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# NECESSARY AND SUFFICIENT CONDITIONS FOR CONVERGENCE OF FIRST-RARE-EVENT-TIME PROCESSES FOR PERTURBED SEMI-MARKOV PROCESSES

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Abstract. Necessary and sufficient conditions for convergence in distribution of first-rare-event times and convergence in Skorokhod J-topology of first-rare-event-time processes for perturbed semi-Markov processes with finite phase space are obtained.

## 1. Introduction

Random functionals similar with first-rare-event times are known under different names such as first hitting times, first passage times, absorption times, in theoretical studies, and as lifetimes, first failure times, extinction times, etc., in applications. Limit theorems for such functionals for Markov-type processes have been studied by many researchers.

The main features for the most previous results is that they give sufficient conditions of convergence for such functionals. As a rule, those conditions involve assumptions, which imply convergence in distribution for sums of i.i.d. random variables distributed as inter-jump times to some infinitely divisible laws plus some ergodicity condition for the imbedded Markov chain plus condition of vanishing probabilities of occurring a rare event during one transition step for the corresponding processes.

There is a huge bibliography of works related to limit theorems for first-rare-event times and related functionals. Here, I would like to refer to the originating paper by Korolyuk (1969), where convergence in distribution for hitting times to an exponentially distributed random variable has been proved for linearly perturbed finite Markov chains and, then, to mention books, which contain results of further theoretical and applied studies in the area. These are, Silvestrov (1974, 1980), Korolyuk and Turbin (1976, 1978), Keilson (1979), Anisimov (1988, 2008), Korolyuk and Swishchuk (1992), Kartashov (1996), Kalashnikov (1997), Korolyuk V.S. and Korolyuk V.V. (1999), Asmussen (2003), Koroliuk and Limnios (2005), Gyllenberg and Silvestrov (2008), and Asmussen and Albrecher (2010). I also refer to the books by Silvestrov (2004) and Gyllenberg and Silvestrov (2008) and papers by Kovalenko (1994) and Silvestrov D. and Silvestrov S. (2016), where one can find comprehensive bibliographies of works in the area.

In the context of necessary and sufficient conditions of convergence in distribution for first-rare-event-time type functionals, we would like to point out the paper by Kovalenko (1965) and the books by Gnedenko and Korolev (1996) and Bening and Korolev (2002), where one can find some related results for geometric sums of random variables, and the papers by Korolyuk, D. and Silvestrov (1983, 1984) and Silvestrov and Velikii (1988),

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where one can find some related results for first-rare-event-time type functionals defined on Markov chains and semi-Markov processes with arbitrary phase space.

The results of the present paper relate to the model of perturbed semi-Markov processes with a finite phase space. Instead of conditions based on "individual" distributions of inter-jump times, more general and weaker conditions imposed on distributions sojourn times averaged by stationary distributions of the corresponding imbedded Markov chains are used. Moreover, it is shown that these conditions are not only sufficient but also necessary conditions for convergence in distribution of first-rare-event times and convergence in Skorokhod J-topology of first-rare-event-time processes. These results give some kind of a "final solution" for limit theorems for first-rare-event-time processes for perturbed semi-Markov process with a finite phase space.

The paper generalize and improve results concerned necessary and sufficient conditions of convergence in distribution for first-rare-event times for semi-Markov processes obtained in papers by Silvestrov and Drozdenko (2006) and Drozdenko (2007, 2009).

First, a weaker model ergodic condition is imposed on the corresponding embedded Markov chains. Second, the results of the above papers about convergence in distribution for first-rare-event times are extended, in Theorem 1, to the form of corresponding functional limit theorem for first-rare-event-time processes, with necessary and sufficient conditions of convergence. Third, new proofs, based on general limit theorems for randomly stopped stochastic processes, developed and extensively presented in Silvestrov (2004), are given, instead of more traditional proofs based on cyclic representations of first-rare-event times if the form of geometric-type random sums. This actually made it possible to get more advanced results in the form of functional limit theorem. I would like also to mention Lemmas 1 - 8, which give some useful supplementary information about asymptotic properties of first-rare-event-time processes and step-sum reward processes.

I would like to conclude the introduction with the remark that the present paper is a shorten version of the research report by Silvestrov (2016), where one can find some additional details of proofs, comments and references.

## 2. Main results

Let $(\eta_{\varepsilon,n}, \kappa_{\varepsilon,n}, \zeta_{\varepsilon,n})$, $n = 0, 1, \dots$ be, for every $\varepsilon \in (0, \varepsilon_0]$, a Markov renewal process, i.e., a homogenous Markov chain with a phase space $\mathbb{Z} = \{1, 2, \dots, m\} \times [0, \infty) \times \{0, 1\}$, an initial distribution $\bar{q}_\varepsilon = \langle q_{\varepsilon,i} = \mathsf{P}\{\eta_{\varepsilon,0} = i, \kappa_{\varepsilon,0} = 0, \zeta_{\varepsilon,0} = 0\} = \mathsf{P}\{\eta_{\varepsilon,0} = i\}, i \in \mathbb{X}\rangle$ and transition probabilities,

$$\begin{aligned}
&\mathsf{P}\{\eta_{\varepsilon,n+1} = j, \kappa_{\varepsilon,n+1} \leq t, \zeta_{\varepsilon,n+1} = \jmath / \eta_{\varepsilon,n} = i, \xi_{\varepsilon,n} = s, \zeta_{\varepsilon,n} = \imath\} \\
&\quad = \mathsf{P}\{\eta_{\varepsilon,n+1} = j, \kappa_{\varepsilon,n+1} \leq t, \zeta_{\varepsilon,n+1} = \jmath / \eta_{\varepsilon,n} = i\} \\
&\quad = Q_{\varepsilon,ij}(t, \jmath), \ i, j \in \mathbb{X}, \ s, t \geq 0, \ \imath, \jmath = 0, 1.
\end{aligned} \tag{1}$$

As it is known, the first component $\eta_{\varepsilon,n}$ of the above Markov renewal process is also a homogenous Markov chain, with the phase space $\mathbb{X} = \{1, 2, \dots, m\}$, the initial distribution $\bar{q}_\varepsilon = \langle q_{\varepsilon,i} = \mathsf{P}\{\eta_{\varepsilon,0} = i\}, i \in \mathbb{X}\rangle$ and the transition probabilities, $p_{\varepsilon,ij} = Q_{\varepsilon,ij}(+\infty, 0) + Q_{\varepsilon,ij}(+\infty, 1)$, $i, j \in \mathbb{X}$.

Random variables $\kappa_{\varepsilon,n}$, $n = 1, 2, \dots$ can be interpreted as sojourn times and random variables $\tau_{\varepsilon,n} = \kappa_{\varepsilon,1} + \cdots + \kappa_{\varepsilon,n}$, $n = 1, 2, \dots, \tau_{\varepsilon,0} = 0$ as moments of jumps for a semi-Markov process $\eta_\varepsilon(t)$, $t \geq 0$ defined by the following relation, $\eta_\varepsilon(t) = \eta_{\varepsilon,n}$ for $\tau_{\varepsilon,n} \leq t < \tau_{\varepsilon,n+1}$, $n = 0, 1, \dots$.

As far as random variables $\zeta_{\varepsilon,n}$, $n = 1, 2, \dots$ are concerned, they are interpreted as so-called, "flag variables" using to record events $\{\zeta_{\varepsilon,n} = 1\}$ which can be interpreted as "rare" events. Let us introduce first-rare-event-time process,

$$\xi_\varepsilon(t) = \sum_{n=1}^{[t\nu_\varepsilon]} \kappa_{\varepsilon,n}, \ t \geq 0, \text{ where } \nu_\varepsilon = \min(n \geq 1 : \zeta_{\varepsilon,n} = 1). \tag{2}$$

In the present paper, we describe class $\mathcal{F}$ of all possible càdlàg processes $\xi_0(t)$, $t \geq 0$, which can appear in the corresponding functional limit theorem given in the form of the asymptotic relation, $\xi_\varepsilon(t)$, $t \geq 0 \stackrel{\mathsf{J}}{\longrightarrow} \xi_0(t)$, $t \geq 0$ as $\varepsilon \to 0$, and give necessary and sufficient conditions for holding of the above asymptotic relation with the specific (by its finite-dimensional distributions) limiting stochastic process $\xi_0(t)$, $t \geq 0$ from class $\mathcal{F}$.

