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ASYMPTOTIC PROPERTIES OF NON-STANDARD DRIFT PARAMETER ESTIMATORS IN THE MODELS INVOLVING FRACTIONAL BROWNIAN MOTION

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MERIEM BEL HADJ KHLIFA, YULIYA MISHURA, AND MOUNIR ZILI

Abstract. We investigate the problem of estimation of the unknown drift parameter in the stochastic differential equations driven by fractional Brownian motion, with the coefficients supplying standard existence–uniqueness demands. We consider a particular case when the ratio of drift and diffusion coefficients is non-random, and establish the asymptotic strong consistency of the estimator with different ratios, from many classes of non-random standard functions. Simulations are provided to illustrate our results, and they demonstrate the fast rate of convergence of the estimator to the true value of a parameter.

Анотація. Статтю присвячено задачі оцінювання невідомого параметра зсуву в стохастичному диференціальному рівнянні, що містить дробовий броунівський рух, і коефіцієнти якого задовольняють стандартні умови існування та єдиності розв'язку. Розглянуто частковий випадок, коли відношення коефіцієнтів зсуву та дифузії є невипадковим, і доведено сильну конзистентність оцінки для різних відношень коефіцієнтів, що належать до стандартних класів функцій. Результати супроводжуються обчисленнями, які показують швидку збіжність одержаних оцінок до справжнього значення параметра.

Аннотация. Статья посвящена задаче оценивания неизвестного параметра сдвига в стохастическом дифференциальном уравнении, содержащем дробное броуновское движение, с коэффициентами, удовлетворяющими стандартные условия существования и единственности решения. Рассмотрен частный случай, когда отношение коэффициентов сдвига и диффузии неслучайно, и доказана сильная состоятельность оценки при различных отношениях коэффициентов, принадлежащих стандартным классам функций. Результаты сопровождаются вычислениями, которые показывают быструю сходимость полученных оценок к истинному значению параметра.

1. Introduction

The paper is devoted to the drift parameter estimation in the diffusion models involving fractional Brownian motion. Such important problem was studied originally in the papers [3] and [4], where the authors investigated the fractional Ornstein-Uhlenbeck process with unknown drift parameter. After that, several works were interested in the same statistical problem with many different methods (see. for instance [1], [6] and [10]). These methods are well described and compared in the paper [5]. One of them is the construction of maximum likelihood estimator, but it is necessary to apply Girsanov theorem for fractional Brownian motion, and it leads to complicated calculations and moreover it is hard to discretize this estimator (see [4] & [9]). Instead, some more simple estimator is proposed that is easy to discretize but in order to establish its strong consistency, one needs to bound the integral w.r.t. fractional Brownian motion, that is non-trivial problem. However, in some particular cases especially when the ratio of the coefficients of the initial equation is non-random, the bounds can be significantly simplified. In this paper, we investigate this case in detail and illustrate the strong consistency of the estimator

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with the help of different classes of non-random ratios: power, trigonometric, exponential and logarithmic functions. It is pleasant for the investigator that in all these examples we have strong consistency, but with the different rate of convergence of the remainder term to zero. Theoretical results are illustrated by simulations.

Let $(\Omega, \mathcal{F}, \overline{\mathcal{F}}, P)$ be a complete probability space with filtration $\overline{\mathcal{F}} = \{\mathcal{F}_t, t \in \mathbb{R}^+\}$ satisfying the standard assumptions. It is assumed that all processes under consideration are adapted to filtration $\overline{\mathcal{F}}$.

Definition 1.1. Fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$ is a Gaussian process $B^H = \{B_t^H, t \in \mathbb{R}^+\}$ on $(\Omega, \mathcal{F}, P)$ featuring the properties

(a) $B_0^H = 0$;
(b) $E(B_t^H) = 0$, $t \in \mathbb{R}^+$;
(c) $E(B_t^H B_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$, $s, t \in \mathbb{R}^+$.

We consider the continuous modification of B^H whose existence is guaranteed by the classical Kolmogorov theorem. In what follows we consider the case when $H > \frac{1}{2}$.

To describe the statistical model, we need to introduce the pathwise integrals w.r.t. fBm.

2. Elements of fractional calculus and fractional integration

At first we give the basic facts on fractional integration; for more detail, see [8, 12]. Consider functions $f, g : [0,T] \to \mathbb{R}$, and let $[a,b] \subset [0,T]$. For $\alpha \in (0,1)$ define Riemann-Liouville fractional derivatives on the finite interval

$$\begin{aligned}\big(\mathcal{D}_{a+}^{\alpha} f\big)(x) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\int_a^x f(t)(x-t)^{-\alpha} dt \right) 1_{(a,b)}(x) \\ &= \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}} du \right) 1_{(a,b)}(x),\end{aligned}$$

$$\big(\mathcal{D}_{b-}^{\alpha} g\big)(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{g(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{g(x)-g(u)}{(u-x)^{1+\alpha}} du \right) 1_{(a,b)}(x). \tag{1}$$

