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CONSISTENCY AND ASYMPTOTIC NORMALITY OF PERIODOGRAM ESTIMATOR OF HARMONIC OSCILLATION PARAMETERS

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Abstract. The problem of detection of hidden periodicities is considered in the paper. In the capacity of useful signal model the harmonic oscillation observed on the background of random noise being a local functional of Gaussian strongly dependent stationary process is taken. For estimation of unknown angular frequency and amplitude of harmonic oscillation periodogram estimator is chosen, for which sufficient conditions of asymptotic normality are obtained and limit normal distribution is found.

Анотація. У статті розглядається задача виявлення прихованої періодичності. В якості моделі корисного сигналу взято гармонічне коливання, що спостерігається на фоні випадкового шуму, який є локальним функціоналом від гаусівського сильно залежного стаціонарного процесу. Для оцінки невідомих кутової частоти та амплітуди гармонічного коливання була обрана періодограмна оцінка, для якої отримано достатні умови асимптотичної нормальності та знайдено граничний нормальний розподіл.

Аннотация. В статье рассматривается задача выявления скрытой периодичности. В качестве модели полезного сигнала взято гармоническое колебание, которое наблюдается на фоне случайного шума, являющимся локальным функционалом от гауссовского сильно зависимого стационарного процесса. Для оценки неизвестных угловой частоты и амплитуды гармонического колебания была выбрана периодограмная оценка, для которой получены достаточные условия асимптотической нормальности и найдено предельное нормальное распределение.

1. Introduction

Detection of hidden periodicities is a problem that has a long history started by Lagrange in XVIII century [1].

In statistical setting the detection of hidden periodicities is the estimation of unknown amplitudes and angular frequencies, generally speaking, of the sum of harmonic oscillations by observation of this sum on the background of a random noise masking these oscillations.

There are many publications on the subject. Among them first of all we have to mention the works by Whittle [2], Walker [3], Hannan [4], Dorogovtsev, Grechka [5], Ivanov [6], Knopov [7], Quinn and Hannan [8], etc. A good survey of the topic one can find in [9].

In the paper the problem of detecting hidden periodicities is considered in the case when we observe the only harmonic oscillation on the background of random noise being a local functional of Gaussian stationary process with strong dependence. For estimation of unknown parameters the periodogram estimator is chosen.

In the proofs we use approach of the paper [4] where the case of weakly dependent Gaussian stationary noise has been considered.

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2. The main result

Suppose the observed random process is of the form

$$X(t) = A_0 \cos\varphi_0 t + \varepsilon(t), \tag{1}$$

where $A_0 > 0$, $\varphi_0 \in (\underline{\varphi}, \overline{\varphi})$, $0 < \underline{\varphi} < \overline{\varphi} < \infty$, and the random noise $\varepsilon(t)$ satisfies the following conditions:

A1. $\varepsilon(t)$, $t \in \mathbb{R}^1$, is a local functional of a Gaussian stationary process $\xi(t)$, that is $\varepsilon(t) = G(\xi(t))$, $G(x)$, $x \in \mathbb{R}^1$, is a Borel function such that $\mathsf{E}\,\varepsilon(0) = 0$, $\mathsf{E}\,\varepsilon^2(0) < \infty$.

A2. $\xi(t)$, $t \in \mathbb{R}^1$, is a real mean square continuous measurable Gaussian stationary process defined on the probability space (Ω, F, P), $\mathsf{E}\,\xi(0) = 0$.

Assume also that one of the next conditions is fulfilled:

A3. Covariance function (c.f.) of the process $\xi(t)$ is $\mathsf{E}\,\xi(t)\xi(0) = B(t) = L(|t|)|t|^{-\alpha}$, $\alpha \in (0,1)$, where $L(t)$, $t \geq 0$, is a nondecreasing slowly varying at infinity function, $\mathsf{E}\,\xi^2(0) = B(0) = 1$.

A4. C.f. of the process $\xi(t)$ is $B(t) = \cos\psi t (1+t^2)^{-\alpha/2}$, $\alpha \in (0,1)$, $\psi > 0$ is some number, $\varphi_0 \neq \psi$.

Suppose that for a function $G(x) \in L_2\left(\mathbb{R}^1, \varphi(x)\,dx\right)$,

$$\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2},$$

$C_1(G) \neq 0$ or $C_1(G) = \cdots = C_{m-1}(G) = 0$, $C_m(G) \neq 0$, where

$$C_k(G) = \int_{-\infty}^{+\infty} G(t) H_k(t) \varphi(t)\,dx, \qquad k \geq 0,$$

and $H_k(t)$ are Hermite polynomials. Then the number $m \geq 1$ is said to be Hermite rank of G.