Here and henceforth, symbol $\stackrel{\mathsf{J}}{\longrightarrow}$ denotes convergence in Skorokhod J-topology for real-valued càdlàg stochastic processes defined on time interval $[0, \infty)$.

The problems formulated above are solved under three general model assumptions.

Let us introduce the probabilities of occurrence of rare event during one transition step of the semi-Markov process $\eta_\varepsilon(t)$,

$$p_{\varepsilon,i} = \mathsf{P}_i\{\zeta_{\varepsilon,1} = 1\}, \ i \in \mathbb{X}.$$

The first model assumption **A** specifies interpretation of the event $\{\zeta_{\varepsilon,n} = 1\}$ as "rare" and guarantees the possibility for such event to occur:

**A**: $0 < \max_{1 \leq i \leq m} p_{\varepsilon,i} \to 0$ as $\varepsilon \to 0$.

The second model assumption is a condition of asymptotically uniform ergodicity for the embedded Markov chains $\eta_{\varepsilon,n}$:

**B**: There exists a ring chain of states $i_0, i_1, \ldots, i_N = i_0$ which contains all states from the phase space $\mathbb{X}$ and such that $\underline{\lim}_{\varepsilon \to 0} p_{\varepsilon, i_{k-1} i_k} > 0$, for $k = 1, \ldots, N$.

Let us introduce random variables,

$$\mu_{\varepsilon,i}(n) = \sum_{k=1}^{n} I(\eta_{\varepsilon,k-1} = i), \ n = 0, 1, \ldots, \ i \in \mathbb{X}.$$

According Lemma 1 given below, condition **B** guarantees that there exists $\varepsilon'_0 \in (0, \varepsilon_0]$ such that, for every $\varepsilon \in (0, \varepsilon'_0]$, the phase space $\mathbb{X}$ of Markov chain $\eta_{\varepsilon,n}$ is one class of communicative states, and, thus, the Markov chain $\eta_{\varepsilon,n}$ is ergodic, i.e., the following asymptotic relation holds, for any initial distribution $\bar{q}_\varepsilon$,

$$\frac{\mu_{\varepsilon,i}(n)}{n} \stackrel{a.s.}{\longrightarrow} \pi_{\varepsilon,i} > 0 \text{ as } n \to \infty, \text{ for } i \in \mathbb{X}. \tag{3}$$

As well known, the stationary distribution $\pi_{\varepsilon,i}$, $i \in \mathbb{X}$ is the unique solution for the system of linear equations,

$$\pi_{\varepsilon,i} = \sum_{j \in \mathbb{X}} \pi_{\varepsilon,j} p_{\varepsilon,ji}, \ i \in \mathbb{X}, \ \sum_{i \in \mathbb{X}} \pi_{\varepsilon,i} = 1. \tag{4}$$

Note that condition **B** does not require convergence of transition probabilities and, in sequel, do not imply convergence of stationary probabilities $\pi_{\varepsilon,i}$ as $\varepsilon \to 0$.

Finally, the following condition guarantees that the last summand $\kappa_{\varepsilon,\nu_\varepsilon}$ in the random sum $\xi_\varepsilon(1)$ is asymptotically negligible:

**C**: $\mathsf{P}_i\{\kappa_{\varepsilon,1} > \delta / \zeta_{\varepsilon,1} = 1\} \to 0$ as $\varepsilon \to 0$, for $\delta > 0$, $i \in \mathbb{X}$.

Let us define a probability which is the result of averaging of the probabilities of occurrence of rare event in one transition step by the stationary distribution of the imbedded Markov chain $\eta_{\varepsilon,n}$,

$$p_\varepsilon = \sum_{i=1}^{m} \pi_{\varepsilon,i} p_{\varepsilon,i} \text{ and } v_\varepsilon = p_\varepsilon^{-1}. \tag{5}$$

Let us introduce the distribution functions of a sojourn times $\kappa_{\varepsilon,1}$ for the semi-Markov processes $\eta_\varepsilon(t)$,

$$G_{\varepsilon,i}(t) = \mathsf{P}_i\{\kappa_{\varepsilon,1} \leq t\}, t \geq 0,\ i \in \mathbb{X} \text{ and } G_\varepsilon(t) = \sum_{i=1}^{m} \pi_{\varepsilon,i} G_{\varepsilon,i}(t),\ t \geq 0.$$

Let $\theta_{\varepsilon,n},\ n = 1, 2, \dots$ be i.i.d. random variables with distribution $G_\varepsilon(t)$, which is a result of averaging of distribution functions of sojourn times by the stationary distribution of the imbedded Markov chain $\eta_{\varepsilon,n}$,

Now, we can formulate the necessary and sufficient condition for J-convergence of first-rare-event-time processes:

**D**: $\theta_\varepsilon = \sum_{n=1}^{[v_\varepsilon]} \theta_{\varepsilon,n} \xrightarrow{d} \theta_0$ as $\varepsilon \to 0$, where $\theta_0$ is a non-negative random variable with distribution not concentrated in zero.

As it is well known, ($\mathbf{d}_1$) the limiting random variable $\theta_0$ penetrating condition **D** should be infinitely divisible and, thus, its Laplace transform has the form, $\mathsf{E}e^{-s\theta_0} = e^{-A(s)}$, where $A(s) = gs + \int_0^\infty (1 - e^{-sv}) G(dv),\ s \geq 0$, $g$ is a non-negative constant and $G(dv)$ is a measure on interval $(0, \infty)$ such that $\int_{(0,\infty)} \frac{v}{1+v} G(dv) < \infty$; ($\mathbf{d}_2$) $g + \int_{(0,\infty)} \frac{v}{1+v} G(dv) > 0$ (this is equivalent to the assumption that $\mathsf{P}\{\theta_0 = 0\} < 1$).

Let us define the Laplace transforms,

$$\varphi_{\varepsilon,i}(s) = \mathsf{E}_i e^{-s\kappa_{\varepsilon,1}}, i \in \mathbb{X} \text{ and } \varphi_\varepsilon(s) = \mathsf{E}e^{-s\theta_{\varepsilon,1}} = \sum_{\in \mathbb{X}} \pi_{\varepsilon,i} \varphi_{\varepsilon,i}(s),\ s \geq 0.$$

Condition **D** can be reformulated in the equivalent form, in terms of the above Laplace transforms:

$\mathbf{D}_1$: $v_\varepsilon(1 - \varphi_\varepsilon(s)) \to A(s)$ as $\varepsilon \to 0$, for $s > 0$, where the limiting function $A(s) > 0$, for $s > 0$ and $A(s) \to 0$ as $s \to 0$.

In this case, ($\mathbf{d}_3$) $A(s)$ is a cumulant of non-negative random variable with distribution not concentrated in zero. Moreover, ($\mathbf{d}_4$) $A(s)$ should be the cumulant of infinitely divisible distribution of the form given in the above conditions ($\mathbf{d}_1$) and ($\mathbf{d}_2$).

The main result of the paper is the following theorem.