Assuming that $\mathcal{D}_{a+}^{\alpha} f \in L_1[a,b]$, $\mathcal{D}_{b-}^{1-\alpha} g_{b-} \in L_{\infty}[a,b]$, where $g_{b-}(x) = g(x) - g(b)$, the generalized Lebesgue-Stieltjes integral is defined as

$$\int_a^b f(x) dg(x) = \int_a^b \Big(\mathcal{D}_{a+}^{\alpha} f\Big)(x) \Big(\mathcal{D}_{b-}^{1-\alpha} g_{b-}\Big)(x) dx.$$

Introduce the norm

$$\|f\|_{\alpha,a,b,\infty} = \sup_{t \in [a,b]} \left(|f(t)| + \int_a^t \frac{|f(t)-f(z)|}{(t-z)^{1+\alpha}} dz \right),$$

and denote $W_{\alpha,a,b,\infty}$ the class of functions for which this norm is finite. Let the function g have Hölder trajectories, namely, $g \in C^{\theta}[a,b]$ with $\theta \in (\frac{1}{2}, 1)$. In order to integrate w.r.t. the function g and get the appropriate upper estimate for the integral, fix some $\alpha \in (1-\theta, 1/2)$ and introduce the following norm:

$$\|f\|_{\alpha,[a,b]} = \int_a^b \left(\frac{|f(s)|}{(s-a)^{\alpha}} + \int_a^s \frac{|f(s)-f(z)|}{(s-z)^{1+\alpha}} dz \right) ds.$$

For simplicity we will abbreviate $\|\cdot\|_{\alpha,t} = \|\cdot\|_{\alpha,[0,t]}$. Denote

$$\Lambda_{\alpha}(g) := \sup_{0 \le s < t \le T} |\mathcal{D}_{t-}^{1-\alpha} g_{t-}(s)|.$$

In view of Hölder continuity, $\Lambda_{\alpha}(g) < \infty$.

Then for any $t \in (0,T]$ and any f such that $\|f\|_{\alpha,t} < \infty$, the integral $\int_0^t f(s)dg(s)$ is well defined as a generalized Lebesgue–Stieltjes integral, and the following bound is evident:

$$\Big|\int_0^t f(s)dg(s)\Big| \le \frac{\Lambda_\alpha(g)}{\Gamma(1-\alpha)}\|f\|_{\alpha,t}. \tag{2}$$

It is well known that in the case when both functions, f and g are Hölder, more precisely, $f \in C^\beta[a,b]$, $g \in C^\theta[a,b]$ with $\beta+\theta > 1$, the generalized Lebesgue–Stieltjes integral $\int_a^b f(x)dg(x)$ exists, equals the limit of Riemann sums and admits upper bound (2) for any $\alpha \in (1-\theta, \beta \wedge 1/2)$.

3. Description of the model and properties of the estimator

Consider the equation

$$X_t = x_0 + \theta\int_0^t a(s,X_s)ds + \int_0^t b(s,X_s)dB_s^H,\ t \in \mathbb{R}^+, \tag{3}$$

where $x_0 \in \mathbb{R}$ is the initial value and θ is the unknown drift parameter to be estimated. First, we formulate existence–uniqueness conditions for the solution of equation (3).

Let $K > 0$ be a constant and let the following assumptions hold on any interval $[0,T]$:

(A_1) Linear growth of a and b in space: there exists such $K > 0$ that for any $s \in [0,T]$ and any $x \in \mathbb{R}$

$$|b(s,x)| \le K(1+|x|),$$

and for any $s \in [0,T]$ and any $x \in \mathbb{R}$

$$|a(s,x)| \le a_0(s) + K|x|,$$

where the non-negative function $a_0 = a_0(s) \in L_\rho[0,T]$ for some $\rho \ge 2$.

(A_2) Local Lipschitz continuity of a and Lipschitz continuity of b in space: for any $N > 0$ there exists such K_N that for any $t \in [0,T]$ and $|x|,|y| \le N$

$$|a(t,x) - a(t,y)| \le K_N|x-y|,$$

and there exists $K > 0$ such that for any $t \in [0,T]$ and $x, y \in \mathbb{R}$

$$|b(t,x) - b(t,y)| \le K|x-y|,$$

(A_3) Hölder continuity in time: function $b(t,x)$ is differentiable in x and there exists $\beta \in (1-H,1)$ such that for any $s,t \in [0,T]$ and any $x \in \mathbb{R}$

$$|a(s,x)-a(t,x)| + |b(s,x)-b(t,x)| + |\partial_x b(s,x) - \partial_x b(t,x)| \le K|s-t|^\beta.$$

(A_4) Hölder continuity of $\partial_x b(t,x)$ in space: there exists such $\delta \in (\frac{1}{H}-1, 1)$ that for any $t \in [0,T]$ and $x,y \in \mathbb{R}$

$$|\partial_x b(t,x) - \partial_x b(t,y)| \le D|x-y|^\delta,$$

Also, let

$$\alpha_0 = \frac{1}{2} \wedge \beta \wedge \frac{\delta}{1+\delta}.$$

Then according to Theorem 2.1 from [7], if $\alpha \in (1-H, \alpha_0)$ and $\rho \ge \frac{1}{\alpha}$, there exists unique solution of equation (3) with trajectories a.s. belonging to the space $W_{\alpha,0,T,\infty}$, and moreover with a.s. Hölder trajectories up to order H.

