Теорія ймовір. та матем. статист.
Вип. 89, 2013, стор. 40–50

Teor. Ĭmovir. ta Matem. Statyst.
No. 89, 2013, pp. 40–50

QUANTITATIVE AND QUALITATIVE LIMITS FOR EXPONENTIAL ASYMPTOTICS OF HITTING TIMES FOR BIRTH-AND-DEATH CHAINS IN A SCHEME OF SERIES

UDC 519.21

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Abstract. We consider time-homogeneous discrete birth-and-death Markov chain (X_t) and investigate the asymptotics of the hitting time $\tau_n = \inf(t \geq 1\colon X_t \geq n)$ as well as the chain position before this time in the scheme of series as $n \to \infty$. In our case one-step probabilities of the chain vary simultaneously with n. The proofs are based on the explicit two-side inequalities with numerical bounds for the survival probability $\mathsf{P}(\tau_n > t)$. These inequalities can be used also for the pre-limit finite-time schemes. We have applied the results obtained for construction the uniform asymptotic representations of the corresponding risk function.

Анотація. Ми розглядаємо однорідний за часом дискретний ланцюг народження та загибелі (X_t) та вивчаємо асимптотику моменту досягнення $\tau_n = \inf(t \geq 1\colon X_t \geq n)$ і стану ланцюга до цього моменту у схемі серій, де $n \to \infty$ та одночасно змінюються перехідні імовірності ланцюга за один крок. Доведення спираються на відповідні двобічні явні нерівності для ймовірності виживання $\mathsf{P}(\tau_n > t)$ з числовими границями. Останні можна використати і у допграничних схемах. Наведено застосування у вигляді асимптотичних розвинень для відповідної функції ризику.

Аннотация. Мы рассматриваем однородную дискретную цепь рождения и гибели (X_t) и изучаем асимптотику момента достижения $\tau_n = \inf(t \geq 1\colon X_t \geq n)$ и положения цепи до этого момента, в схеме серий, где $n \to \infty$ и одновременно изменяются переходные вероятности цепи за один шаг. Доказательства основаны на соответствующих двусторонних неравенствах для вероятности выживания $\mathsf{P}(\tau_n > t)$ с явными числовыми ограничениями. Последние можно использовать и в допредельных схемах. Приведены применения в виде асимптотичных представлений для соответствующих функций риска.

1. Introduction

The task of investigation of the distribution stability for general Markov chains under the broad assumptions about the nature of jumps is expounded in details in the author's monograph [2]. Some applications of the theory are included there as well. The proofs are based on the analytical operator methods. The book includes some new inequalities for the renewal process asymptotics and the solutions of the renewal equation.

Foundations of the stability theory for stochastic models are set in the monograph by Zolotarev [5]. Important achievements in the stability theory are included in the book by Mayn and Tweedie [4].

This paper is based on the author's results placed in [2, Ch.7]. These results were obtained earlier but they have not been published. The comparison with paper [3] can be useful. The similar but not identical results were obtained earlier in [6].

2. Main results

Let us consider the time-homogeneous birth-and-death Markov chain

$$X = (X_t, t = 0, 1, \dots)$$

2000 *Mathematics Subject Classification.* Primary 60J45; Secondary 60A05, 60K05.

Key words and phrases. Birth and death Markov chains, hitting times, exponential asymptotics.

with values in a discrete space $E = \mathbb{Z}_+$. A matrix of one-step transition probabilities $P = (p_{ij}, i, j \in E)$ has entries $p_{i,i-1} = q_i$, $p_{i,i+1} = p_i$, $p_{ii} = r_i = 1 - p_i - q_i$ when $i \geq 1$, and $p_{01} = p_0$, $p_{00} = 1 - p_0 = q_0$. We assume that the chain is not reducible: p_i, $q_i > 0$. The symbols $\mathsf{P}_i(\cdot)$ and $\mathsf{E}_i(\cdot)$ will be used to denote the conditional probability and expectation given $\{X_0 = i\}$.

Let us define the hitting moment of the "distant" level as

$$\tau_n = \inf(t \geq 1\colon X_t \geq n). \tag{1}$$

We investigate the asymptotics of the time τ_n in a scheme of series where $n \to \infty$. In our case the one-step transition probabilities (p_i, q_i) could change. For instance, they could depend on n.

Let's introduce the following notation for $t \geq 0$

$$\theta_t = \prod_{1 \leq i \leq t} (q_i/p_i), \qquad \theta_0 = 1, \qquad \sigma_t = \sum_{0 \leq i < t} \theta_i, \qquad \varkappa_t = 1/(p_t\theta_t), \qquad t \geq 0. \tag{2}$$

Consider the aggregate parameters

$$\begin{aligned} \lambda_n &= \left(1 + \sum_{i \leq j \in E_n} \varkappa_i \theta_j\right)^{-1} = \left(1 + \sum_{i \in E_n} \varkappa_i(\sigma_n - \sigma_i)\right)^{-1}, \\ \omega_n &= \lambda_n - \lambda_n^2 + \lambda_n^2 \sum_{i \leq j < k \leq l \in E_n} \varkappa_i \theta_j \varkappa_k \theta_l \\ &= \lambda_n - \lambda_n^2 + \lambda_n^2 \sum_{i < k \in E_n} \varkappa_i(\sigma_k - \sigma_i)\varkappa_k(\sigma_n - \sigma_k). \end{aligned} \tag{3}$$

Hereafter we will use the summation sign without upper and lower indexes assuming summation on the hole index set $E_n \equiv \{0, 1, \ldots, n-1\}$. It worth to mention that the process continuity implies the entire determination for the distribution of the time τ_n by $(p_i, q_i, i \in E_n)$.