**Theorem 1**. *Let conditions* **A**, **B** *and* **C** *hold. Then,* **(i)** *condition* **D** *is necessary and sufficient for holding (for some or any initial distributions $\bar{q}_\varepsilon$, respectively, in statements of necessity and sufficiency) of the asymptotic relation $\xi_\varepsilon = \xi_\varepsilon(1) \xrightarrow{d} \xi_0$ as $\varepsilon \to 0$, where $\xi_0$ is a non-negative random variable with distribution not concentrated in zero. In this case,* **(ii)** *the limiting random variable $\xi_0$ has the Laplace transform $\mathsf{E}e^{-s\xi_0} = \frac{1}{1+A(s)}$, where $A(s)$ is a cumulant of infinitely divisible distribution defined in condition* **D**. *Moreover,* **(iii)** *the stochastic processes $\xi_\varepsilon(t),\ t \geq 0 \xrightarrow{\mathsf{J}} \xi_0(t) = \theta_0(t\nu_0),\ t \geq 0$ as $\varepsilon \to 0$, where* **(a)** *$\nu_0$ is a random variable, which has the exponential distribution with parameter* 1, **(b)** *$\theta_0(t),\ t \geq 0$ is a non-negative càdlàg Lévy process with the Laplace transforms $\mathsf{E}e^{-s\theta_0(t)} = e^{-tA(s)},\ s, t \geq 0$,* **(c)** *the random variable $\nu_0$ and the process $\theta_0(t),\ t \geq 0$ are independent.*

## 3. Asymptotics of first-rare-event times for Markov chains

We split the proof of Theorem 1 in series of lemmas, which themselves are of some independent interest.

Let $\tilde{\eta}_{\varepsilon,n}$ be, for every $\varepsilon \in (0, \varepsilon_0]$ a Markov chain with the phase space $\mathbb{X}$ and a matrix of transition probabilities $\|\tilde{p}_{\varepsilon,ij}\|$.

We shall use the following condition:

**E**: $p_{\varepsilon,ij} - \tilde{p}_{\varepsilon,ij} \to 0$ as $\varepsilon \to 0$, for $i, j \in \mathbb{X}$.

If transition probabilities $\tilde{p}_{\varepsilon,ij} \equiv p_{0,ij}$, $i,j \in \mathbb{X}$ do not depend on $\varepsilon$, then condition **E** reduces to the following condition:

**F**: $p_{\varepsilon,ij} \to p_{0,ij}$ as $\varepsilon \to 0$, for $i,j \in \mathbb{X}$.

**Lemma 1**. *Let condition* **B** *holds for the Markov chains* $\eta_{\varepsilon,n}$. *Then:* **(i)** *There exists* $\varepsilon'_0 \in (0,\varepsilon_0]$ *such that the Markov chain* $\eta_{\varepsilon,n}$ *is ergodic, for every* $\varepsilon \in (0,\varepsilon'_0]$ *and* $0 < \underline{\lim}_{\varepsilon\to 0} \pi_{\varepsilon,i} \leq \overline{\lim}_{\varepsilon\to 0} \pi_{\varepsilon,i} < 1$, *for* $i \in \mathbb{X}$. **(ii)** *If, together with* **B**, *condition* **E** *holds, then, there exists* $\varepsilon''_0 \in (0,\varepsilon'_0]$ *such that Markov chain* $\tilde{\eta}_{\varepsilon,n}$ *is ergodic, for every* $\varepsilon \in (0,\varepsilon''_0]$, *and its stationary distribution* $\tilde{\pi}_{\varepsilon,i}$, $i \in \mathbb{X}$ *satisfy the asymptotic relation,* $\pi_{\varepsilon,i} - \tilde{\pi}_{\varepsilon,i} \to 0$ *as* $\varepsilon \to 0$, *for* $i \in \mathbb{X}$. **(iii)** *If condition* **F** *holds, then matrix* $\|p_{0,ij}\|$ *is stochastic, condition* **B** *is equivalent to the assumption that a Markov chain* $\eta_{0,n}$, *with the matrix of transition probabilities* $\|p_{0,ij}\|$, *is ergodic and the following asymptotic relation holds,* $\pi_{\varepsilon,i} \to \pi_{0,i}$ *as* $\varepsilon \to 0$, *for* $i \in \mathbb{X}$, *where* $\pi_{0,i}$, $i \in \mathbb{X}$ *is the stationary distribution of the Markov chain* $\eta_{0,n}$.

**Proof**. Let us first prove proposition **(iii)**. Condition **F** obviously implies that matrix $\|p_{0,ij}\|$ is stochastic. Conditions **B** and **F** imply that $\lim_{\varepsilon\to 0} p_{\varepsilon,i_{k-1}i_k} = p_{0,i_{k-1}i_k} > 0$, $k = 1,\ldots,N$, for the ring chain penetrating condition **B**. Thus, the Markov chain *case*, $\eta_{0,n}$ with the matrix of transition probabilities $\|p_{0,ij}\|$ is ergodic. Vise versa, the assumption that a Markov chain $\eta_{0,n}$ with the matrix of transition probabilities $\|p_{0,ij}\|$ is ergodic implies that there exists a ring chain of states $i_0,\ldots,i_N = i_0$ which contains all states from the phase space $\mathbb{X}$ and such that $p_{0,i_{k-1}i_k} > 0$, $k = 1,\ldots,N$. In this case, condition **F** implies that

$$\lim_{\varepsilon\to 0} p_{\varepsilon,i_{k-1}i_k} = p_{0,i_{k-1}i_k} > 0, \qquad k = 1,\ldots,N,$$

and, thus, condition **B** holds. Let us assume that the convergence relation for stationary distributions penetrating proposition **(iii)** does not hold. In this case, there exist $\delta > 0$ and a sequence $0 < \varepsilon_n \to 0$ as $n \to \infty$ such that $\underline{\lim}_{n\to\infty} |\pi_{\varepsilon_n,i'} - \pi_{0,i'}| \geq \delta$, for some $i' \in \mathbb{X}$. Since, the sequences $\pi_{\varepsilon_n,i}$, $n = 1,2,\ldots$, $i \in \mathbb{X}$ are bounded, there exists a subsequence $0 < \varepsilon_{n_k} \to 0$ as $k \to 0$ such that $\pi_{\varepsilon_{n_k},i} \to \pi'_{0,i}$ as $k \to \infty$, for $i \in \mathbb{X}$. This relation, condition **F** and relation (4) imply that numbers $\pi'_{0,i}$, $i \in \mathbb{X}$ satisfy the system of linear equation given in (4). This is impossible, since inequality $|\pi'_{0,i'} - \pi_{0,i'}| \geq \delta$ should hold, while the stationary distribution $\pi_{0,i}$, $i \in \mathbb{X}$ is the unique solution of system (4).

Let us now prove proposition **(i)**. Condition **B** implies that there exist $\varepsilon'_0 \in (0,\varepsilon_0]$ such that $p_{\varepsilon,i_{k-1}i_k} > 0$, $k = 1,\ldots,N$, for the ring chain penetrating condition **B**, for $\varepsilon \in (0,\varepsilon'_0]$. Thus, the phase space $\mathbb{X}$ is one class of communicative states for the Markov chain $\eta_{\varepsilon,n}$ and, therefore, this Markov chain is ergodic, for every $\varepsilon \in (0,\varepsilon'_0]$. Let us now assume that $\underline{\lim}_{\varepsilon\to 0} \pi_{\varepsilon,i'} = 0$, for some $i' \in \mathbb{X}$. In this case, there exists a sequence $0 < \varepsilon_n \to 0$ as $n \to \infty$ such that $\pi_{\varepsilon_n,i'} \to 0$ as $n \to \infty$. Since, the sequences $p_{\varepsilon_n,ij}$, $n = 1,2,\ldots$, $i,j \in \mathbb{X}$ are bounded, there exists a subsequence $0 < \varepsilon_{n_k} \to 0$ as $k \to 0$ such that $p_{\varepsilon_{n_k},ij} \to p_{0,ij}$ as $k \to \infty$, for $i,j \in \mathbb{X}$. By proposition **(iii)**, the matrix $\|p_{0,ij}\|$ is stochastic, the Markov chain $\eta_{0,n}$ with the matrix of transition probabilities $\|p_{0,ij}\|$ is ergodic and its stationary distribution $\pi_{0,i}$, $i \in \mathbb{X}$ satisfies the asymptotic relation, $\pi_{\varepsilon_{n_k},i} \to \pi_{0,i}$ as $k \to \infty$, for $i \in \mathbb{X}$. This is impossible since equality $\pi_{0,i'} = 0$ should hold, while all stationary probabilities $\pi_{0,i}$, $i \in \mathbb{X}$ are positive. Thus, $\underline{\lim}_{\varepsilon\to 0} \pi_{\varepsilon,i} > 0$, for $i \in \mathbb{X}$. This implies that, also, $\overline{\lim}_{\varepsilon\to 0} \pi_{\varepsilon,i} < 1$, for $i \in \mathbb{X}$, since $\sum_{i\in\mathbb{X}} \pi_{\varepsilon,i} = 1$, for $\varepsilon \in (0,\varepsilon'_0]$.