We also assume that function $G(\cdot)$ from condition A1 satisfies assumption

B1. $m\alpha > 1$, where α is a parameter of c.f. B.

We need in a result proved in [10].

Lemma 1. *If conditions **A1**, **A2**, and **A3** or **A4** are satisfied, then*

$$\mathsf{E}\left(\sup_{\lambda \in \mathbb{R}^1} \frac{1}{T} \left| \int_0^T e^{-i\lambda t} \varepsilon(t)\,dt \right| \right)^2 \to 0, \qquad T \to \infty.$$

Consider the functional

$$Q_T(\varphi) = \left| \frac{2}{T} \int_0^T X(t) e^{i\varphi t}\,dt \right|^2. \tag{2}$$

The periodogram estimator of the frequency φ_0 is said to be any random variable (r.v.) $\varphi_T \in [\underline{\varphi}, \overline{\varphi}]$ such that $Q_T(\varphi_T) = \max_{\varphi \in [\underline{\varphi}, \overline{\varphi}]} Q_T(\varphi)$.

Theorem 1. *If conditions of Lemma 1 are satisfied, then $\varphi_T \xrightarrow{\mathsf{P}} \varphi_0$, $T \to \infty$.*

Proof. For any fixed φ consider a behavior, as $T \to \infty$, of the value

$$\begin{aligned} Q_T(\varphi) &= \frac{4}{T^2} \left(A_0^2 \left| \int_0^T \cos\varphi_0 t e^{i\varphi t}\,dt \right|^2 + \left| \int_0^T \varepsilon(t) e^{i\varphi t}\,dt \right|^2 \right) \\ &\quad + \frac{4}{T^2} \left(2A_0 \operatorname{Re}\left[\int_0^T \cos\varphi_0 t e^{i\varphi t}\,dt \int_0^T \varepsilon(t) e^{-i\varphi t}\,dt \right] \right). \end{aligned} \tag{3}$$

As
$$T^{-1}\left|\int_0^T \cos\varphi_0 t e^{i\varphi t}\,dt\right| \le 1,$$
then due to lemma 1, the 2nd and the 3rd summands in the right-hand side of (3) tend to 0, as $T\to\infty$, in probability. Next we have for $\varphi\in[\underline{\varphi},\overline{\varphi}]$
$$\frac{2}{T}\left|\int_0^T \cos\varphi_0 t e^{i\varphi t}\,dt\right| = \begin{cases} \frac{e^{i(\varphi-\varphi_0)T}-1}{i(\varphi-\varphi_0)T}+\frac{e^{i(\varphi+\varphi_0)T}-1}{i(\varphi+\varphi_0)T}, & \varphi\ne\varphi_0,\\ \frac{e^{i\varphi_0 T}-1}{2i\varphi_0 T}+1, & \varphi=\varphi_0.\end{cases} \tag{4}$$
From (3) and (4) it follows that
$$Q_T(\varphi_0)\xrightarrow{\mathsf{P}} A_0^2, \qquad T\to\infty, \tag{5}$$
$$Q_T(\varphi)\xrightarrow{\mathsf{P}} 0, \qquad T\to\infty, \tag{6}$$
uniformly on any set
$$\Phi_\delta=\left\{\varphi\in[\underline{\varphi},\overline{\varphi}] : |\varphi-\varphi_0|\ge\delta\right\}, \qquad \delta>0.$$
By definition of φ_T
$$\begin{aligned}\mathsf{P}(|\varphi_T-\varphi_0|\ge\delta) &= \mathsf{P}(|\varphi_T-\varphi_0|\ge\delta, Q_T(\varphi_T)\ge Q_T(\varphi_0))\\ &\le \mathsf{P}\left(\sup_{\varphi\in\Phi_\delta} Q_T(\varphi)\ge Q_T(\varphi_0)\right)\to 0, \qquad T\to\infty,\end{aligned}$$
according to (5) and (6). □

We define the periodogram estimator of amplitude A_0 as $A_T=Q_T^{1/2}(\varphi_T)$.