The following estimation can be applicable to any scheme of series as well as for the fixed n.

Theorem 2.1. *The following inequality holds true*

$$\sup_{t \geq 0} \left| \mathsf{P}_0(\tau_n > t) - \left(1 - m_n^{-1}\right)^t \right| \leq 2\omega_n(1 + \lambda_n)p_0/\lambda_n\sigma_n(1 - \omega_n), \tag{4}$$

where

$$m_n^{-1} = \lambda_n/(1 + \omega_n).$$

Remark 2.1. It follows from the definitions (3) that $0 < \omega_n \leq 1/2$ in (4).

Corollary 2.1. *Let $n \to \infty$ in a scheme of series in such a way that $\lambda_n \to 0$ and*

$$\omega_n p_0 = o(\lambda_n \sigma_n), \qquad n \to \infty. \tag{5}$$

Then

$$\sup_{x \geq 0} |\mathsf{P}_0(\tau_n/m_n > x) - \exp(-x)| \to 0, \qquad n \to \infty.$$

This convergence is uniform in the scheme of series if the relation (5) *is uniform too.*

Corollary 2.2. *Let the chain X be unchangeable for the scheme of series, irreducible and ergodic, and $n \to \infty$. Then $\lambda_n \to 0$, $\omega_n \to 0$, and the following representation is true*

$$\sup_{x \geq 0} |\mathsf{P}_0(\lambda_n \tau_n > x) - \exp(-x)| = O(\omega_n), \qquad n \to \infty. \tag{6}$$

Corollary 2.3. *Let n and the distribution of the chain X be fixed excepting $p_0 \to 0$. Then the following representation holds true*

$$\sup_{x\geq 0}|\mathsf{P}_0(p_0\tau_n > x) - \exp(-x)| = O(p_0), \qquad p_0 \to 0. \tag{7}$$

We can obtain from (4) the limit results for the specially structured schemes of series at one time. Here are some examples.

Corollary 2.4. *Let the transition probabilities in a scheme of series for the birth-and-death chain satisfy the relationship*

$$p_i = \varepsilon_n v_i + o(\varepsilon_n), \qquad q_i = \varepsilon_n u_i + o(\varepsilon_n), \qquad i \geq 1,\ n \to \infty, \tag{8}$$

for some $\varepsilon_n \to 0$, and $v_i, u_i > 0$. Let us use the denotations

$$\theta_t = \prod_{i=1}^{t}(u_i/v_i), \qquad \sigma_t = \sum_{s=1}^{t-1}\theta_s, \qquad \chi_t = 1/(v_t\theta_t), \qquad t \geq 1. \tag{9}$$

We assume that in a scheme of series

$$\sigma_n \to \infty, \qquad \sum_{t\geq 1}\chi_t \equiv \chi = O(1), \qquad n \to \infty. \tag{10}$$

Then, subject to

$$\overline{\omega}_n \equiv \sigma_n^{-1}\sum_{1\leq i<k<n}\chi_i(\sigma_k - \sigma_i)\chi_k = o(1), \qquad n \to \infty,$$

the uniform convergence is true

$$\sup_{t\geq 0}\left|\mathsf{P}_0(\tau_n > t) - \left(1 - m_n^{-1}\right)^t\right| = O\left(\overline{\omega}_n + \sigma_n^{-1}\right) = o(1), \qquad n \to \infty.$$

Remark 2.2. If the coefficients v_i, u_i are bounded and separated from zero, then the conditions (10) are equivalent to the ergodicity of the the birth-and-death chain with jump probabilities $(u_i/(u_i+v_i), v_i/(u_i+v_i))$, $i \geq 1$.

To analyze the asymptotics of joint distribution of the time τ_n and the chain value X till this time (the comparison can be made with [6]) we additionally assume that there is a systematic shift to zero

$$q_i > p_i, \qquad i \geq 1, \tag{11}$$

and state 0 in a scheme of series is asymptotically positive and attainable:

$$\varliminf_{n\to\infty}\lambda_n\sigma_n > 0, \qquad 0 < \varliminf_{n\to\infty} p_0 \leq \varlimsup_{n\to\infty} p_0 < 1. \tag{12}$$

It was established in the proof of the Corollary 2.2 (see the limit relation (38)) that the conditions (12) hold for any fixed irreducible ergodic birth-and-death chain.

