect to P and possesses an everywhere continuous distribution function we conclude

(i) $\{\omega\colon Y'(\omega) \in [a - \overline{A}, b - \overline{A}]\} = \Omega'_0$ with $P'(\Omega_0 \bigtriangleup \Omega'_0) = 0$,
(ii) $P'(\{\omega\colon Y'(\omega) \in [a - \overline{A}, a - \underline{A}]\}) \le P'(\{\omega\colon Y'(\omega) \ne Y''(\omega)\}) = 0$ and
(iii) $P'(\{\omega\colon Y'(\omega) \in [a - \underline{A}, b - \overline{A}]\}) = c_0$.

Observe, that (iii) implies that

$$a - \underline{A} < b - \overline{A}.$$

Since $\mathsf{P}(\{\omega\colon Y(\omega) \in [a - \overline{A}, a - \underline{A}]\})$ exists it must be zero by (ii), i.e.

$$\mathsf{P}\left(\left\{\omega\colon Y(\omega) \in [a - \overline{A}, a - \underline{A}]\right\}\right) = 0. \tag{4.3}$$

In the same way we show

$$\mathsf{P}\left(\left\{\omega : Y(\omega) \in [b - \overline{A}, b - \underline{A}]\right\}\right) = 0. \tag{4.4}$$

(4.3) and (4.4) contradict the monotonicity of $G(x)$, and thus the assertion of Theorem 1 holds.

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