Lemma 2. *If conditions of Lemma 1 are satisfied, then*
$$Q_T(\varphi_T)\xrightarrow{\mathsf{P}} A_0^2, \qquad T\to\infty.$$

Proof. Using (3), one can write
$$\begin{aligned} 0 &\le Q_T(\varphi_T) - Q_T(\varphi_0)\\ &= \frac{4A_0^2}{T^2}\left|\int_0^T \cos\varphi_0 t e^{i\varphi_T t}\,dt\right|^2 - \frac{4A_0^2}{T^2}\left|\int_0^T \cos\varphi_0 t e^{i\varphi_0 t}\,dt\right|^2 + \eta_T, \end{aligned} \tag{7}$$
$$\eta_T\xrightarrow{\mathsf{P}} 0, \qquad T\to\infty.$$
As from (4) we have
$$\sup_{\varphi\in[\underline{\varphi},\overline{\varphi}]}\frac{1}{T}\left|\int_0^T \cos\varphi_0 t e^{i\varphi t}\,dt\right|\le 1 \quad\text{and}\quad \lim_{T\to\infty}\frac{1}{T}\left|\int_0^T \cos\varphi_0 t e^{i\varphi_0 t}\,dt\right| = 1,$$
then
$$\overline{\lim_{T\to\infty}}\sup_{\varphi\in[\underline{\varphi},\overline{\varphi}]}\left\{\frac{4A_0^2}{T^2}\left|\int_0^T \cos\varphi_0 t e^{i\varphi t}\,dt\right|^2 - \frac{4A_0^2}{T^2}\left|\int_0^T \cos\varphi_0 t e^{i\varphi_0 t}\,dt\right|^2\right\}\le 0. \tag{8}$$
Taking into account relations (7), (8), we get
$$Q_T(\varphi_T) - Q_T(\varphi_0)\xrightarrow{\mathsf{P}} 0, \qquad T\to\infty.$$
According to (5) $Q_T(\varphi_0)\xrightarrow{\mathsf{P}} A_0^2$, $T\to\infty$, so
$$Q_T(\varphi_T)\xrightarrow{\mathsf{P}} A_0^2 \quad \text{as } T\to\infty.$$
□

Theorem 2. *If conditions of Lemma 1 are satisfied, then*

$$T\left(\varphi_T - \varphi_0\right) \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty.$$

Proof. From lemma 2 and (7) it follows

$$\frac{2}{T^2}\left|\int_0^T \cos\varphi_0 t e^{i\varphi_T t}\, dt\right|^2 - \frac{2}{T^2}\left|\int_0^T \cos\varphi_0 t e^{i\varphi_0 t}\, dt\right|^2 \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty. \tag{9}$$

In order to satisfy (9), it is necessary and sufficient that (see. (4))

$$\left|\frac{e^{i(\varphi_T-\varphi_0)T}-1}{i\left(\varphi_T-\varphi_0\right)T} + \frac{e^{i(\varphi_T+\varphi_0)T}-1}{i\left(\varphi_T+\varphi_0\right)T}\right|^2 - \left|\frac{e^{i\varphi_0 T}-1}{2i\varphi_0 T}+1\right|^2 \xrightarrow{\mathsf{P}} 0, \tag{10}$$

or

$$\frac{\sin\frac{1}{2}(\varphi_T-\varphi_0)T}{\frac{1}{2}(\varphi_T-\varphi_0)T} \xrightarrow{\mathsf{P}} 1 \quad \text{as } T \to \infty.$$

But the latter is possible if and only if

$$T(\varphi_T - \varphi_0) \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty. \qquad \square$$

Consider a vector function

$$a(t) = (a_1(t), a_2(t), \ldots, a_q(t))', \qquad t \geq 0, \tag{11}$$

and family of matrix measures $\mu_T\left(d\lambda\right) = \left(\mu_T^{jl}(d\lambda)\right)_{j,l=1}^q$,

$$\mu_T^{jl}\left(d\lambda\right) = \left(a_T^j(\lambda)\overline{a_T^i(\lambda)}\right)\left(\int_{-\infty}^{+\infty}\left|a_T^j(\lambda)\right|^2 d\lambda\right)^{-1/2}\left(\int_{-\infty}^{+\infty}\left|a_T^l(\lambda)\right|^2 d\lambda\right)^{-1/2} d\lambda,$$

$$a_T^j(\lambda) = \int_0^T e^{i\lambda t} a_j(t)\, dt, \qquad j,l = 1,\ldots,q.$$

Assume that $\mu_T(d\lambda)$ weakly converges, as $T \to \infty$, to a matrix measure $\mu(d\lambda)$, that is for any continuous bounded function $b(\lambda)$, $\lambda \in \mathbb{R}^1$,

$$\int_{-\infty}^{+\infty} b(\lambda)\, \mu_T(d\lambda) \to \int_{-\infty}^{+\infty} b(\lambda)\, \mu(d\lambda), \qquad T \to \infty.$$

Then the measure $\mu(d\lambda)$ is said to be spectral measure of vector function (11).