We define the speed-of-mixing indicator as

$$\delta_n = \min_{1\leq i<n}(q_i - p_i) > 0. \tag{13}$$

Theorem 2.2. *Let the conditions* (11), (12) *and $\lambda_n \to 0$, $\omega_n \to 0$ hold true in a scheme of series as $n \to \infty$ so that*

$$\lambda_n \ln\lambda_n^{-1} = o(\delta_n^4), \qquad n \to \infty. \tag{14}$$

Then, for every $s_0 > 0$ the uniform representation holds true

$$\begin{aligned}\sup_{s\geq s_0, B\subset E}&\left|\mathsf{P}_0(\lambda_n\tau_n > s,\ X_{s/\lambda_n} \in B) - \pi^n(B)\exp(-s)\right| \\ &= O(\omega_n + \lambda_n\delta_n^{-4}\ln(1/\lambda_n\delta_n)) = o(1), \qquad n \to \infty,\end{aligned} \tag{15}$$

where the discrete distribution $\pi^n = (\pi_i^n, i \in E^n)$ *can be defined through* (2), (3) *by equalities*

$$\pi_n^n = \lambda_n, \qquad \pi_i^n = \lambda_n \varkappa_i(\sigma_n - \sigma_i), \qquad i < n, \qquad \pi^n(B) = \sum_{i \in B} \pi_i^n. \tag{16}$$

Corollary 2.5. *Let the chain* X *do not change in a scheme of series and be irreducible and ergodic. Then, the sufficient condition for the convergence to zero of the left-hand part of* (15) *is*

$$\varliminf_{n\to\infty} n\delta_n / \ln n > 3/2. \tag{17}$$

Remark 2.3. For comparison with (17) we remark that

(a) in a class of birth-and-death chains satisfying the conditions (13) and $\delta_n \to 0$, $n \to \infty$ the sufficient condition of ergodicity is

$$\varliminf_{n\to\infty} n\delta_n / \ln n > 1,$$

(b) under additional assumptions $r_i \equiv 0$ and $q_i - p_i \downarrow 0$, $i \to \infty$, it follows from the condition

$$\varlimsup_{n\to\infty} n\delta_n < 1/2$$

that the chain is not ergodic.

3. Proofs

The proofs in this section are based on the Corollary 7.5 [2, Ch.VII].

In order to use it we consider the auxiliary finite chain $X^n = (X_t^n, t \geq 0)$ with the set of states $E^n \equiv \{0, 1, \ldots, n\} = E_n \cup \{n\}$ and the transition probabilities

$$P_n = (p_{ij}(n), i, j \in E^n),$$

where $p_{ij}(n) = p_{ij}$ as $i \in E_n$ and

$$p_{n0}(n) = 1.$$

It is evident that the distributions for the time τ_n in (1) for chains (X_t) and (X_t^n) are equal.

Lemma 3.1. *A chain* X^n *has the unique invariant probability* $\pi^n = (\pi_i^n, i \in E^n)$ *where*

$$\pi_n^n = \lambda_n, \qquad \pi_j^n = \lambda_n \varkappa_j(\sigma_n - \sigma_j) = \lambda_n \sum_{i < j \in E_n} \theta_i \varkappa_j, \qquad j < n. \tag{18}$$

Proof. The system of equations for $x_i \equiv \pi_i^n$ has a form

$$\begin{gathered} x_0 q_0 + x_1 q_1 + x_n = x_0, \\ x_{i-1} p_{i-1} + x_i r_i + x_{i+1} q_{i+1} = x_i, \qquad 1 \leq i < n-1, \\ x_{n-2} p_{n-2} + x_{n-1} r_{n-1} = x_{n-1}, \\ x_{n-1} p_{n-1} = x_n. \end{gathered} \tag{19}$$

We obtain the following equations from the first and the second rows

$$x_{i-1} p_{i-1} - x_i q_i = x_i p_i - x_{i+1} q_{i+1} = x_n, \qquad 1 \leq i < n-1. \tag{20}$$

And finally, using the third, the forth rows of (19) and from (20) we recurrently calculate when $0 \leq k < n$

$$\begin{aligned} x_k &= x_n q_k^{-1} \theta_{k-1}^{-1} \left[\sum_{i=k}^{n-3} \theta_i + \theta_{n-3} q_{n-2} (p_{n-1} + q_{n-1}) / p_{n-2} p_{n-1} \right] \\ &= x_n q_k^{-1} \theta_{k-1}^{-1} \left[\sum_{i=k}^{n-3} \theta_i + \theta_{n-2} (1 + q_{n-1}/p_{n-1}) \right] = x_n \sum_{i=k}^{n-1} \varkappa_k \theta_i. \end{aligned} \tag{21}$$

The condition of normalization $\sum_{k=0}^{n} x_k = 1$ implies (18). □

In order to prove the Theorem 2.1 we shall use the Corollary 7.5 [2, Ch.VII] for the chain X^n with a set E^n and invariant probability π^n. Let us mention that every transition kernel $Q(x, A)$ and the corresponding linear operator in the descrete space can be defined by the matrix $Q(x, A) = \sum_{y \in A} Q_{xy}$, $Q_{xy} = Q(x, \{y\})$. Operation of multiplication kernels by measures, functions and kernels corresponds to the multiplication of the matrices by rows, columns and matrices.

In particular, the system for the kernel $\overline{R} = (\overline{R}_{xy}, x, y \in E^n)$ in the formulation of the Corollary 7.5 [2, Ch.VII] is as following

$$\overline{R}_{xy} = \sum_{k \in E^n} P_{xk}(n)\overline{R}_{ky} + P_{xy}(n) - \pi^n_y,$$
$$\sum_{k \in E^n} \pi^n_k \overline{R}_{ky} = 0, \qquad x, y \in E^n, \tag{22}$$

where the last equation arises from the Lemma 3.1 since π^n is the eigenvector for the matrix P_n.