Now, suppose that the following assumption holds:

(B_1) $b(t,X_t) \ne 0$, $t \in [0,T]$ and $\frac{a(t,X_t)}{b(t,X_t)}$ is a.s. Lebesgue integrable on $[0,T]$ for any $T > 0$.

Denote $\psi(t,x) = \frac{a(t,x)}{b(t,x)}$, $\varphi(t) := \psi(t, X_t)$ and introduce the process

$$Y_t = \int_0^t b^{-1}(s, X_s)dX_s = \theta \int_0^t \varphi(s)ds + B_t^H.$$

According to [5], where different estimators of unknown drift parameter θ were created and compared, we consider the following estimator of θ, that is simple in its structure and is easy to discretize, but needs some additional assumptions to be strongly consistent. Similar approach was applied in [3] to the fractional Ornstein-Uhlenbeck process with constant coefficients. More precisely, consider the estimator

$$\theta_T = \frac{\int_0^T \varphi(s)dY_s}{\int_0^T \varphi^2(s)ds} = \theta + \frac{\int_0^T \varphi(s)dB_s^H}{\int_0^T \varphi^2(s)ds}. \tag{4}$$

Theorem 3.1. [5] *Let assumptions (A_1)–(A_4) and (B_1) hold and let function φ satisfy the following assumption: There exists such $\alpha > 1 - H$ and $p > 1$ that*

$$\varrho_{\alpha,p,T} := \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}_{0+}^\alpha \varphi)(s)|ds}{\int_0^T \varphi_s^2 ds} \to 0 \quad a.s. \ as \quad T \to \infty. \tag{5}$$

Then estimate θ_T is correctly defined and strongly consistent as $T \to \infty$.

Proof of Theorem 3.1 is based on the representation (4) and is reduced to the estimation of the remainder term $\varrho_T = \frac{\int_0^T \varphi(s)dB_s^H}{\int_0^T \varphi^2(s)ds}$. More precisely, it is proved that under condition (5), the remainder term tends to zero a.s., because

$$\Big|\int_0^T \varphi(s)dB_s^H\Big| \le \sup_{0\le t\le T} |(\mathcal{D}_{T-}^{1-\alpha} B_{T-}^H)(t)| \int_0^T |(\mathcal{D}_{0+}^\alpha \varphi)(s)|ds,$$

and furthermore, for any $p > 1$ there exists a non-negative random variable $\xi = \xi(p)$ independent of T such that for any $T > 0$

$$\sup_{0\le t\le T} |(\mathcal{D}_{T-}^{1-\alpha} B_{T-}^H)(t)| \le \xi(p) T^{H+\alpha-1}(\log T)^p.$$

Summarizing, for $T > 1$

$$\begin{aligned} |\varrho_T| &\le \left| \frac{\xi(p)T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}_{0+}^\alpha \varphi)(s)|ds}{\int_0^T \varphi^2(s)ds} \right| \\ &= \frac{\xi(p)T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}_{0+}^\alpha \varphi)(s)|ds}{\int_0^T \varphi^2(s)ds} = \xi(p)\varrho_{\alpha,p,T} \end{aligned} \tag{6}$$

a.s. Generally speaking, assumption (5) is not very easy to check. However, the situation is simplified substantially if the function φ is non-random. In this case it is possible, for the selected classes of φ, to establish not only the convergence to zero, but the rate of convergence as well.

4. Examples of the remainder terms with the estimation of the rate of convergence to zero

We start with the simplest case when φ is a power function, $\varphi(t) = t^a$, $a \ge 0, t \ge 0$. It means that $a(t,x) = b(t,x)t^a$. If the coefficient $b(t,x)$ satisfies assumptions (A_1)–(A_4) and $b(t, X_t) \ne 0, t \in [0,T]$, then $a(t,x)$ satisfies assumptions (A_1)–(A_4) on any interval $[0,T]$, condition (B_1) holds, then the main equation has the unique solution, the estimator θ_T is correctly defined and we can study the properties of the remainder term $\varrho_{\alpha,p,T}$.

Lemma 4.1. *Let $\varphi(t) = t^a$, $a \geq 0, t \geq 0$. Then $\varrho_{\alpha,p,T} = C_a T^{H-a-1}(\log T)^p \to 0$ as $T \to \infty$, where*

$$C_a = \frac{(2a+1)\Gamma(a+1)}{\Gamma(a-\alpha+2)}.$$

Proof. Let $B(\alpha, \beta)$ stands for Beta function. For the power function $\varphi(t) = t^a$ we can use the definition of fractional derivative and get that for any $0 < \alpha < 1$

$$(\mathcal{D}^\alpha_{0+}\varphi)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x t^a (x-t)^{-\alpha} dt$$
$$= B(a+1, 1-\alpha) \frac{1}{\Gamma(1-\alpha)} \left(x^{a+1-\alpha}\right)' = \frac{\Gamma(a+1)}{\Gamma(a+1-\alpha)} x^{a-\alpha}.$$

Therefore, for $p > 1$ we have that

$$\varrho_{\alpha,p,T} = \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}^\alpha_{0+}\varphi)(x)| dx}{\int_0^T \varphi^2(x) dx} = \frac{T^{H+\alpha-1}(\log T)^p \int_0^T \frac{\Gamma(a+1)}{\Gamma(a+1-\alpha)} x^{a-\alpha} dx}{\frac{T^{2a+1}}{2a+1}}$$
$$= \frac{(2a+1)\Gamma(a+1)}{\Gamma(a+2-\alpha)} T^{H-a-1}(\log T)^p.$$

Thus

$$\varrho_{\alpha,p,T} = C_a \frac{(\log T)^p}{T^{a+1-H}} \to 0 \text{ as } T \to +\infty.$$

Remark 4.1. As to the rate of convergence to zero, we can say that

$$\varrho_{\alpha,p,T} = O(T^{H-1-a+\varepsilon})$$

as $T \to \infty$ for any $\varepsilon > 0$.