To determine the spectral measure of vector (11) one can use the relations [11]

$$\lim_{T\to\infty} d_{iT}^{-1} d_{jT}^{-1} \int_0^T a_i(t+s)a_j(t)\, dt = \int_{-\infty}^{+\infty} e^{i\lambda s}\, \mu_{ij}\left(d\lambda\right), \qquad i,j = 1,\ldots,q,$$

with

$$d_{iT}^2 = \int_0^T a_i^2(t)\, dt, \qquad i = 1,\ldots,q.$$

Let for $j \geq m$

$$f^{*j}(\lambda) = \int_{\mathbb{R}^{j-1}} f(\lambda - \lambda_2 - \cdots - \lambda_j) \prod_{i=2}^{j} f\left(\lambda_i\right)\, d\lambda_2 \ldots d\lambda_j$$

be the j-th convolution of the spectral density $f(\lambda)$ of the random process ξ. Note that $B^k(\cdot) \in L_1\left(\mathbb{R}^1\right)$, $k \geq m$, so all the $f^{*j}(\lambda)$, $k \geq m$, are continuous bounded functions.

Further we formulate the general theorem on asymptotic normality of certain vector integrals [12] and will use in the paper partial cases of this result.

Theorem 3. *Suppose assumptions* ***A1***, ***A2***, ***B1***, *and* ***A3*** *or* ***A4*** *are fulfilled. In addition the vector function* (11) *possesses spectral measure* $\mu(d\lambda)$ *and*

$$\sup_{t\in[0,T]} d_{iT}^{-1}\,|a_i(t)| \le k_i \cdot T^{-1/2}, \qquad i=1,\dots,q, \tag{12}$$

$k_i < +\infty$, $i=1,\dots,q$, *are some constants.*

Then the vector

$$b_T = d_T^{-1}\int_0^T G(\xi(t))\,a(t)\,dt, \qquad d_T = \operatorname{diag}(d_{iT})_{i=1}^q.$$

is asymptotically, as $T\to\infty$, *normal* $N(0,K)$ *where*

$$K = 2\pi\sum_{k=m}^{\infty}\frac{C_k^2}{k!}\int_{-\infty}^{\infty} f^{*k}(\lambda)\,\mu(d\lambda,\theta)\,. \tag{13}$$

Corollary. *If the conditions of Lemma 1 and* ***B1*** *are satisfied, then the random vector*

$$\left(d_{1T}^{-1}\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt,\, d_{2T}^{-1}\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt\right)'$$

is asymptotically, as $T\to\infty$, *normal* $N(0,K_1)$ *with*

$$K_1 = 2\pi\sum_{j=m}^{\infty}\frac{C_j^2}{j!}f^{*j}(\varphi_0)\begin{pmatrix}1 & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & 1\end{pmatrix}.$$

It follows from this fact that the vector

$$\left(T^{-1/2}\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt,\, T^{-3/2}\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt\right)' \tag{14}$$

is asymptotically, as $T\to\infty$, normal $N(0,K_2)$ with

$$K_2 = 2\pi\sum_{j=m}^{\infty}\frac{C_j^2}{j!}f^{*j}(\varphi_0)\begin{pmatrix}\frac{1}{2} & \frac{1}{4}\\ \frac{1}{4} & \frac{1}{6}\end{pmatrix}.$$

Similarly, one can obtain asymptotic normality of the vector

$$\left(T^{-1/2}\int_0^T \varepsilon(t)\cos\varphi_0 t\,dt,\, T^{-3/2}\int_0^T \varepsilon(t)t\cos\varphi_0 t\,dt\right)' \tag{15}$$

with the same covariance matrix K_2.

Lemma 3. *If the conditions of Lemma 1 and* ***B1*** *are satisfied, then* $T^{-1/2}Q_T'(\varphi_0)$ *is asymptotically, as* $T\to\infty$, *normal* $N(0,K_3)$ *with*

$$K_3 = \frac{4}{3}\pi A_0^2\sum_{j=m}^{\infty}\frac{C_j^2}{j!}f^{*j}(\varphi_0).$$

Proof. Obviously

$$Q_T(\varphi) = \frac{4}{T^2}\left[\int_0^T X(t)\cos\varphi t\,dt\right]^2 + \frac{4}{T^2}\left[\int_0^T X(t)\sin\varphi t\,dt\right]^2.$$