Moreover, it follows from the defining $\overline{R}$ as a sum of series of powers of P_n (Corollary 7.5 [2, Ch.VII]) that the operators $\overline{R}$ and P_n commutate so the equations (22) are equivalent to the system

$$\overline{R}_{xy} = \sum_{k \in E^n} \overline{R}_{xk} P_{ky}(n) + P_{xy}(n) - \pi^n_y,$$
$$\sum_{k \in E^n} \overline{R}_{xk} = 0, \qquad x, y \in E^n. \tag{23}$$

Lemma 3.2. *The solutions of systems* (22), (23) *for* $x = n$ *or* $y = n$ *are as following*

$$\overline{R}_{nn} = -\omega_n, \tag{24}$$

$$\overline{R}_{kn} = \lambda_n \sum_{i \le j < k} \varkappa_i \theta_j + \lambda_n - \omega_n, \qquad k < n, \tag{25}$$

$$\overline{R}_{nk} = \varkappa_k(\sigma_n - \sigma_k)(\lambda_n - \omega_n) + \lambda_n \sum_{k<i<n} \varkappa_k(\sigma_n - \sigma_i)\varkappa_i(\sigma_i - \sigma_k), \qquad k < n. \tag{26}$$

Proof. Denote $x_k = \overline{R}_{kn}$. Taking into account (18) we put $y = n$ into (22) and obtain the system

$$x_0 = q_0 x_0 + p_0 x_1 - \lambda_n,$$
$$x_i = q_i x_{i-1} + r_i x_i + p_i x_{i+1} - \lambda_n, \qquad 0 < i < n-1,$$
$$x_{n-1} = q_{n-1} x_{n-2} + r_{n-1} x_{n-1} + p_{n-1} x_n + p_{n-1} - \lambda_n,$$
$$x_n = x_0 - \lambda_n. \tag{27}$$

The following equalities are deduced from the first and the second rows

$$p_0(x_1 - x_0) = \lambda_n,$$
$$(x_{i+1} - x_i)/\theta_i = \lambda_n \varkappa_i + (x_i - x_{i-1})/\theta_{i-1}, \qquad 1 \le i < n-1.$$

By recurrent calculation we get

$$x_{k+1} - x_k = \lambda_n \sum_{i=0}^{k} \varkappa_i \theta_k, \qquad 0 \le k < n-1,$$

$$x_k = x_0 + \lambda_n \sum_{j<k} \sum_{i=0}^{j} \varkappa_i \theta_j, \qquad 0 \le k < n. \tag{28}$$

Putting the equalities (28) and (27) into the second equation (22) we obtain that

$$\begin{aligned}0 &= \sum_{k\in E^n} \pi_k^n x_k = x_0 - \lambda_n^2 + \lambda_n \sum_{k\in E_n}\sum_{j<k}\sum_{i=0}^{j} \varkappa_i\theta_j \\ &= x_0 - \lambda_n^2 + \lambda_n^2 \sum_{i\le j<k\in E_n} \varkappa_i\theta_j\varkappa_k(\sigma_n-\sigma_k) = x_0 - \lambda_n^2 + \omega_n - \lambda_n + \lambda_n^2 \\ &= x_0 + \omega_n - \lambda_n,\end{aligned}$$

and deduce from (27) the identities in (24), (25).

For proving (26) we use the denote $x_k \equiv \overline{R}_{nk}$ and use (23) when $x = n$

$$\begin{aligned} x_0 &= q_0x_0 + q_1x_1 + x_n + 1 - \pi_0^n, \\ x_k &= p_{k-1}x_{k-1} + r_kx_k + q_{k+1}x_{k+1} - \pi_k^n, \qquad 1\le k< n-1, \\ x_{n-1} &= p_{n-2}x_{n-2} + r_{n-1}x_{n-1} - \pi_{n-1}^n, \\ x_n &= p_{n-1}x_{n-1} - \pi_n^n, \end{aligned} \tag{29}$$

where the probabilities π_k^n are defined in the Lemma 3.1.

Using the first two equations (29) and the recurrent calculations we deduce that

$$p_kx_k - q_{k+1}x_{k+1} = -\sum_{i=0}^{k}\pi_i^n + 1 + x_n = \sum_{i=k+1}^{n}\pi_i^n + x_n, \qquad 0\le k< n-2. \tag{30}$$

Multiplying (30) by θ_k and summing over $k = 0,\dots,n-3$ we obtain

$$x_k\theta_kp_k = x_{n-2}\theta_{n-2}p_{n-2} + \sum_{j=k}^{n-3}\theta_j\left(x_n + \sum_{i=j+1}^{n}\pi_i^n\right).$$

Taking into account the last two equations in (29) and the identity

$$\theta_{n-2}(p_{n-1}+q_{n-1})/p_{n-1} = \theta_{n-2} + \theta_{n-1}$$

we deduce that

$$x_k = x_n(\sigma_n - \sigma_k) + \sum_{j=k}^{n-1}\varkappa_k\theta_j\sum_{i=j+1}^{n}\pi_i^n, \qquad 0\le k<n. \tag{31}$$

And finally, putting there the values π_i^n from (16) and $x_n = \overline{R}_{nn}$ from (24) concludes the proof of the Lemma 3.2. □

Proof of Theorem 2.1. Let us utilize the inequality (7.40) from the Corollary 7.5 [2, Ch.VII] to the chain $X = X^n$ on $E = E^n$ with time $\tau_H = \tau_n$ and the set $H = \{n\}$. An invariant measure and the chain potential are calculated in the Lemmas 3.1 and 3.2.