Finally, let us now prove proposition **(ii)**. Conditions **B** and **E** obviously imply that $\underline{\lim}_{\varepsilon\to 0} \tilde{p}_{\varepsilon,i_{k-1}i_k} = \underline{\lim}_{\varepsilon\to 0} p_{\varepsilon,i_{k-1}i_k} > 0$, $k = 1,\ldots,N$, for the ring chain penetrating condition **B**. Thus, condition **B** holds also for the Markov chains $\tilde{\eta}_{\varepsilon,n}$ and there exist $\varepsilon''_0 \in (0,\varepsilon'_0]$ such that Markov chain $\tilde{\eta}_{\varepsilon,n}$ is ergodic, for every $\varepsilon \in (0,\varepsilon''_0]$. Let assume that the convergence relation for stationary distributions penetrating proposition **(ii)** does not hold. In this case, there exist here exist $\delta > 0$ and a sequence $0 < \varepsilon_n \to 0$ as

$n \to \infty$ such that $\underline{\lim}_{n\to\infty} |\pi_{\varepsilon_n,i'} - \tilde{\pi}_{\varepsilon_n,i'}| \geq \delta$, for some $i' \in \mathbb{X}$. Since, the sequences $p_{\varepsilon_n,ij}, n = 1, 2, \ldots, i, j \in \mathbb{X}$ are bounded, there exists a subsequence $0 < \varepsilon_{n_k} \to 0$ as $k \to 0$ such that $p_{\varepsilon_{n_k},ij} \to p_{0,ij}$ as $k \to \infty$, for $i, j \in \mathbb{X}$. This relations and condition **E** imply that, also, $\tilde{p}_{\varepsilon_{n_k},ij} \to p_{0,ij}$ as $k \to \infty$, for $i, j \in \mathbb{X}$. By proposition **(iii)**, the matrix $\|p_{0,ij}\|$ is stochastic, the Markov chain $\eta_{0,n}$ with the matrix of transition probabilities $\|p_{0,ij}\|$ is ergodic and its stationary distribution $\pi_{0,i},\ i \in \mathbb{X}$ satisfies the asymptotic relations, $\pi_{\varepsilon_{n_k},i} \to \pi_{0,i}$ as $k \to \infty$, for $i \in \mathbb{X}$ and $\tilde{\pi}_{\varepsilon_{n_k},i} \to \pi_{0,i}$ as $k \to \infty$, for $i \in \mathbb{X}$. This is impossible, since relation $\underline{\lim}_{k\to\infty} |\pi_{\varepsilon_{n_k},i'} - \tilde{\pi}_{\varepsilon_{n_k},i'}| \geq \delta$ should hold. □

Proposition **(iii)** of Lemma 1 implies that, in the case, where the transition probabilities $p_{\varepsilon,ij} = p_{0,ij},\ i, j \in \mathbb{X}$ do not depend on parameter $\varepsilon$ or $p_{\varepsilon,ij} \to p_{0,ij}$ as $\varepsilon \to 0$, for $i, j \in \mathbb{X}$, condition **B** reduces to the standard assumption that the Markov chain $\eta_{0,n}$, with the matrix of transition probabilities $\|p_{0,ij}\|$, is ergodic.

These simpler variants of asymptotic ergodicity condition, based on condition **F** and the assumption of ergodicity of the Markov chain $\eta_{0,n}$ combined with averaging of characteristic in condition **D** by its stationary distribution $\pi_{0,i},\ i \in \mathbb{X}$, have been used in the mentioned above works by Silvestrov and Drozdenko (2006) and Drozdenko (2007) for proving simper analogues of propositions **(i)** and **(ii)** of Theorem 1. In this case, the averaging of characteristics in the necessary and sufficient condition **D**, in fact, relates mainly to distributions of sojourn times.

Condition **B**, used in the present paper, balances in a natural way averaging of characteristics in condition **D** between distributions of sojourn times and stationary distributions of the corresponding embedded Markov chains.

**Lemma 2**. *Let condition* **B** *hold. Then,*

$$\mu^*_{\varepsilon,i}(t) = \frac{\mu_{\varepsilon,i}([tv_\varepsilon])}{\pi_{\varepsilon,i} v_\varepsilon} \xrightarrow{\mathsf{P}} t \text{ as } \varepsilon \to 0, \text{ for } t \geq 0, i \in \mathbb{X}. \tag{6}$$

**Proof**. Let $\alpha_{\varepsilon,j} = \min(n > 0 : \eta_{\varepsilon,n} = j)$ be the moment of first hitting to the state $j \in \mathbb{X}$ for the Markov chain $\eta_{\varepsilon,n}$. Condition **B** implies that there exist $p \in (0, 1)$ and $\varepsilon_p \in (0, \varepsilon_0]$ such that $\prod_{k=1}^{N} p_{\varepsilon,i_{k-1}i_k} > p$, for $\varepsilon \in (0, \varepsilon_p]$. The following inequalities are obvious, $\mathsf{P}_i\{\alpha_{\varepsilon,j} > kN\} \leq (1-p)^k,\ k \geq 1,\ i, j \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_p]$. These inequalities imply that there exists $K_p \in (0, \infty)$ such that $\max_{i,j\in\mathbb{X}} \mathsf{E}_i\alpha^2_{\varepsilon,j} \leq K_p < \infty,\ i, j \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_p]$. Also, as well known, $\mathsf{E}_i\alpha_{\varepsilon,i} = \pi^{-1}_{\varepsilon,i},\ i \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_p]$.

Let $\alpha_{\varepsilon,i,n} = \min(k > \alpha_{\varepsilon,i,n-1} : \eta_{\varepsilon,k} = i),\ n = 1, 2, \ldots$ be sequential moments of hitting to state $i \in \mathbb{X}$, for the Markov chain $\eta_{\varepsilon,n}$, and $\beta_{\varepsilon,i,n} = \alpha_{\varepsilon,i,n} - \alpha_{\varepsilon,i,n-1},\ n = 1, 2, \ldots,$ where $\alpha_{\varepsilon,i,0} = 0$. The random variables $\beta_{\varepsilon,i,n},\ n \geq 1$ are independent and identically distributed for $n \geq 2$. The above relations for moments of random variables $\alpha_{\varepsilon,i}$ imply that $\alpha_{\varepsilon,i,1}/v_\varepsilon \xrightarrow{\mathsf{P}} 0$ as $\varepsilon \to 0$, for $i \in \mathbb{X}$. Also, $\mathsf{P}_i\{v_\varepsilon^{-1}|\alpha_{\varepsilon,i,[tv_\varepsilon]} - \pi^{-1}_{\varepsilon,i}[tv_\varepsilon]| > \delta\} \leq tK_p/\delta^2 v_\varepsilon,\ \delta > 0,\ t \geq 0,\ i \in \mathbb{X}$, for $\varepsilon \in (0, \varepsilon_p]$. These relations obviously implies that random variables $\alpha_{\varepsilon,i,[tv_\varepsilon]}/\pi^{-1}_{\varepsilon,i} v_\varepsilon \xrightarrow{\mathsf{P}} t$ as $\varepsilon \to 0$, for $t \geq 0$. The dual identities $\mathsf{P}\{\mu_{\varepsilon,i}(r) \geq k\} = \mathsf{P}\{\alpha_{\varepsilon,i,k} \leq r\},\ r, k = 0, 1, \ldots$ let one, in standard way, convert the latter asymptotic relation to the equivalent relation $\mu^*_{\varepsilon,i}(t) = \mu_{\varepsilon,i,[tv_\varepsilon]}/\pi_{\varepsilon,i}v_\varepsilon \xrightarrow{\mathsf{P}} t$ as $\varepsilon \to 0$, for $t \geq 0$. □