Then

$$
\begin{aligned}
T^{-1/2}Q'_T(\varphi_0) &= -\frac{8}{T^{5/2}}\int_0^T X(t)\cos\varphi_0 t\,dt\int_0^T X(t)t\sin\varphi_0 t\,dt \\
&\quad+\frac{8}{T^{5/2}}\int_0^T X(t)\sin\varphi_0 t\,dt\int_0^T X(t)t\cos\varphi_0 t\,dt \\
&= -\frac{8}{T^{5/2}}\Bigg[\frac{1}{2}\int_0^T A_0\cos^2\varphi_0 t\,dt\int_0^T A_0 t\sin 2\varphi_0 t\,dt \\
&\qquad+\int_0^T A_0\cos^2\varphi_0 t\,dt\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt \\
&\qquad+\frac{1}{2}\int_0^T A_0 t\sin 2\varphi_0 t\,dt\int_0^T \varepsilon(t)\cos\varphi_0 t\,dt \\
&\qquad\quad+\int_0^T \varepsilon(t)\cos\varphi_0 t\,dt\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt\Bigg] \\
&\quad+\frac{8}{T^{5/2}}\Bigg[\frac{1}{2}\int_0^T A_0\sin 2\varphi_0 t\,dt\int_0^T A_0 t\cos^2\varphi_0 t\,dt \\
&\qquad+\frac{1}{2}\int_0^T A_0\sin 2\varphi_0 t\,dt\int_0^T \varepsilon(t)t\cos\varphi_0 t\,dt \\
&\qquad+\int_0^T A_0 t\cos^2\varphi_0 t\,dt\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt \\
&\qquad\quad+\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt\int_0^T \varepsilon(t)t\cos\varphi_0 t\,dt\Bigg] \\
&= \sum_{i=1}^{8} S_i.
\end{aligned}
$$

Evidently $S_1, S_5 \to 0$, $T \to \infty$. Using lemma 1, we have $S_3 \xrightarrow{\mathsf{P}} 0$, $T \to \infty$. Convergence to 0 in probability of summands S_4, S_6 and S_8 arises from asymptotic normality of integrals

$$
T^{-3/2}\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt \quad\text{and}\quad T^{-3/2}\int_0^T \varepsilon(t)t\cos\varphi_0 t\,dt.
$$

So,

$$
T^{-1/2}Q'_T(\varphi_0) = S_2 + S_7 + \eta_T^{(2)}, \qquad \eta_T^{(2)} \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty,
$$

and

$$
\begin{aligned}
T^{-\frac{1}{2}}Q'_T(\varphi_0) &= 2A_0T^{-\frac{1}{2}}\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt - 4A_0T^{-\frac{3}{2}}\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt + \eta_T^{(3)} \\
&= 2A_0(b_{1T} - 2b_{2T}) + \eta_T^{(3)}, \qquad \eta_T^{(3)} \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty.
\end{aligned}
\tag{16}
$$

Using asymptotic normality of vector (14), find the variance $4A_0^2 D\,(b_{1T} - 2b_{2T})$. Let $\lambda = (\lambda_1, \lambda_2)$, $b_T = (b_{1T}, b_{2T})$. It is easily seen that

$$
\mathsf{E}\,e^{i\langle\lambda, b_T\rangle} \to \exp\left\{-\frac{1}{2}\langle K_2\lambda, \lambda\rangle\right\}, \qquad T \to \infty.
$$

Taking $\lambda = (\tau, -2\tau)$ we obtain

$$
\mathsf{E}\,e^{i\tau(b_{1T} - 2b_{2T})} \to \exp\left\{-\frac{\tau^2}{2}\left(K_2^{11} - 4K_2^{12} + 4K_2^{22}\right)\right\},
$$

that is r.v. $b_{1T} - 2b_{2T}$ is asymptotically normal $N\left(0, \frac{1}{6}\right)$. So, $T^{-1/2}Q'_T(\varphi_0)$ is asymptotically, as $T \to \infty$, normal $N(0, K_3)$. □

Lemma 4. *If conditions of Lemma 1 and **B1** are fulfilled, then for any r.v. $\widetilde{\varphi}_T$ satisfying inequality $|\widetilde{\varphi}_T - \varphi_0| \le |\varphi_T - \varphi_0|$ with probability 1, for all $T > 0$,*

$$\frac{1}{T^2}Q''_T(\widetilde{\varphi}_T) \xrightarrow{\mathsf{P}} -\frac{1}{6}A_0^2, \qquad T \to \infty.$$