In the notations of (7.39) [2, Ch.VII]

$$r_{HH} = \sup_{x\in H}\left|\overline{R}\right|(x,H) = \left|\overline{R}_{nn}\right| = \omega_n, \tag{32}$$

under (24) since $\omega_n > 0$. It follows from (25), (18), (3)

$$
\begin{aligned}
r_{\pi H} &= \int \pi(dx) \left|\overline{R}\right|(x,H) = \sum_{k=0}^{n} \pi_k^n \left|\overline{R}_{kn}\right| \\
&\le \lambda_n \left|\overline{R}_{nn}\right| + \sum_{k=0}^{n-1} \lambda_n \varkappa_k (\sigma_n - \sigma_k) \max\left(\omega_n, \lambda_n + \lambda_n \sum_{i\le j<k} \varkappa_i \theta_j\right) \\
&= \lambda_n \omega_n + \max\left[\lambda_n \omega_n \sum_{k=0}^{n-1} \varkappa_k (\sigma_n - \sigma_k),\right. \qquad (33) \\
&\qquad \left.\lambda_n^2 \sum_{k=0}^{n-1} \varkappa_k (\sigma_n - \sigma_k)\left(1 + \sum_{i\le j<k} \varkappa_i \theta_j\right)\right] \\
&\le \lambda_n \omega_n + \max\left[\lambda_n \omega_n (\lambda_n^{-1} - 1), \lambda_n^2 (\lambda_n^{-1} - 1)\right] + \omega_n - \lambda_n + \lambda_n^2 \\
&= \lambda_n \omega_n + \max[\omega_n (1 - \lambda_n), \omega_n] = \omega_n (1 + \lambda_n).
\end{aligned}
$$

Furthermore, according to the equality (7.41) [2, Ch. VII]

$$
m_H^{-1} = (E_\pi \tau_H)^{-1} = \pi_n^n \sum_{t\ge 0} (-1)^t (\overline{R}_{nn})^t = \lambda_n (1 + \omega_n)^{-1} = m_n^{-1}. \qquad (34)
$$

And finally, the constant a in the Corollary 7.5 [2, Ch.VII] is the upper limit for the density of the initial distribution of α (it is concentrated in 0) regarding the measure π^n

$$
a = 1/\pi_0^n = 1/\lambda_n \varkappa_0 \sigma_n = p_0/\lambda_n \sigma_n. \qquad (35)
$$

Putting the relations (32), (33), (34) and (35) into the inequality (7.40) of the Corollary 7.5 [2, Ch.VII] we have proved the estimate (4) in the Theorem 1. □

Proof of Remark 1. The positiveness of $\omega_n > 0$ follows from condition $\lambda_n < 1$ in definition (3). Let us denote as

$$
s_n = \sum_{i\le j\in E_n} \varkappa_i \theta_j > 0
$$

the sum included in (3). Using the last definition

$$
\lambda_n = (1 + s_n)^{-1},
$$

$$
\begin{aligned}
\omega_n &= \left(s_n + \sum_{i\le j<k\le l\in E_n} \varkappa_i \theta_j \varkappa_k \theta_l\right) / (1 + s_n)^2 \\
&\le \left(s_n + 1/2 \sum_{i\le j\in E_n} \varkappa_i \theta_j \sum_{k\le l\in E_n} \varkappa_k \theta_l\right) / (1 + s_n)^2 \qquad (36) \\
&= (s_n + s_n^2/2)/(1 + s_n)^2 \le 1/2 < 1.
\end{aligned}
$$

□

Proof of Corollary 2.1. The proof can be concluded from the inequality (4) since the right-hand part of (4) equals to $O(p_0 \omega_n / \lambda_n \sigma_n)$ in view of (36). From the other side, the relation in the left-hand part after the substitution $t = [x m_n]$ is equivalent to

$$
(1 - m_n^{-1})^{[x m_n]} \to \exp(-x), \qquad n \to \infty, \qquad (37)
$$

uniformly on $x \ge 0$ since $m_n^{-1} \le \lambda_n \to 0$. □

Proof of Corollary 2.2. The well-known recurrence and positivity criteria for the birth-and-death chain [1] correspond to the divergence of $\sigma_n \to \infty$ and convergence of $\varkappa = \sum_{i\ge 0} \varkappa_i < \infty$.

Let us calculate

$$(\lambda_n\sigma_n)^{-1} = \sigma_n^{-1}\left(1+\sum_{i\le j\in E_n}\varkappa_i\theta_j\right) = \sigma_n^{-1} + \sum_{i\ge 0}\varkappa_i\left(1-\sigma_i\sigma_n^{-1}\right)\mathbb{1}_{i<n}$$
$$\to \sum_{i\ge 0}\varkappa_i = \varkappa \in (0,\infty), \qquad n\to\infty, \tag{38}$$

using the Lebesgue theorem on majorized convergence.