Let $f_{\varepsilon,i} \geq 0,\ i \in \mathbb{X}$ are non-random non-negative numbers, and $f_\varepsilon = \sum_{i\in\mathbb{X}} f_{\varepsilon,i}\pi_{\varepsilon,i}$. Let us. also, introduce stochastic process,

$$\bar{\kappa}_\varepsilon(t) = \sum_{n=1}^{[tv_\varepsilon]} f_{\varepsilon,\eta_{\varepsilon,n-1}}, t \geq 0. \tag{7}$$

Let us formulate two conditions imposed on function $f_\varepsilon$:

**G**: **(a)** $f_\varepsilon > 0$ for $\varepsilon \in (0, \varepsilon''_0]$, where $\varepsilon''_0 \in (0, \varepsilon'_0]$.

and

**H**: $f_\varepsilon \to f_0 \in [0, \infty]$ as $\varepsilon \to 0$.

**Lemma 3**. *Let conditions* **B** *and* **G** *hold. Then,* **(i)** $f_\varepsilon^{-1}\bar{\kappa}_\varepsilon(t) \xrightarrow{\mathsf{P}} t$ *as* $\varepsilon \to 0$, *for* $t \geq 0$. **(ii)** *Condition* **H** *is necessary and sufficient condition for holding (for some or any initial distributions* $\bar{q}_\varepsilon$, *respectively, in statements of necessity and sufficiency) of the asymptotic relation,* $\bar{\kappa}_\varepsilon(1) \xrightarrow{d} \theta_0$ *as* $\varepsilon \to 0$, *where* $\theta_0$ *is a non-negative proper or improper random variable. In this case,* **(iii)** *The random variable* $\theta_0 \stackrel{d}{=} f_0$, *i.e., it is a constant, and* **(iv)** $\bar{\kappa}_\varepsilon(t) \xrightarrow{\mathsf{P}} f_0 t$ *as* $\varepsilon \to 0$, *for* $t > 0$.

**Proof**. Let us use the following representation,

$$\bar{\kappa}_\varepsilon(t) = \sum_{i \in \mathbb{X}} \mu^*_{\varepsilon,i}(t) v_\varepsilon \pi_{\varepsilon,i} f_{\varepsilon,i}, \ t \geq 0. \tag{8}$$

For any sequence $0 < \varepsilon_n \to 0$ as $n \to \infty$, there exists a subsequence $0 < \varepsilon_{n_k} \to 0$ as $k \to \infty$ such that $\frac{v_{n_k}\pi_{\varepsilon_{n_k},i} f_{\varepsilon_{n_k},i}}{f_{\varepsilon_{n_k}}} \to g_i \in [0,1]$ as $k \to \infty$, for $i \in \mathbb{X}$. Constants $g_i$, $i \in \mathbb{X}$ can depend on the choice of subsequence $\varepsilon_{n_k}$, but, obviously satisfy the following relation, $\sum_{i \in \mathbb{X}} g_i = 1$. These relations and Lemma 2 imply that $f^{-1}_{\varepsilon_{n_k}} \bar{\kappa}_{\varepsilon_{n_k}}(t) \xrightarrow{\mathsf{P}} \sum_{i \in \mathbb{X}} g_i t = t$ as $k \to \infty$, for $t \geq 0$, since the limiting processes in relations (6) given in Lemma 2 are non-random functions. Since the limits $t$ is the same for all subsequences $\varepsilon_{n_k}$ described above, the above asymptotic relation implies the asymptotic relation given in proposition **(i)** of Lemma 3.

This relation implies that the random variables $\bar{\kappa}_\varepsilon(1) = f_\varepsilon \cdot (f_\varepsilon^{-1}\bar{\kappa}_\varepsilon(1))$ converge in distribution as $k \to \infty$, if and only if $f_\varepsilon \to f_0 \in [0, \infty]$ as $\varepsilon \to 0$. Moreover, in this case, the limiting (possibly improper) random variable is constant $f_0$. Also, this relation implies that, in this case, $\bar{\kappa}_\varepsilon(t) = f_\varepsilon \cdot (f_\varepsilon^{-1}\bar{\kappa}_\varepsilon(t)) \xrightarrow{\mathsf{P}} f_0 t$ as $\varepsilon \to 0$, for $t > 0$. □

The following lemma describe asymptotics for first-rare-event times $\nu_\varepsilon$ for Markov chains $\eta_{\varepsilon,n}$.

**Lemma 4**. *Let conditions* **A** *and* **B** *hold. Then, the random variables* $\nu^*_\varepsilon = p_\varepsilon \nu_\varepsilon \xrightarrow{d} \nu_0$ *as* $\varepsilon \to 0$, *where* $\nu_0$ *is a random variable exponentially distributed with parameter* 1.

**Proof**. Let us define probabilities, for $\varepsilon \in (0, \varepsilon_0]$,

$$P_{\varepsilon,ij} = \mathsf{P}_i\{\eta_{\varepsilon,1} = j, \zeta_{\varepsilon,1} = 0\}, \ \tilde{p}_{\varepsilon,ij} = \frac{P_{\varepsilon,ij}}{\sum_{r \in \mathbb{X}} P_{\varepsilon,ir}} = \frac{P_{\varepsilon,ij}}{1 - p_{\varepsilon,i}}, \ i,j \in \mathbb{X}.$$

Let also $\tilde{\eta}_{\varepsilon,n}$, $n = 0, 1, \ldots$ be a homogeneous Markov chain with the phase space $\mathbb{X}$, the initial distribution $\bar{q}_\varepsilon = \langle q_{\varepsilon,i}, i \in \mathbb{X}\rangle$ and the matrix of transition probabilities $\|\tilde{p}_{\varepsilon,ij}\|$.

Let us also introduce stochastic processes,

$$\kappa^*_\varepsilon(t) = \sum_{n=1}^{[tv_\varepsilon]} - \ln(1 - p_{\varepsilon,\tilde{\eta}_{\varepsilon,n-1}}), t \geq 0.$$

The following relation takes place, for $t \geq 0$,

$$\begin{aligned} \mathsf{P}\{\nu^*_\varepsilon > t\} &= \sum_{i \in \mathbb{X}} q_{\varepsilon,i} \sum_{i = i_0, i_1, \ldots, i_{[tv_\varepsilon]} \in \mathbb{X}} \prod_{k=1}^{[tv_\varepsilon]} P_{\varepsilon, i_{k-1} i_k} \\ &= \mathsf{E}\exp\{-\sum_{k=1}^{[tv_\varepsilon]} - \ln(1 - p_{\varepsilon,\tilde{\eta}_{\varepsilon,k-1}})\} = \mathsf{E}e^{-\kappa^*_\varepsilon(t)}. \end{aligned} \tag{9}$$

Conditions **A** and **B** imply that condition **B** holds for transition probabilities of the Markov chains $\tilde{\eta}_{\varepsilon,n}$, since,

$$|p_{\varepsilon,ij} - \tilde{p}_{\varepsilon,ij}| \leq \frac{2p_{\varepsilon,i}}{1 - p_{\varepsilon,i}} \to 0 \text{ as } \varepsilon \to 0, \text{ for } i,j \in \mathbb{X}. \tag{10}$$

Thus, by Lemma 1, there exist $\varepsilon_0'' \in (0, \varepsilon_0']$ such that the Markov chain $\tilde{\eta}_{\varepsilon,n}$ is ergodic, for every $\varepsilon \in (0, \varepsilon_0'']$, and its stationary probabilities $\tilde{\pi}_{\varepsilon,i}$, $i \in \mathbb{X}$ satisfy relation, $\tilde{\pi}_{\varepsilon,i} - \pi_{\varepsilon,i} \to 0$ as $\varepsilon \to 0$, for $i \in \mathbb{X}$. This relation, implies that, $f_\varepsilon = -v_\varepsilon \sum_{i \in \mathbb{X}} \tilde{\pi}_{\varepsilon,i} \ln(1 - p_{\varepsilon,i}) \sim v_\varepsilon \sum_{i \in \mathbb{X}} \pi_{\varepsilon,i}\, p_{\varepsilon,i} = v_\varepsilon p_\varepsilon = 1$ as $\varepsilon \to 0$.