Proof. Write

$$\begin{aligned}\frac{1}{T^2}Q''_T(\widetilde{\varphi}_T) &= 8\left[\int_0^T t\sin\widetilde{\varphi}_T tX(t)\,dt\right]^2 - \frac{8}{T}\int_0^T \cos\widetilde{\varphi}_T tX(t)\,dt \frac{1}{T^3}\int_0^T t^2\cos\widetilde{\varphi}_T tX(t)\,dt \\ &\quad + 8\left[\frac{1}{T^2}\int_0^T t\cos\widetilde{\varphi}_T tX(t)\,dt\right]^2 \\ &\quad - \frac{8}{T}\int_0^T \sin\widetilde{\varphi}_T tX(t)\,dt \frac{1}{T^3}\int_0^T t^2\sin\widetilde{\varphi}_T tX(t)\,dt \\ &= \sum_{i=1}^4 Q_i.\end{aligned}$$

Then the integral

$$\begin{aligned}\frac{1}{T}\int_0^T \cos\widetilde{\varphi}_T tX(t)\,dt &= \frac{1}{T}\int_0^T \cos\widetilde{\varphi}_T t\left[A_0\cos\varphi_0 t + \varepsilon(t)\right]\,dt \\ &= \frac{A_0}{T}\int_0^T \cos\widetilde{\varphi}_T t\cos\varphi_0 t\,dt + \frac{1}{T}\int_0^T \cos\widetilde{\varphi}_T t\varepsilon(t)\,dt \\ &= \frac{1}{2T}\int_0^T A_0\left(\cos\left(\widetilde{\varphi}_T - \varphi_0\right)t + \cos\left(\widetilde{\varphi}_T + \varphi_0\right)t\right)\,dt + \eta_T^{(4)} \\ &= \frac{A_0}{2T}\int_0^T \cos\left(\widetilde{\varphi}_T - \varphi_0\right)t\,dt + \eta_T^{(5)}, \\ &\quad \eta_T^{(4)}, \eta_T^{(5)} \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty.\end{aligned}$$

Using the lemma conditions and the result of the theorem 2 we obtain

$$\frac{1}{T}\int_0^T \cos\widetilde{\varphi}_T tX(t)\,dt \xrightarrow{\mathsf{P}} \frac{A_0}{2}, \qquad T \to \infty.$$

Using similar calculations we get

$$\frac{1}{T^3}\int_0^T t^2\cos\widetilde{\varphi}_T tX(t)\,dt \xrightarrow{\mathsf{P}} \frac{A_0}{6}, \qquad T \to \infty,$$

so, $Q_2 \xrightarrow{\mathsf{P}} -\frac{2}{3}A_0^2$. Similarly, $Q_3 \xrightarrow{\mathsf{P}} A_0^2/2$, and $Q_1, Q_4 \xrightarrow{\mathsf{P}} 0$, $T \to \infty$. Then

$$\frac{1}{T^2}Q''_T(\widetilde{\varphi}_T) \xrightarrow{\mathsf{P}} -\frac{2A_0^2}{3} + \frac{A_0^2}{2} = -\frac{A_0^2}{6}, \qquad T \to \infty.$$

□

Theorem 4. *If the conditions of Lemma 1 and **B1** are satisfied, then $T^{3/2}(\varphi_T - \varphi_0)$ is asymptotically, as $T \to \infty$, normal with zero mean and variance*

$$48\pi\sum_{j=m}^{\infty}\frac{c_j^2}{j!}f^{*j}(\varphi_0).$$

Proof. As $Q'_T(\varphi_T) = 0$, then

$$Q'_T(\varphi_0) + Q''_T(\widetilde{\varphi}_T)(\varphi_T - \varphi_0) = 0 \tag{17}$$

with some r.v. $\widetilde{\varphi}_T$, satisfying

$$|\widetilde{\varphi}_T - \varphi_0| \le |\varphi_T - \varphi_0|, \qquad T \to \infty.$$

From (17)

$$T^{3/2}(\varphi_T - \varphi_0) = -\frac{T^{-1/2}Q'_T(\varphi_0)}{T^{-2}Q''_T(\widetilde{\varphi}_T)}.$$

The theorem follows now from lemmas 3 and 4. □

Theorem 5. *If the conditions of Lemma 1 and* ***B1*** *are satisfied, then the normed estimator* $T^{1/2}(A_T - A_0)$ *is asymptotically, as* $T \to \infty$*, normal with zero mean and variance*

$$4\pi \sum_{j=m}^{\infty} \frac{c_j^2}{j!} f^{*j}(\varphi_0).$$

Proof. Write

$$T^{1/2}(A_T - A_0) = T^{1/2}\left[Q_T^{1/2}(\varphi_T) - A_0\right] = T^{1/2}\left[Q_T(\varphi_T) - A_0^2\right]\left[Q_T^{1/2}(\varphi_T) + A_0\right]^{-1}.$$