Now, we can consider φ that is a polynomial function. In this case, similarly to monomial case, the solution of the equation (3) exists and is unique, and the estimator is correctly defined. As an immediate generalization of the Lemma 4.1, we get the following statement.

Lemma 4.2. *Let $N \in \mathbb{N} \setminus \{0\}$ and $\varphi_N(t) = \sum_{k=0}^{N} \alpha_k t^{a_k}$, $t \geq 0$, (a_k) be a sequence of non-negative power coefficients, $0 \leq a_0 < a_1 < ... < a_N$, and (α_k) be a sequence of non-negative coefficients, $\alpha_N > 0$. Then $\varrho_{\alpha,p,T} \to 0$ as $T \to \infty$, and the rate of convergence to zero is $\varrho_{\alpha,p,T} = O(T^{H-1-a_N+\varepsilon})$ for any $\varepsilon > 0$.*

Proof. The linearity of the operator $\mathcal{D}^\alpha_{0+}$ implies that

$$|\ (\mathcal{D}^\alpha_{0+}\varphi_N)(x)\ | \leq \sum_{k=0}^{N} \alpha_k \mid \mathcal{D}^\alpha_{0+}(x^{a_k}) \mid = \sum_{k=0}^{N} \alpha_k \frac{\Gamma(a_k+1)}{\Gamma(a_k+1-\alpha)} \mid x \mid^{a_k-\alpha}.$$

So, on one hand, similarly to the calculations performed in the proof of the Lemma 4.1, we get

$$\begin{aligned}
\varrho_{\alpha,p,T} &= \left(T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}^{\alpha}_{0+}\varphi_N)(x)|dx\right)\left(\int_0^T \varphi_N^2(x)dx\right)^{-1}\\
&\le \left(T^{H+\alpha-1}(\log T)^p \int_0^T \sum_{k=0}^N \tfrac{\alpha_k\Gamma(a_k+1)}{\Gamma(a_k+1-\alpha)} \mid x\mid^{a_k-\alpha}\, dx\right)\left(\int_0^T \varphi_N^2(x)dx\right)^{-1}\\
&= \left(T^{H+\alpha-1}(\log T)^p \sum_{k=0}^N \tfrac{\alpha_k\Gamma(a_k+1)}{\Gamma(a_k+1-\alpha)(a_k-\alpha+1)} T^{a_k-\alpha+1}\right)\left(\int_0^T \varphi_N^2(x)dx\right)^{-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_0^T \varphi_N^2(x)dx &= \int_0^T \left(\sum_{k=0}^N \alpha_k^2 t^{2a_k} + \sum_{k\neq j}\alpha_k\alpha_j t^{a_k}t^{a_j}\right)dt\\
&= \sum_{k=0}^N \frac{\alpha_k^2}{2a_k+1}T^{2a_k+1} + \sum_{k\neq j}\frac{\alpha_k\alpha_j}{a_k+a_j+1}T^{a_k+a_j+1}\\
&\sim \frac{\alpha_N^2}{2a_N+1}T^{2a_N+1} \quad as\ \ T\to\infty,
\end{aligned}$$

and

$$\sum_{k=0}^N \alpha_k \frac{\Gamma(a_k+1)}{\Gamma(a_k+1-\alpha)(a_k-\alpha+1)}T^{a_k-\alpha+1} \sim \frac{\alpha_N\Gamma(a_N+1)}{\Gamma(a_N+2-\alpha)}T^{a_N-\alpha+1} \quad as\ \ T\to\infty,$$

whence

$$\varrho_{\alpha,p,T} \sim T^{H-a_N-1}(\log T)^p \frac{(2a_N+1)\Gamma(a_N+1)}{\Gamma(a_N+2-\alpha)\alpha_N} \to 0, \quad as\ \ T\to\infty.$$

We clearly see that $\varrho_{\alpha,p,T} = O(T^{H-a_N-1+\varepsilon})$ as $T\to\infty$, for any $\varepsilon>0$. □

Now consider the case of trigonometric function.