Here and henceforth, relation $a(\varepsilon) \sim b(\varepsilon)$ as $\varepsilon \to 0$ means that $a(\varepsilon)/b(\varepsilon) \to 1$ as $\varepsilon \to 0$.

We can now apply sufficiency statement of proposition **(ii)** of Lemma 3 to the non-negative step-sum processes, $\kappa^*_\varepsilon(t)$ and to get relation, $\kappa^*_\varepsilon(t) \stackrel{\mathsf{P}}{\longrightarrow} t$ as $\varepsilon \to 0$, for $t \geq 0$. This relation obviously implies that, for every $t \geq 0$,

$$\mathsf{P}\{\nu^*_\varepsilon > t\} = \mathsf{E}e^{-\kappa^*_\varepsilon(t)} \to e^{-t} \text{ as } \varepsilon \to 0. \tag{11}$$

The proof is completed. □

Let, as in Lemma 3, $f_{\varepsilon,i}$, $i \in \mathbb{X}$ be non-random, non-negative numbers and $f_\varepsilon = v_\varepsilon \sum_{i \in \mathbb{X}} \pi_{\varepsilon,i} f_{\varepsilon,i}$. Let us introduce stochastic processes,

$$\nu_\varepsilon(t) = \sum_{n=1}^{[t\nu_\varepsilon]} f_{\varepsilon,\eta_{\varepsilon,n-1}}, t \geq 0.$$

**Lemma 5**. *Let conditions* **A**, **B** *and* **G** *hold. Then,* **(i)** $f_\varepsilon^{-1}\nu_\varepsilon(t)$, $t \geq 0 \stackrel{d}{\longrightarrow} t\nu_0$, $t \geq 0$ *as* $\varepsilon \to 0$, *where* $\nu_0$ *is a random variable exponentially distributed with parameter* 1. **(ii)** *Condition* **H** *is necessary and sufficient condition for holding* (*for some or any initial distributions* $\bar{q}_\varepsilon$, *respectively, in statements of necessity and sufficiency*) *of the asymptotic relation,* $\nu_\varepsilon(1) \stackrel{d}{\longrightarrow} \nu$ *as* $\varepsilon \to 0$, *where* $\nu$ *is a non-negative random variable with distribution not concentrated in zero.* **(iii)** *The random variable* $\nu \stackrel{d}{=} f_0\nu_0$ *and,* **(iv)** $\nu_\varepsilon(t)$, $t \geq 0 \stackrel{d}{\longrightarrow} f_0\nu_0 t$, $t > 0$ *as* $\varepsilon \to 0$.

**Proof**. The following representation takes place,

$$\nu_\varepsilon(t) = \bar{\kappa}_\varepsilon(t\nu^*_\varepsilon), t \geq 0, \tag{12}$$

where $\bar{\kappa}_\varepsilon(t)$ are processes defined in relation (7) and $\nu^*_\varepsilon$ are random variables introduced in Lemma 4.

Relations given in proposition **(i)** of Lemma 3 and in Lemma 4 imply, by Slutsky theorem, the relation, $(t\nu^*_\varepsilon, f_\varepsilon^{-1}\bar{\kappa}_\varepsilon(t))$, $t \geq 0 \stackrel{d}{\longrightarrow} (t\nu_0, t)$, $t \geq 0$ as $\varepsilon \to 0$. The components of the processes on the left hand side of the above relation are non-decreasing processes and the process on the right hand side is continuous. This let us apply Theorem 3.2.1 from Silvestrov (2004) to processes $f_\varepsilon^{-1}\nu_\varepsilon(t) = f_\varepsilon^{-1}\bar{\kappa}_\varepsilon(t\nu^*_\varepsilon)$, $t \geq 0$ and to get the asymptotic relation given in proposition **(i)** of Lemma 5.

This relation also implies the random variables $\nu_\varepsilon(1) = f_\varepsilon \cdot (f_\varepsilon^{-1}\nu_\varepsilon(1))$ converge in distribution if and only if $f_\varepsilon \to f_0 \in [0, \infty]$ as $\varepsilon \to 0$. Moreover, in this case, the limiting (possibly improper) random variable $\nu \stackrel{d}{=} f_0\nu_0$. Also this relation implies that $\nu_\varepsilon(t) = f_\varepsilon \cdot (f_\varepsilon^{-1}\nu_\varepsilon(t)), t \geq 0 \stackrel{d}{\longrightarrow} f_0\nu_0 t$, $t > 0$ as $\varepsilon \to 0$. □

## 4. Propositions (i) and (ii) of Theorem 1

Let us introduce, for every $\varepsilon \in (0, \varepsilon_0']$, random variables, which are sequential moments of hitting state $i \in \mathbb{X}$ by the Markov chain $\eta_{\varepsilon,n}$,

$$\tau_{\varepsilon,i,n} = \begin{cases} \min(k \geq 0, \eta_{\varepsilon,k} = i) & \text{for } n = 1, \\ \min(k > \tau_{\varepsilon,i,n-1}, \eta_{\varepsilon,k} = i) & \text{for } n \geq 2. \end{cases} \tag{13}$$

Let us also define random variables,

$$\kappa_{\varepsilon,i,n} = \kappa_{\varepsilon,\tau_{\varepsilon,i,n}+1}, n = 1, 2, \ldots, i \in \mathbb{X}. \tag{14}$$

**Lemma 6**. *Let condition* **B** *holds. Then, for every* $\varepsilon \in (0, \varepsilon_0']$, **(i)** *the random variables* $\kappa_{\varepsilon,i,n}$, $n = 1, 2, \ldots$, $i \in \mathbb{X}$ *are independent.* **(ii)** $\mathsf{P}\{\kappa_{\varepsilon,i,n} \leq t\} = G_{\varepsilon,i}(t)$, $t \geq 0$, *for* $n = 1, 2, \ldots$, $i \in \mathbb{X}$; **(iii)** *the following representation takes place,*

$$\kappa_\varepsilon(t) = \sum_{n=1}^{[tv_\varepsilon]} \kappa_{\varepsilon,n} = \sum_{i\in\mathbb{X}} \sum_{n=1}^{\mu_{\varepsilon,i}([tv_\varepsilon])} \kappa_{\varepsilon,i,n}, t \geq 0 \tag{15}$$

**Proof**. Let us take an arbitrary sequence of different pairs $(i_1, k_1), (i_2, k_2), \ldots$ taking values in space $\mathbb{X}\times\{1, 2\ldots\}$. Note that random variables $\tau_{\varepsilon,i_1,k_1}, \tau_{\varepsilon,i_2,k_2}, \ldots$, by definition, all take different values.

Hitting times $\tau_{\varepsilon,i,k}$ are Markov moments for the embedded Markov chain $\eta_{\varepsilon,n}$ as, therefore, also for the Markov renewal process $(\eta_{\varepsilon,n}, \kappa_{\varepsilon,n})$. Also, any event $\{\tau_{\varepsilon,i_1,k_1} < \cdots < \tau_{\varepsilon,i_r,k_r} < \min_{r<l\leq n} \tau_{\varepsilon,i_l,k_l}\}$ is determined by the random variables $\eta_{\varepsilon,0}, \ldots, \eta_{\varepsilon,\tau_{\varepsilon,i_r,k_r}}$.