From Lemma 2

$$Q_T^{1/2}(\varphi_T) + A_0 \xrightarrow{\mathsf{P}} 2A_0, \qquad T \to \infty. \tag{18}$$

We have

$$T^{1/2}\left[Q_T(\varphi_T) - Q_T(\varphi_0)\right] = T^{1/2}Q'_T(\varphi_0)(\varphi_T - \varphi_0) + \frac{1}{2}T^{1/2}Q''_T(\widetilde{\varphi}_T)(\varphi_T - \varphi_0)^2$$

with some $\widetilde{\varphi}_T$ such that $|\widetilde{\varphi}_T - \varphi_0| \le |\varphi_T - \varphi_0|$. The value

$$T^{1/2}Q'_T(\varphi_0)(\varphi_T - \varphi_0) = T^{-1/2}Q'_T(\varphi_0)\,T(\varphi_T - \varphi_0) \tag{19}$$

tends to 0 in probability, according to theorem 2 and lemma 3. The expression

$$\frac{T^{1/2}}{2}Q''_T(\widetilde{\varphi}_T)(\varphi_T - \varphi_0)^2 = \frac{1}{2T^2}Q''_T(\widetilde{\varphi}_T)\,T^{3/2}(\varphi_T - \varphi_0)\,T(\varphi_T - \varphi_0)$$

tends to 0 in probability as it follows from lemma 4 and theorems 2 and 4. Using (18) and (19) we can conclude that asymptotic distribution of $T^{1/2}(A_T - A_0)$ is the same as asymptotic distribution of

$$\frac{T^{1/2}}{2A_0}\left[Q_T(\varphi_0) - A_0^2\right]. \tag{20}$$

From (3) and (4) it is seen that $Q_T(\varphi_0) - A_0^2$ behaves at infinity as

$$Z_T(\varphi_0) = \frac{4}{T^2}\left|\int_0^T \varepsilon(t)e^{i\varphi_0 t}\,dt\right|^2 + \frac{8A_0}{T^2}\operatorname{Re}\left\{\int_0^T \cos\varphi_0 t e^{i\varphi_0 t}\,dt \int_0^T \varepsilon(t)e^{-i\varphi_0 t}\,dt\right\}.$$

Consider

$$\frac{1}{T^{3/2}}\left|\int_0^T \varepsilon(t)e^{i\varphi_0 t}\,dt\right|^2 = \frac{1}{T^{1/2}}\int_0^T \varepsilon(t)e^{i\varphi_0 t}\,dt\,\frac{1}{T}\int_0^T \varepsilon(t)e^{-i\varphi_0 t}\,dt \xrightarrow{\mathsf{P}} 0, \qquad T \to \infty,$$

because the 1st integral is asymptotically normal and the 2nd tends to 0 in probability.

So, it remains to analyze the behavior of

$$\begin{aligned}
&T^{-3/2}\operatorname{Re}\left\{\int_0^T \cos\varphi_0 t e^{i\varphi_0 t}\,dt \int_0^T \varepsilon(t)e^{-i\varphi_0 t}\,dt\right\} \\
&\quad= T^{-3/2}\int_0^T \cos^2\varphi_0 t\,dt \int_0^T \varepsilon(t)\cos\varphi_0 t\,dt \\
&\quad+ T^{-3/2}\int_0^T \cos\varphi_0 t\sin\varphi_0 t\,dt \int_0^T \varepsilon(t)\sin\varphi_0 t\,dt \\
&\quad= \frac{1}{2T^{1/2}}\int_0^T \varepsilon(t)\cos\varphi_0 t\,dt + \eta_T^{(6)}, \qquad \eta_T^{(6)} \xrightarrow{\mathsf{P}} 0, \qquad T\to\infty.
\end{aligned}$$

As it was shown earlier, $T^{-1/2}\int_0^T \varepsilon(t)\cos\varphi_0 t\,dt$ is asymptotically normal with parameters 0 and $\pi\sum_{j=m}^{\infty}\frac{c_j^2}{j!}f^{*j}(\varphi_0)$. Using this fact we obtain that $T^{1/2}\left(Q_T(\varphi_0) - A_0^2\right)$ is asymptotically normal with parameters 0 and $16\pi A_0^2\sum_{j=m}^{\infty}\frac{c_j^2}{j!}f^{*j}(\varphi_0)$, so $T^{1/2}(A_T - A_0)$ is asymptotically normal with zero mean and variance $4\pi\sum_{j=m}^{\infty}\frac{c_j^2}{j!}f^{*j}(\varphi_0)$. □