So, $\lambda_n \sim 1/\varkappa\sigma_n \to 0$, $n\to\infty$.

Similarly, it follows from the representation

$$\omega_n = \lambda_n - \lambda_n^2 + \lambda_n^2 \sum_{0\le i<k<n} \varkappa_i(\sigma_k-\sigma_i)\varkappa_k(\sigma_n-\sigma_k)$$
$$\le \lambda_n + (\lambda_n\sigma_n)^2 \sum_{i,k\in E_n} \varkappa_i\varkappa_k(\sigma_k-\sigma_i)^+\sigma_n^{-1}\mathbb{1}_{i<k<n},$$

and the monotonicity of σ_n, using the Lebesgue theorem on majorized convergence, that $\overline{\lim}_{n\to\infty}\omega_n = 0$.

Taking into account (38) and the Remark 1 we can conclude that the right-hand part of (4) $2\omega_n(1+\lambda_n)p_0/\lambda_n\sigma_n(1-\omega_n)$ is equal to $O(\omega_n)$.

Utilization of approximation (37) in its left-hand part, convergence of $\omega_n \to 0$ and the estimate $|\exp(-x-x\varepsilon)-\exp(-x)| \le \varepsilon$, $x,\varepsilon \ge 0$ result in (6) □

Proof of Corollary 2.3. Let us use the representations (4) of the Theorem 2.1, where n is fixed. Since p_0 is included into (3) only as a part of $\varkappa_0$, then $\lambda_n = 1/(1+L/p_0) \sim p_0/L$, $p_0 \to 0$, $\omega_n = p_0/L + o(p_0)$, $p_0\to 0$, $\sigma_n^2 = C$ for some constants $L, C > 0$. Thus, (7) follows from (4). □

Proof of Corollary 2.4. It follows from the definitions (3), (8), (9) that

$$\lambda_n^{-1} = 1 + \sigma_n/p_0 + \varepsilon_n^{-1}\sum_{1\le i<n}\chi_i(\sigma_n-\sigma_i) \sim \sigma_n\chi/\varepsilon_n \to \infty, \qquad n\to\infty.$$

Simultaneously,

$$\omega_n = \lambda_n - \lambda_n^2 + \lambda_n^2\sum_{1\le i<k<n}\varepsilon_n^{-2}\chi_i(\sigma_k-\sigma_i)\chi_k(\sigma_n-\sigma_k) \sim \lambda_n + \lambda_n^2\sigma_n^2\overline{\omega}_n/\varepsilon_n^2$$
$$\sim \lambda_n + \chi^{-2}\overline{\omega}_n = o(1), \qquad n\to\infty.$$
□

Proof of Theorem 2.2. Let us use the inequality (7.43) of the Corollary 7.5 [2, Ch.VII] to the chain $X = X^n$ on $E = E^n$ with time $\tau_H = \tau_n$ and the set $H = \{n\}$. An invariant probability and the potential of the chain X^n are calculated in the Lemmas 3.1 and 3.2.

The estimate for new variation of the potential in (7.42) can be deduced from the equalities (24), (26) since

$$r_H = 1 + \sup_{x\in H}\left|\overline{R}\right|(x,E) = 1 + \sum_{k=0}^{n}\left|\overline{R}_{nk}\right|$$
$$\le 1 + \left|\overline{R}_{nn}\right| + (\lambda_n+\omega_n)\sum_{k<n}\varkappa_k(\sigma_n-\sigma_k) + \lambda_n\sum_{k<i<n}\varkappa_k(\sigma_i-\sigma_k)\varkappa_i(\sigma_n-\sigma_k) \tag{39}$$
$$= 1 + \omega_n + (\lambda_n+\omega_n)\left(\lambda_n^{-1}-1\right) + \left(\omega_n - \lambda_n + \lambda_n^2\right)\lambda_n^{-1} = 1 + 2\omega_n\lambda_n^{-1}.$$

The relations (3) are also used in the expressions above.

Since $\pi(H) = \pi_n^n = \lambda_n$ then utilizing (32) to the firs term in the right-hand part of (7.43) [2, Ch. VII] we obtain the inequality

$$\pi(H)2r_H/(1-r_{HH}) \le \lambda_n 2\left(1+2\omega_n\lambda_n^{-1}\right)/(1-\omega_n) = O(\omega_n), \qquad n\to\infty, \tag{40}$$

where Remark 2.1 is taken into account as well. The representation $\lambda_n = O(\omega_n)$ is the evident conclusion from (3).

Thus, in order to apply (7.43) we need to find such constants $r_\alpha \in (0,1)$ and $b < \infty$ that

$$|\mathsf{P}_0(X_t^n \in B) - \pi^n(B)| \le b(1-r_\alpha)^t \tag{41}$$

for all $t > 0$, $B \subset E$.

Let us use the Theorem 3.6 [2, Ch.III].