Lemma 4.3. *Let $\varphi(t)=\sin(\lambda t)$, $\lambda\ge 0$. Then estimator θ_T is strongly consistent as $T\to\infty$.*

Proof. In this case we apply Theorem 3.2 from [5]. According to this result, if there exists such $p>0$ that

$$\limsup_{T\to\infty} \frac{T^{2H-1+p}}{\int_0^T \varphi^2(t)dt} < \infty,$$

then the estimator θ_T is strongly consistent as $T\to\infty$. In our case

$$\int_0^T \varphi^2(t)dt = \int_0^T \sin^2(\lambda t)dt = \frac{1}{2}\Big(T - \frac{\sin(2\lambda T)}{2\lambda}\Big),$$

which means that we can choose any $0<p<2-2H$, and the proof follows. □

Remark 4.2. We see that in the case of power and polynomial functions (Remark 4.1 and Lemma 4.2) we can get not only convergence to zero but the rate of convergence, but in the case of trigonometric function we only get convergence. The difference can be seen from the following result.

Lemma 4.4. *Let* $\varphi(t) = \sin(\lambda t)$, $\lambda \geq 0$. *Then*

$$\lim_{T\to+\infty} \varrho_{\alpha,p,T} = \lim_{T\to+\infty} \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}^\alpha_{0+}\varphi)(x)|dx}{\int_0^T \varphi^2(x)dx} = +\infty.$$

Proof. First, consider the fractional derivative. Since φ is absolutely continuous on the interval $[0, T]$, for any $T > 0$, by Lemma 2.2 in [8], we have that

$$\begin{aligned}
(\mathcal{D}^\alpha_{0+}\varphi)(x) &= \frac{1}{\Gamma(1-\alpha)}\left[\frac{\varphi(0)}{(x-0)^\alpha} + \int_0^x \frac{\varphi'(s)\mathrm{d}s}{(x-s)^\alpha}\right] \\
= \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{\lambda\cos(\lambda s)}{(x-s)^\alpha}ds &= \frac{\lambda}{\Gamma(1-\alpha)}\int_0^x \frac{\cos(\lambda(x-v))}{v^\alpha}dv \\
= \frac{\lambda}{\Gamma(1-\alpha)}&\left[\cos(\lambda x)\int_0^x \frac{\cos(\lambda v)}{v^\alpha}dv + \sin(\lambda x)\int_0^x \frac{\sin(\lambda v)}{v^\alpha}dv\right] \\
&= \frac{\lambda}{\Gamma(1-\alpha)}\left[\cos(\lambda x)J_1(x) + \sin(\lambda x)J_2(x)\right],
\end{aligned} \tag{7}$$

where

$$J_1(x) = \int_0^x \frac{\cos(\lambda v)}{v^\alpha}dv, \quad \text{and} \quad J_2(x) = \int_0^x \frac{\sin(\lambda v)}{v^\alpha}dv.$$

According to [2], p.893, for any $0 < \alpha < 1$ and $\lambda > 0$ we have that

$$\int_0^\infty \cos(\lambda t)t^{-\alpha}dt = \Gamma(1-\alpha)\sin\left(\frac{\pi\alpha}{2}\right)\lambda^{\alpha-1} > 0,$$

and

$$\int_0^\infty \sin(\lambda t)t^{-\alpha}dt = \Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\lambda^{\alpha-1} > 0.$$

Denote

$$J(\alpha,\lambda) = \left(\Gamma(1-\alpha)\sin\left(\frac{\pi\alpha}{2}\right)\lambda^{\alpha-1}\right) \wedge \left(\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\lambda^{\alpha-1}\right) > 0.$$

Then there exists $x_0 > 0$ such that for any $x > x_0$ $J_i(x) > \frac{J(\alpha,\lambda)}{2}$, $i = 1, 2$. Additionally, for $x \in [\frac{2\pi k}{\lambda}, \frac{2\pi k + \frac{\pi}{2}}{\lambda}]$, with $k > \left(1 \vee \left(\frac{\lambda x_0}{2\pi}\right)\right)$ we have that

$$\begin{aligned}
|(\mathcal{D}^\alpha_{0+}\varphi)(x)| = \left|\frac{\lambda}{\Gamma(1-\alpha)}\left[\cos(\lambda x)J_1(x) + \sin(\lambda x)J_2(x)\right]\right| &\geq \frac{\lambda}{\Gamma(1-\alpha)}\frac{J(\alpha,\lambda)}{2}(\cos(\lambda x) \\
+\sin(\lambda x)) \geq \frac{\lambda}{\Gamma(1-\alpha)}\frac{J(\alpha,\lambda)}{2}(\cos^2(\lambda x) + \sin^2(\lambda x)) &= \frac{\lambda}{\Gamma(1-\alpha)}\frac{J(\alpha,\lambda)}{2} \\
= \frac{\lambda^\alpha}{2}\sin(\frac{\pi\alpha}{2}) \wedge \cos(\frac{\pi\alpha}{2}) &=: J_1(\alpha,\lambda).
\end{aligned} \tag{8}$$

Now, consider $I(T) = \displaystyle\int_0^T \left|(\mathcal{D}^\alpha_{0+}\varphi)(x)\right|dx$. Evidently, for

$$T > \left(\frac{5\pi}{2\lambda}\right) \vee \left(x_0 + \frac{\pi}{2\lambda}\right) \quad \text{and} \quad \left(1 \vee \left(\frac{\lambda x_0}{2\pi}\right)\right) < k < \frac{\lambda T}{2\pi} - \frac{1}{4}$$

we have from (8) that

$$\int_{\frac{2\pi k}{\lambda}}^{\frac{2\pi k + \frac{\pi}{2}}{\lambda}} \left|(D^\alpha_{0+}\varphi)(x)\right|dx > \frac{\pi}{2\lambda}J_1(\alpha,\lambda).$$