Using the above remarks, we get relation,

$$\begin{aligned}
&\mathsf{P}\{\kappa_{\varepsilon,i_r,k_r} \leq u_r, r = 1, 2, \tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2}\} \\
&\quad = \mathsf{P}\{\kappa_{\varepsilon,\tau_{\varepsilon,i_2,k_2}+1} \leq u_2/\kappa_{\varepsilon,\tau_{\varepsilon,i_1,k_1}+1} \leq u_1, \tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2}\} \\
&\quad\quad \times \mathsf{P}\{\kappa_{\varepsilon,\tau_{\varepsilon,i_1,k_1}+1} \leq u_1, \tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2}\} \\
&\quad = G_{\varepsilon,i_2}(u_2)\mathsf{P}\{\kappa_{\varepsilon,\tau_{\varepsilon,i_1,k_1}+1} \leq u_1, \tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2}\} \\
&\quad = G_{\varepsilon,i_2}(u_2)\mathsf{P}\{\kappa_{\varepsilon,\tau_{\varepsilon,i_1,k_1}+1} \leq u_1/\tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2}\}\mathsf{P}\{\tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2}\} \\
&\quad = G_{\varepsilon,i_1}(u_1)G_{\varepsilon,i_2}(u_2)\mathsf{P}\{\tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2}\}.
\end{aligned}$$

Also, the similar relation, in which random variables $\tau_{\varepsilon,i_1,k_1}, \tau_{\varepsilon,i_2,k_2}$ exchange each other, takes place. By adding the above two relations, we get relation, $\mathsf{P}\{\kappa_{\varepsilon,i_r,k_r} \leq u_r, r = 1, 2\} = G_{\varepsilon,i_1}(u_1)G_{\varepsilon,i_2}(u_2)$, which prove the pair-wise independence of random variables $\kappa_{\varepsilon,i,n}$, $n = 1, 2, \ldots$, $i \in \mathbb{X}$.

In analogous way, we can get relation,

$$\begin{aligned}
&\mathsf{P}\{\kappa_{\varepsilon,i_r,k_r} \leq u_r, r = 1, 2, 3, \tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2} < \tau_{\varepsilon,i_3,k_3}\} \\
&\quad = G_{\varepsilon,i_2}(u_2)G_{\varepsilon,i_3}(u_3)\mathsf{P}\{\kappa_{\varepsilon,i_1,k_1} \leq u_1, \tau_{\varepsilon,i_1,k_1} < \tau_{\varepsilon,i_2,k_2} < \tau_{\varepsilon,i_3,k_3}\}
\end{aligned}$$

and analogous relation, where random variables $\tau_{\varepsilon,i_2,k_2}$ and $\tau_{\varepsilon,i_3,k_3}$ exchange each other. By adding the above two relations, we get relation

$$\begin{aligned}
&\mathsf{P}\{\kappa_{\varepsilon,i_r,k_r} \leq u_r, r = 1, 2, 3, \tau_{\varepsilon,i_1,k_1} < \min(\tau_{\varepsilon,i_2,k_2}, \tau_{\varepsilon,i_3,k_3})\} \\
&\quad = G_{\varepsilon,i_2}(u_2)G_{\varepsilon,i_3}(u_3)\mathsf{P}\{\kappa_{\varepsilon,i_1,k_1} \leq u_1, \tau_{\varepsilon,i_1,k_1} < \min(\tau_{\varepsilon,i_2,k_2}, \tau_{\varepsilon,i_3,k_3})\} \\
&\quad = G_{\varepsilon,i_1}(u_1)G_{\varepsilon,i_2}(u_2)G_{\varepsilon,i_3}(u_3)\mathsf{P}\{\tau_{\varepsilon,i_1,k_1} < \min(\tau_{\varepsilon,i_2,k_2}, \tau_{\varepsilon,i_3,k_3})\}.
\end{aligned}$$

Also, two similar relations, in which random variables $\tau_{\varepsilon,i_1,k_1}$, $\tau_{\varepsilon,i_2,k_2}$, $\tau_{\varepsilon,i_3,k_3}$ are exchanged, respectively, by random variables $\tau_{\varepsilon,i_2,k_2}$, $\tau_{\varepsilon,i_3,k_3}$, $\tau_{\varepsilon,i_1,k_1}$ or by $\tau_{\varepsilon,i_3,k_3}$, $\tau_{\varepsilon,i_1,k_1}$, $\tau_{\varepsilon,i_2,k_2}$, take place. By adding the above three relations, we get relation, $\mathsf{P}\{\kappa_{\varepsilon,i_r,k_r} \leq u_r, r = 1, 2, 3\} = G_{\varepsilon,i_1}(u_1)G_{\varepsilon,i_2}(u_2)G_{\varepsilon,i_3}(u_3)$, which prove the triplet-wise independence of random variables $\kappa_{\varepsilon,i,n}$, $n = 1, 2, \ldots$, $i \in \mathbb{X}$.

Propositions **(ii)** follows from the above remarks. Proposition **(iii)** is obvious since relation (15), just, represents two alternative ways of grouping summands in the same random sums. □

It is useful to note that the families of random variables $\langle \mu_{\varepsilon,i}(n), n = 0, 1, \ldots, i \in \mathbb{X}\rangle$ and $\langle \kappa_{\varepsilon,i,n}, n = 1, 2, \ldots, i \in \mathbb{X}\rangle$ are not independent.

Let us introduce Laplace transforms, $\varphi_{\varepsilon,ij}(\imath, s) = \mathsf{E}_i I(\eta_{\varepsilon,1} = j, \zeta_{\varepsilon,1} = \imath)e^{-s\kappa_{\varepsilon,1}}$, $s \geq 0$, for $i, j \in \mathbb{X}$, $\imath = 0, 1$, and $\varphi_{\varepsilon,i}(\imath, s) = \mathsf{E}_i I(\zeta_{\varepsilon,1} = \imath)e^{-s\kappa_{\varepsilon,1}}$, $s \geq 0$, for $i \in \mathbb{X}$, $\imath = 0, 1$, and define probabilities, for $s \geq 0$,

$$p_{\varepsilon,s,ij} = \frac{\varphi_{\varepsilon,ij}(0, s)}{\sum_{r\in\mathbb{X}} \varphi_{\varepsilon,ij}(0, s)} = \frac{\varphi_{\varepsilon,ij}(0, s)}{\varphi_{\varepsilon,i}(0, s)}, \ i, j \in \mathbb{X}.$$

Let $(\eta_{\varepsilon,s,n}, \zeta_{\varepsilon,s,n})$, $n = 0, 1, \ldots$ be, for every $s \geq 0$ and $\varepsilon \in (0, \varepsilon'_0]$, a Markov renewal process, with the phase space $\mathbb{X} \times \{0, 1\}$, the initial distribution $\bar{q}_\varepsilon = \langle q_{\varepsilon,i} = \mathsf{P}\{\eta_{\varepsilon,0} =$

$i, \zeta_{\varepsilon,s,0} = 0\} = \mathsf{P}\{\eta_{\varepsilon,s,0} = i\}, i \in \mathbb{X}\rangle$ and the transition probabilities,

$$\begin{aligned} &\mathsf{P}\{\eta_{\varepsilon,s,n+1} = j, \zeta_{\varepsilon,s,n+1} = \jmath / \eta_{\varepsilon,s,n} = i, \zeta_{\varepsilon,s,n} = \imath\} \\ &\quad = \mathsf{P}\{\eta_{\varepsilon,s,n+1} = j, \zeta_{\varepsilon,s,n+1} = \jmath / \eta_{\varepsilon,s,n} = i\} \\ &\quad = p_{\varepsilon,s,ij}(p_{\varepsilon,i}\,\jmath + (1 - p_{\varepsilon,i})(1 - \jmath)), \ i, j \in \mathbb{X}, \ \imath, \jmath = 0, 1. \end{aligned} \tag{16}$$

Note that the first component of the Markov renewal process, $\eta_{\varepsilon,s,n}$, $n = 0, 1, \ldots$ is a homogeneous Markov chain with the phase space $\mathbb{X}$, an initial distribution $\bar{q}_\varepsilon = \langle q_{\varepsilon,i}, i \in \mathbb{X}\rangle$ and the matrix of transition probabilities $\|p_{\varepsilon,s,ij}\|$.