We define the following norm on the space of measures $\mu = (\mu_i, i \in E^n)$

$$\|\mu\| = \sum_{i\in E^n} v^i |\mu_i|, \tag{42}$$

where the constant $v > 1$ will be choose later. The form of the corresponding operator norm on $L(E^n)$ is placed in [2, p. 1.1]. Let us mention that since $v > 1$:

$$\begin{aligned}|\mathsf{P}_0(X_t^n \in B) - \pi^n(B)| &\le \left|\alpha P_n^t - \pi^n\right|(E^n) \le \left\|\alpha P_n^t - \pi^n\right\| = \left\|\alpha(P_n^t - \Pi_n)\right\| \\ &\le \|\alpha\|\left\|P_n^t - \Pi_n\right\| = \left\|P_n^t - \Pi_n\right\|,\end{aligned} \tag{43}$$

where $P_n^t \equiv (P_n)^t$ and $\alpha_i = \delta_{i0}$ is the initial distribution of the matrix Π_n that has equal rows of the type π^n.

Let us transform the matrix P_n as $P_n = T_n + h\circ\beta$, where the function

$$h = (\delta_{i0}, i \in E^n),$$

the measure $\beta = (p_0, 1-p_0, 0, \ldots, 0) = (P_{0j}(n), j \in E^n)$, and the matrix

$$T_n = (P_{ij}(n)1_{i>0}, i, j \in E^n).$$

So, the condition (C) [2, p.3.3] is true when $n = 1$ (in denotations of [2]).

Let us calculate the operator norm $\rho_n \equiv \|T_n\|$:

$$\begin{aligned}\rho_n &= \max_{i\in E^n} v^{-i}\sum_{j\in E^n} v^j P_{ij}(n)1_{i>0} \\ &= \max\left\{\max_{1\le i<n} v^{-i}\left(q_i v^{i-1} + r_i v^i + p_i v^{i+1}\right), v^{-n}\right\} \\ &= \max_{1\le i<n}\left(1-(v-1)\left(q_i v^{-1} - p_i\right)\right) = 1-(v-1)v^{-1}\min_{1\le i<n}(q_i - p_i - (v-1)p_i) \\ &\le 1-(v-1)v^{-1}(\delta_n - (v-1)/2),\end{aligned} \tag{44}$$

taking into account (13) and the condition (11) under which $\delta_n > 0$ and $p_i < 1/2$.

Let us put $v = 1+\delta_n$. Then (44) implies the following inequalities

$$\rho_n \le 1-\delta_n^2/2(1+\delta_n) < 1-\delta_n^2/4.$$

The condition (T) from [2, p. 3.3] is fulfilled when $m = 1$ (in denotations of [2]) and the following representation holds true uniformly in a scheme of series

$$(1-\rho_n)^{-1} = O\left(\delta_n^{-2}\right), \qquad n\to\infty. \tag{45}$$

Thus, all the conditions of the Theorem 3.6 [2, Ch. III] are true and in the denotations of the Theorem: $n = m = 1$, $\alpha = \beta$, $h = h$, $P = P_n$, $\pi = \pi_n$, $\Pi = \Pi_n$, $T = T_n$, $\rho = \rho_n$, and the norm $\|\cdot\|$ is defined in (42). In particular, for the parameter σ in (3.31) [2] we get the estimate

$$\sigma \le k\|\alpha\|/(1-\rho) = O\left(\delta_n^{-2}\right), \qquad n\to\infty. \tag{46}$$

In order to applied (3.30) we choose (3.29) according to (3.32)

$$\omega \leq \omega_1 = 2\exp\bigl(-(1-\pi h)\ln(\alpha h)/\pi h(1-(\alpha h))\bigr) - 1 = O(1), \qquad n \to \infty, \tag{47}$$

where we used the equalities $\pi h = \pi_0^n = \lambda_n \sigma_n / p_0$, $\alpha h = p_0$ and the condition of distancing from zero (12).

From (45), (46), (47) we can calculate the asymptotics for the parameter θ_0 in (3.29) [2]:

$$(1-\theta_0)^{-1} = O\left((1-\rho_n)^{-1}\right) O(\sigma\omega) = O\left(\delta_n^{-4}\right), \qquad n \to \infty. \tag{48}$$

Let us choose in (3.30) [2, p. 3.3] the parameter $\theta = (1+\theta_0)/2$.

Since $\theta - \theta_0 = (1-\theta_0)/2$, $1-\theta = (1-\theta_0)/2$ so from (46), (48) and from (3.30) [2, p. 3.3] we deduce the inequality (41) in the form

$$\left\| P_n^t - \Pi_n \right\| \leq b_n (1-r_n)^t, \tag{49}$$

where

$$r_n^{-1} = \max\left((1-\rho_n)^{-1}, (1-\theta)^{-1}\right) = O\left(\delta_n^{-4}\right), \qquad n \to \infty, \tag{50}$$

$$b_n = (1+\sigma)/(\theta - \theta_0) = O\left(\delta_n^{-6}\right), \qquad n \to \infty. \tag{51}$$

Finally, we deduce the following inequality for the second term in the right-hand part of (7.43) [2, Ch. VII]

$$\begin{aligned} \pi(H) a r_\alpha^{-1} \ln(1 + be/a\pi(H)) &= \lambda_n a r_n^{-1} \ln(1 + b_n e/a\lambda_n) \\ &\leq \lambda_n O(1) O\left(\delta_n^{-4}\right) \ln\left(O\left(\delta_n^{-6}\right)\lambda_n^{-1}\right) \\ &= O\left(\lambda_n \delta_n^{-4} \ln(1/\delta_n\lambda_n)\right), \qquad n \to \infty, \end{aligned} \tag{52}$$

taking into account the identity (35) under the boundedness conditions (12) and the estimates (50), (51).