Finally, denoting $[a]$ the entire part of the number a, we get for $T > \left(\frac{5\pi}{2\lambda}\right) \vee \left(x_0 + \frac{\pi}{2\lambda}\right)$ the following bound from below:

$$I(T) \geq \sum_{k=\left[\left(1\vee\left(\frac{\lambda x_0}{2\pi}\right)\right)\right]+1}^{\left[\frac{\lambda T}{2\pi}-\frac{1}{4}\right]} \int_{\frac{2\pi k}{\lambda}}^{\frac{2\pi k+\frac{\pi}{2}}{\lambda}} \left|(\mathcal{D}_{0+}^{\alpha}\varphi)(x)\right| dx$$
$$> \left(\left[\frac{\lambda T}{2\pi} - \frac{1}{4}\right] - \left[\left(1 \vee \left(\frac{\lambda x_0}{2\pi}\right)\right)\right]\right) \frac{\pi}{2\lambda} J_1(\alpha, \lambda) \sim CT$$

as $T \to \infty$ for some $C > 0$. Recall that $\alpha + H > 1$ and we immediately get that

$$\lim_{T\to+\infty} \varrho_{\alpha,p,T} \geq \lim_{T\to+\infty} \frac{T^{H+\alpha-1}(\log T)^p CT}{T - \frac{\sin(2\lambda T)}{2\lambda}} = +\infty,$$

and lemma is proved. □

Remark 4.3. Note for completeness that for $\frac{T^{H+\alpha-1}(\log T)^p \int_0^T (\mathcal{D}_{0+}^{\alpha}\varphi)(x)dx}{\int_0^T \varphi_x^2 dx}$ situation is different, more precisely,

$$\lim_{T\to+\infty} \frac{T^{H+\alpha-1}(\log T)^p \int_0^T (\mathcal{D}_{0+}^{\alpha}\varphi)(x)dx}{\int_0^T \varphi^2(x)dx} = 0.$$

Indeed, it follows from (7) and from Fubini theorem that

$$\begin{aligned}
\int_0^T (\mathcal{D}_{0+}^{\alpha}\varphi)(x)dx &= \frac{\lambda}{\Gamma(1-\alpha)} \left(\int_0^T \frac{\cos(\lambda v)}{v^{\alpha}} \int_v^T \cos(\lambda x)dx \right. \\
&\quad \left. + \int_0^T \frac{\sin(\lambda v)}{v^{\alpha}} \int_v^T \sin(\lambda x)dx \right) \\
&= \frac{1}{\Gamma(1-\alpha)} \left(\int_0^T \frac{\cos(\lambda v)}{v^{\alpha}} (\sin(\lambda T) - \sin(\lambda v))\, dv \right. \\
&\quad \left. + \int_0^T \frac{\sin(\lambda v)}{v^{\alpha}} (\cos(\lambda v) - \cos(\lambda T))\, dv \right),
\end{aligned} \tag{9}$$

and all integrals in the right-hand side of (9) obviously are bounded. Therefore,

$$\lim_{T\to+\infty} \left| \frac{T^{H+\alpha-1}(\log T)^p \int_0^T (\mathcal{D}_{0+}^{\alpha}\varphi)(x)dx}{\int_0^T \varphi^2(x)dx} \right| \leq \lim_{T\to+\infty} CT^{H+\alpha-2}(\log T)^p = 0.$$

Lemma 4.5. *Let $\varphi(t) = \exp(-\lambda t)$, $\lambda > 0$. Then*

$$\lim_{T\to+\infty} \varrho_{\alpha,p,T} = \lim_{T\to+\infty} \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}_{0+}^{\alpha}\varphi)(x)|dx}{\int_0^T \varphi^2(x)dx} = 0.$$

Proof. Note that the function

$$G(x) = \int_0^x e^{-\lambda t}(x-t)^{-\alpha}dt = e^{-\lambda x} \int_0^x e^{\lambda t} t^{-\alpha} dt$$

has the derivative

$$\begin{aligned}
g(x) = G'(x) = x^{-\alpha} - \lambda e^{-\lambda x} \int_0^x e^{\lambda z} z^{-\alpha} dz &= e^{-\lambda x}\Big(e^{\lambda x} x^{-\alpha} \\
-\lambda \int_0^x e^{\lambda z} z^{-\alpha} dz \Big) &=: e^{-\lambda x} g_1(x).
\end{aligned}$$

Obviously, $\lim_{x\to 0} g(x) = +\infty$, $\lim_{x\to+\infty} g(x) = 0$ and the derivative

$$g_1'(x) = -\alpha e^{\lambda x} x^{-1-\alpha} < 0.$$

It means that g is the product of two decreasing non-negative functions, so, it is decreasing and therefore is non-negative. Then it follows that

$$\begin{aligned}\int_0^T |(\mathcal{D}_{0+}^{\alpha}\varphi)(x)|dx &= \int_0^T (\mathcal{D}_{0+}^{\alpha}\varphi)(x)dx = \frac{1}{\Gamma(1-\alpha)}\int_0^T e^{-\lambda t}(T-t)^{-\alpha}dt \\ &= \frac{1}{\Gamma(1-\alpha)} e^{-\lambda T}\int_0^T e^{\lambda t}t^{-\alpha}dt.\end{aligned} \tag{10}$$