Theorem 6. *If conditions **A1**, **A2**, **B1**, and **A3** or **A4** are satisfied, then the random vector*

$$\left(T^{1/2}(A_T - A_0),\ T^{3/2}(\varphi_T - \varphi_0)\right)'$$

is asymptotically normal, as $T\to\infty$, with zero mean and covariance matrix

$$2\pi\sum_{j=m}^{\infty}\frac{C_j^2}{j!}f^{*j}(\varphi_0)\begin{pmatrix} 2 & 0 \\ 0 & 24A_0^{-2}\end{pmatrix}.$$

Proof. In the proofs of lemma 3, theorems 4 and 5 it was shown that

$$\begin{aligned}
T^{\frac{3}{2}}(\varphi_T - \varphi_0) &= 12A_0^{-1}T^{-\frac{1}{2}}\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt \\
&\quad - 24A_0^{-1}T^{-3/2}\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt + \eta_T^{(7)},
\end{aligned} \tag{21}$$

$$T^{1/2}(A_T - A_0) = 2T^{-1/2}\int_0^T \varepsilon(t)\cos\varphi_0 t dt + \eta_T^{(6)}, \qquad T\to\infty; \tag{22}$$

$\eta_T^{(6)} \xrightarrow{\mathsf{P}} 0$, $\eta_T^{(7)} \xrightarrow{\mathsf{P}} 0$, $T\to\infty$.

We have, for any u_1, u_2,

$$\begin{aligned}
&u_1 T^{1/2}(A_T - A_0) + u_2 T^{3/2}(\varphi_T - \varphi_0) \\
&\quad= u_1 2T^{-1/2}\int_0^T \varepsilon(t)\cos\varphi_0 t\,dt + u_2 12A_0^{-1}T^{-1/2}\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt \\
&\quad - u_2 24A_0^{-1}T^{-3/2}\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt + \eta_T^{(8)} \\
&= v_1\xi_{1T} + v_2\xi_{2T} + v_3\xi_{3T} + \eta_T^{(8)},
\end{aligned} \tag{23}$$

where $v_1 = u_1\frac{2}{\sqrt{2}}$, $v_2 = u_2\frac{12}{\sqrt{2}}A_0^{-1}$, $v_3 = -u_2\frac{24}{\sqrt{6}}A_0^{-1}$, $\xi_{1T} = \sqrt{2}T^{-1/2}\int_0^T \varepsilon(t)\cos\varphi_0 t\,dt$, $\xi_{2T} = \sqrt{2}T^{-1/2}\int_0^T \varepsilon(t)\sin\varphi_0 t\,dt$, $\xi_{3T} = \sqrt{6}T^{-3/2}\int_0^T \varepsilon(t)t\sin\varphi_0 t\,dt$, $\eta_T^{(8)} \xrightarrow{\mathsf{P}} 0$, $T\to\infty$.

Note that the spectral measure $\mu(d\lambda)$ of the vector $(\cos\varphi_0 t, \sin\varphi_0 t, t\sin\varphi_0 t)$ is

$$\mu(d\lambda) = \begin{pmatrix} \alpha & -i\beta & -i\beta \\ -i\beta & \alpha & \frac{\sqrt{3}}{2}\alpha \\ -i\beta & \frac{\sqrt{3}}{2}\alpha & \alpha \end{pmatrix}, \tag{24}$$

where α is a measure concentrated at $\pm\varphi_0$, and $\alpha\left(\{\pm\varphi_0\}\right) = \frac{1}{2}$, β is a signed measure concentrated at $\pm\varphi_0$ and $\beta\left(\{\pm\varphi_0\}\right) = \pm\frac{1}{2}$.

Using result of the Theorem 3, we obtain

$$\mathsf{E}\exp\left\{i\left(v_1\xi_{1T} + v_2\xi_{2T} + v_3\xi_{3T}\right)\right\} \to \exp\left\{-\frac{1}{2}\left\langle Kv, v\right\rangle\right\},$$

where, from (13) and (24) it follows

$$\begin{aligned}\langle Kv, v\rangle &= 2\pi \sum_{k=m}^{\infty} \frac{C_k^2}{k!} f^{*k}(\varphi_0)\left(v_1^2 + v_2^2 + v_3^2 + \sqrt{3}v_2v_3\right)\\ &= 2\pi \sum_{k=m}^{\infty} \frac{C_k^2}{k!} f^{*k}(\varphi_0)\left(2u_1^2 + 24A_0^{-2}u_2^2\right). \qquad \square\end{aligned}$$

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