The relation (14) $\lambda_n \ln \lambda_n^{-1} = o(\delta_n^4)$, $n \to \infty$ is equivalent to the convergence to zero of the last term in (15): $\lambda_n \delta_n^{-4} \ln(\lambda_n^{-1}\delta_n^{-1}) \to 0$, $n \to \infty$. Really, (14) follows from (15) since $\lambda_n \to 0$, $\delta_n \to 0$. From the other hand, from (14) we deduce that $\delta_n^{-4} = o(1/\lambda_n \ln \lambda_n^{-1})$ implying

$$\lambda_n \delta_n^{-4} \ln \delta_n^{-1} = \lambda_n o\left(\left(\lambda_n \ln \lambda_n^{-1}\right)^{-1} \ln(\lambda_n \ln \lambda_n^{-1})^{-1}\right) = o(1), \qquad n \to \infty,$$

which concludes (15).

Since after putting in (7.43) [2, Ch. VII] $t = \lambda_n^{-1} s$ it follows from the inequality $s > s_0 > 0$ that

$$\lambda_n^{-1} s_0 \geq t_0 \equiv r_\alpha^{-1} \ln^+(b/a\pi(H)) = O\left(\delta_n^{-4} \ln(1/\delta_n\lambda_n)\right), \qquad n \to \infty,$$

as the consequence from the convergence to zero of the right-hand part of (15) and therefore $t \geq t_0$ in the Corollary 7.5.

This substitution and taking into account (40) and (52) prove (15). □

Proof of Corollary 2.5. The convergence $\lambda_n \to 0$, $\omega_n \to 0$ was proved in the Corollary 2.2. Correctness of (12) follows from (38).

If $\lim \delta_n > 0$ then the uniform ergodicity holds true and the Corollary statement will be evident since under the condition $\lambda_n + \omega_n \to 0$, $n \to \infty$.

So, we can assume that $\delta_n \to 0$.

From the relation (38) $\lambda_n \sim \varkappa \sigma_n^{-1}$, $0 < \varkappa < \infty$ we find

$$\lambda_n \ln \lambda_n^{-1} \sim \sigma_n^{-1} \ln \sigma_n, \qquad n \to \infty. \tag{53}$$

Furthermore, it follows from the definition (13) that

$$\theta_t = \prod_{i=1}^{t}(1 + (q_i - p_i)/p_i) \geq (1 + 2\delta_n)^t,$$
$$\sigma_n = \sum_{t<n} \theta_t \geq (2\delta_n)^{-1}((1 + 2\delta_n)^n - 1),$$

These relations and (17) imply that

$$(1 + 2\delta_n)^n \geq \exp((2 - \varepsilon)n\delta_n) \geq \exp((3 + \varepsilon)\ln n) = n^{3+\varepsilon},$$
$$\sigma_n^{-1} = O(\delta_n n^{-3-\varepsilon}),$$

for some $\varepsilon > 0$ starting from some number.

So, in the consequence of (53) the condition (14) hold true:

$$\lambda_n \ln \lambda_n^{-1} = O\left(\delta_n n^{-3-\varepsilon} \ln(\delta_n^{-1} n^{3+\varepsilon})\right) = o\left(\delta_n^4\right), \qquad n \to \infty,$$

since $\delta_n^{1-\alpha} n^{1+\varepsilon/3} \to \infty$ for all sufficiently small $\alpha, \varepsilon > 0$ given (17). Hereof,

$$\left(\delta_n n^{-3-\varepsilon} \ln \delta_n^{-1}\right) / \delta_n^4 = \left(\delta_n n^{1+\varepsilon/3} \left(\ln \delta_n^{-1}\right)^{-1/3}\right)^{-3} \to 0, \qquad n \to \infty. \qquad \square$$

Proof of Remark 2.3. According to [1] the ergodicity of the chain is equivalent to the convergence of the series $\sum \varkappa_t$, which corresponds to the convergence of the series $\sum_{n\geq 1} \prod_{i=1}^{n}(p_i/q_i)$. By the definition (13) the convergence of the last series follows from the convergence of the following series

$$\sum_{n\geq 1} \prod_{i=1}^{n}(1 - \delta_n/q_i) \leq \sum_{n\geq 1}(1 - \delta_n)^n < \infty.$$

In the conditions (b) the equality $\delta_{i+1} = q_i - p_i$ hold true, so $q_i > 1/2$ and

$$\sum_{n\geq 1} \prod_{i=1}^{n}(p_i/q_i) \geq \sum_{n\geq 1} \prod_{i=1}^{n}(1 - 2\delta_{i+1}) = \infty. \qquad \square$$

We are grateful to the anonymous referee for the insightful comments that have significantly improved the paper.

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Received 04/09/2013