Calculate

$$\lim_{T\to+\infty} \frac{\int_0^T e^{\lambda t}t^{-\alpha}dt}{e^{\lambda T}T^{-\alpha}} = \lim_{T\to+\infty} \frac{e^{\lambda T}T^{-\alpha}}{\lambda e^{\lambda T}T^{-\alpha} - \alpha e^{\lambda T}T^{-\alpha-1}} = \frac{1}{\lambda}.$$

Moreover, $\displaystyle\int_0^T \varphi^2(x)dx = \frac{1-e^{-2\lambda T}}{2\lambda} \sim \frac{1}{2\lambda}$ as $T\to\infty$. Therefore

$$\lim_{T\to+\infty} \varrho_{\alpha,p,T} = \lim_{T\to+\infty} \frac{2}{\Gamma(1-\alpha)} T^{H-1}(\log T)^p = 0.$$

Lemma is proved. □

Remark 4.4. It is easy to deduce from the previous calculations that in the latter case

$$\varrho_{\alpha,p,T} = O(T^{H-1+\varepsilon})$$

as $T\to\infty$ for any $\varepsilon > 0$.

Lemma 4.6. *Let $\varphi(t) = \exp(\lambda t)$, $\lambda > 0$. Then*

$$\lim_{T\to+\infty} \varrho_{\alpha,p,T} = \lim_{T\to+\infty} \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}_{0+}^{\alpha}\varphi)(x)|dx}{\int_0^T \varphi^2(x)dx} = 0.$$

Proof. It is easy to check that for every $x > 0$ we have the relations

$$\left(\mathcal{D}_{0+}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(1-\alpha)}\left(x^{-\alpha} + \lambda e^{\lambda x}\int_0^x e^{-\lambda z}z^{-\alpha}dz\right) \geq 0.$$

Since for any $T > 0$ $\int_0^T e^{-\lambda t}t^{-\alpha}dt \leq \lambda^{\alpha-1}\Gamma(1-\alpha)$, then it follows that

$$\begin{aligned}\int_0^T |(\mathcal{D}_{0+}^{\alpha}\varphi)(x)|dx &= \int_0^T (\mathcal{D}_{0+}^{\alpha}\varphi)(x)dx = \frac{1}{\Gamma(1-\alpha)}\int_0^T e^{\lambda t}(T-t)^{-\alpha}dt \\ &= \frac{1}{\Gamma(1-\alpha)} e^{\lambda T}\int_0^T e^{-\lambda t}t^{-\alpha}dt \leq \lambda^{\alpha-1}e^{\lambda T}.\end{aligned} \tag{11}$$

Thus,

$$\varrho_{\alpha,p,T} \quad = \quad \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}_{0+}^{\alpha}\varphi)(x)|dx}{\int_0^T \varphi^2(x)dx} \leq \frac{\lambda^{\alpha-1}T^{H+\alpha-1}(\log T)^p e^{\lambda T}}{\int_0^T \varphi^2(x)dx}.$$

Moreover, $\int_0^T \varphi^2(x)dx = \frac{e^{2\lambda T}-1}{2\lambda} \sim \frac{1}{2\lambda}e^{2\lambda T}$ as $T\to\infty$. Therefore

$$\lim_{T\to+\infty} \varrho_{\alpha,p,T} = \lim_{T\to+\infty} \frac{2\lambda^{\alpha}T^{H+\alpha-1}(\log T)^p}{e^{\lambda T}} = 0.$$

Lemma is proved. □

Remark 4.5. In this case it is easy to deduce from the previous calculations that

$$\varrho_{\alpha,p,T} = O(e^{-(\lambda-\varepsilon)T}) = o(T^{-\epsilon})$$

as $T \to \infty$ for any $\varepsilon > 0$.

Lemma 4.7. *Let $\varphi(t) = \log(1+t)$. Then*

$$\lim_{T\to+\infty} \rho_{\alpha,p,T} = \lim_{T\to+\infty} \frac{T^{H+\alpha-1}(\log T)^p \int_0^T |(\mathcal{D}^\alpha_{0+}\varphi)(x)|dx}{\int_0^T \varphi^2(x)dx} = 0.$$

Proof. By integration by parts, it is easy to get that for every $x > 0$

$$\left(\mathcal{D}^\alpha_{0+}\varphi\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{(x-z)^{-\alpha}}{1+z} dz \geq 0.$$

Thus,

$$\begin{aligned} \int_0^T \mid (\mathcal{D}^\alpha_{0+}\varphi)(x) \mid dx &= \int_0^T (\mathcal{D}^\alpha_{0+}\varphi)(x)dx = \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^x \frac{(x-z)^{-\alpha}}{1+z} dzdx \\ = \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_z^T (x-z)^{-\alpha} dx \frac{1}{1+z} dz &= \frac{1}{\Gamma(2-\alpha)} \int_0^T (T-z)^{1-\alpha} \frac{1}{1+z} dz \\ &\leq \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^T \frac{1}{1+z} dz = \frac{T^{1-\alpha}\log(1+T)}{\Gamma(2-\alpha)}. \end{aligned}$$

On the other hand we have that

$$\begin{aligned} \int_0^T \varphi^2(t)dt &= \int_0^T \log^2(1+t)dt \\ &= (T+1)\log^2(1+T) - 2(T+1)\log(1+T) + 2T \\ &\sim T(\log T)^2 \end{aligned}$$

as $T \to \infty$.

Therefore

$$\begin{aligned} \rho_{\alpha,p,T} &\leq \frac{T^H(\log T)^p \log(T+1)}{\Gamma(2-\alpha)((T+1)\log^2(1+T) - 2(T+1)\log(1+T) + 2T)} \\ &\sim \frac{T^{H-1}(\log T)^{p-1}}{\Gamma(2-\alpha)} \; as\; T \to \infty, \end{aligned}$$

which allows to deduce that $\lim_{T\to\infty} \rho_{\alpha,p,T} = 0$. □

Remark 4.6. In this case

$$\varrho_{\alpha,p,T} = O(T^{H-1+\varepsilon})$$

as $T \to \infty$ for any $\varepsilon > 0$.

Now, we illustrate our results by some simulations. For some fixed step $h = 0.005$, we simulate 10 paths of the process Y on the interval $[0,T]$, for different values of T, with $\theta = 1$ then $\theta = -1$, $H = 0.6$ then $H = 0.75$, and with some polynomial, logarithmic, trigonometric and exponential particular expressions of φ. Simulated results for unknown parameter θ are given in the tables below.

From these tables we see that with increasing of T the estimator tends to the real value of θ. This clearly illustrates the strong consistency of our estimator. In the particular case of logarithmic form of φ, it is obvious that the rate of convergence to the true value of θ is not very high.

Table 1. $\theta = 1$

H	$a(t,x)$	$b(x)$	T				
			50	100	300	500	1000
0.6	$t^2\sqrt{x^2+1}$	$\sqrt{x^2+1}$	1.0043	1.00169	1.00047	1.00027	1.00013
	$e^t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	1.00501	1.00501	1.00501	1.00501	1.00501
	$\sin t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	0.97704	1.00815	1.04516	1.04630	1.00408
	$\ln(1+t)(2+\sin x)$	$2+\sin x$	1.19095	1.13984	1.09693	1.08801	1.07777
	$\cos t(2+\sin x)$	$2+\sin x$	0.88322	0.96351	0.98973	1.00807	1.00474
0.75	$t^2\sqrt{x^2+1}$	$\sqrt{x^2+1}$	1.00363	1.00153	1.00045	1.00026	1.00013
	$e^t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	1.00501	1.00501	1.00501	1.00501	1.00501
	$\sin t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	1.03370	1.01917	1.02307	1.02432	1.00383
	$\ln(1+t)(2+\sin x)$	$2+\sin x$	1.18920	1.14153	1.09654	1.08931	1.07922
	$\cos t(2+\sin x)$	$2+\sin x$	0.84565	0.92377	0.98510	0.99827	1.00263

Table 2. $\theta = -1$

H	$a(t,x)$	$b(x)$	T				
			50	100	300	500	1000
0.6	$t^2\sqrt{x^2+1}$	$\sqrt{x^2+1}$	−1.00076	−1.00081	−1.00037	−1.00023	−1.00012
	$e^t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	−1.00501	−1.00501	−1.00501	−1.00501	−1.00501
	$\sin t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	−1.00951	−1.00288	−0.95999	−0.94766	−0.98908
	$\ln(1+t)(2+\sin x)$	$2+\sin x$	−0.88573	−0.90086	−0.90960	−0.91602	−0.92648
	$\cos t(2+\sin x)$	$2+\sin x$	−1.08132	−1.01240	−0.99469	−1.00395	−1.00607
0.75	$t^2\sqrt{x^2+1}$	$\sqrt{x^2+1}$	−1.00137	−1.00097	−1.00039	−1.00024	−1.00012
	$e^t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	−1.00501	−1.00501	−1.00501	−1.00501	−1.00501
	$\sin t\sqrt{x^2+1}$	$\sqrt{x^2+1}$	−1.00180	−0.99471	−0.97423	−0.97601	−1.00121
	$\ln(1+t)(2+\sin x)$	$2+\sin x$	−0.89430	−0.90122	−0.91074	−0.91552	−0.92564
	$\cos t(2+\sin x)$	$2+\sin x$	−1.08129	−1.07844	−1.05366	−1.02158	−1.01580

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Faculty of Sciences of Monastir, Department of Mathematics, Avenue de l'Environnement, 5000, Monastir, Tunisia
E-mail address: meriem.bhk17121988@outlook.fr

01601 Ukraine Kyiv Volodymyrska 64 Taras Shevchenko National University of Kyiv, Department of Probability, Statistics and Actuarial Mathematics
E-mail address: myus@univ.kiev.ua

Faculty of Sciences of Monastir, Department of Mathematics, Avenue de l'Environnement, 5000, Monastir, Tunisia
E-mail address: Mounir.Zili@fsm.rnu.tn

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