Let us prove that condition **D** or conditions **A**, **B** and the asymptotic relation penetrating proposition **(i)** of Theorem 1 imply that, for every $s \geq 0$, condition **B** holds for transition probabilities of the Markov chain $\eta_{\varepsilon,s,n}$.

Condition **D** obviously, implies that, for $i \in \mathbb{X}$,

$$\varphi_{\varepsilon,i}(s) \to 1 \text{ as } \varepsilon \to 0, \text{ for } s \geq 0, \tag{17}$$

Let us show that conditions **A**, **B** and the asymptotic relation penetrating proposition **(i)** of Theorem 1 also imply that relation (17) holds.

Let us now assume that relation (17) does not holds. This means that there exists $i \in \mathbb{X}$ such that for some $\delta, p > 0$ and $\varepsilon_{\delta,p} \in (0, \varepsilon'_0]$ probability $\mathsf{P}\{\kappa_{\varepsilon,i,1} \geq \delta\} \geq p$, for $\varepsilon \in (0, \varepsilon_{\delta,p}]$. This obviously implies that random variables $\tilde{\kappa}_{\varepsilon,i}(t) = \sum_{n=1}^{[t\pi_{\varepsilon,i}v_\varepsilon]} \kappa_{\varepsilon,i,n} \stackrel{\mathsf{P}}{\longrightarrow} \infty$ as $\varepsilon \to 0$, for $t > 0$, and, thus, stochastic processes $\min(T, \tilde{\kappa}_{\varepsilon,i}(t))$, $t > 0 \stackrel{d}{\longrightarrow} h_T(t) = T$, $t > 0$ as $\varepsilon \to 0$. Since, the processes $\tilde{\kappa}_{\varepsilon,i}(t)$, $t > 0$ are non-decreasing and the limiting function $h_T(t) = T$, $t > 0$ is continuous, the latter relation implies (see, for example, Lemma 3.2.2 from Silvestrov (2004)) that $\min(T, \tilde{\kappa}_{\varepsilon,i}(t))$, $t > 0 \stackrel{\mathsf{J}}{\longrightarrow} h_T(t) = T$, $t \geq 0$ as $\varepsilon \to 0$. By Lemma 5, applied to the model with functions $f_{\varepsilon,j} = I(j = i)(\pi_{\varepsilon,i}v_\varepsilon)^{-1}$, $j \in \mathbb{X}$, the following relation takes place, $\mu^*_{\varepsilon,i}(\nu^*_\varepsilon) \stackrel{d}{\longrightarrow} \nu_0$ as $\varepsilon \to 0$, where $\nu_0$ is a random variable exponentially distributed with parameter 1. The latter two relations imply, by Slutsky theorem, that $(\mu^*_{\varepsilon,i}(\nu^*_\varepsilon), \min(T, \tilde{\kappa}_{\varepsilon,i}(t)))$, $t > 0 \stackrel{d}{\longrightarrow} (\nu_0, h_T(t))$, $t > 0$ as $\varepsilon \to 0$. Now we can apply Theorem 2.2.1 from Silvestrov (2004) that yields the following relation, $\min(T, \tilde{\kappa}_{\varepsilon,i}(\mu^*_{\varepsilon,i}(\nu^*_\varepsilon))) \stackrel{d}{\longrightarrow} T$ as $\varepsilon \to 0$, for any $T > 0$. This is possible only if $\tilde{\kappa}_{\varepsilon,i}(\mu^*_{\varepsilon,i}(\nu^*_\varepsilon)) \stackrel{\mathsf{P}}{\longrightarrow} \infty$ as $\varepsilon \to 0$. Thus, random variables $\tilde{\kappa}_{\varepsilon,i}(\mu^*_{\varepsilon,i}(\nu^*_\varepsilon)) = \sum_{n=1}^{\mu_{\varepsilon,i}(\nu_\varepsilon)} \kappa_{\varepsilon,i,n} \leq \xi_\varepsilon(1) = \sum_{i\in\mathbb{X}} \sum_{n=1}^{\mu_{\varepsilon,i}(\nu_\varepsilon)} \kappa_{\varepsilon,i,n} \stackrel{\mathsf{P}}{\longrightarrow} \infty$ as $\varepsilon \to 0$. This relation contradicts to the asymptotic relation given in proposition **(i)** of Theorem 1.

Relation (17) and condition **A** imply that,

$$\varphi_{\varepsilon,i}(s, 0) = \mathsf{E}_i I(\zeta_{\varepsilon,1} = 0)e^{-s\kappa_{\varepsilon,1}} \to 1 \ \text{ as } \varepsilon \to 0, \ \text{for } s \geq 0, i, j \in \mathbb{X}. \tag{18}$$

that in sequel implies the following relation,

$$|p_{\varepsilon,ij} - p_{\varepsilon,s,ij}| \leq \frac{2(1 - \varphi_{\varepsilon,i}(0, s))}{\varphi_{\varepsilon,i}(0, s)} \to 0 \text{ as } \varepsilon \to 0, \ \text{for } s \geq 0, i, j \in \mathbb{X}. \tag{19}$$

Relation (19) implies, by Lemma 1, that, for every $s \geq 0$, there exist $\varepsilon'_{0,s} \in (0, \varepsilon'_0]$ such that the Markov chain $\tilde{\eta}_{\varepsilon,n,s}$ is ergodic, for every $\varepsilon \in (0, \varepsilon'_{0,s}]$, and its stationary probabilities $\pi_{\varepsilon,s,i}$, $i \in \mathbb{X}$ satisfy the relation, $\pi_{\varepsilon,s,i} - \pi_{\varepsilon,i} \to 0$ as $\varepsilon \to 0$, for $i \in \mathbb{X}$.

Let us introduce conditional Laplace transforms,

$$\phi_{\varepsilon,ij}(\imath, s) = \mathsf{E}_i\{I(\eta_{\varepsilon,1} = j)e^{-s\kappa_{\varepsilon,1}} / \zeta_{\varepsilon,1} = \imath\}, \qquad s \geq 0,$$

for $i, j \in \mathbb{X}$, $\imath = 0, 1$, and $\phi_{\varepsilon,i}(\imath, s) = \mathsf{E}_i\{e^{-s\kappa_{\varepsilon,1}} / \zeta_{\varepsilon,1} = \imath\}$, $s \geq 0$, for $i \in \mathbb{X}$, $\imath = 0, 1$.

Relation (18) and condition **A** imply relation, $\phi_{\varepsilon,i}(0, s) = \frac{\varphi_{\varepsilon,i}(0,s)}{1 - p_{\varepsilon,i}} \to 1$ as $\varepsilon \to 0$, for $s \geq 0, i \in \mathbb{X}$. Also condition **C** is equivalent to the following relation, $\phi_{\varepsilon,i}(1, s) = \mathsf{E}_i\{e^{-s\kappa_{\varepsilon,1}} / \zeta_{\varepsilon,1} = 1\} \to 1$ as $\varepsilon \to 0$, for $s \geq 0, i \in \mathbb{X}$.

Proposition **(i)** of Lemma 1 and the above asymptotic relations for stationary probabilities $\pi_{\varepsilon,s,i}$ and $\pi_{\varepsilon,i}$ and conditional Laplace transforms $\phi_{\varepsilon,i}(0, s)$ and $\phi_{\varepsilon,i}(1, s)$ imply that,

$$A_\varepsilon(s) = -v_\varepsilon \sum_{i\in\mathbb{X}} \pi_{\varepsilon,s,i} \ln \phi_{\varepsilon,i}(0,s)$$
$$\sim v_\varepsilon \sum_{i\in\mathbb{X}} \pi_{\varepsilon,i}(1-\phi_{\varepsilon,i}(0,s)) \text{ as } \varepsilon\to 0, \text{ for } s>0. \tag{20}$$

Let us assume that Markov chains $\eta_{\varepsilon,n}$ and $\eta_{\varepsilon,n,s}$ has the same initial distribution $\bar{q}_\varepsilon$. Let us also introduce random variables,

$$\nu_{\varepsilon,s} = \min(n\geq 1 : \zeta_{\varepsilon,s,n} = 1). \tag{21}$$

The following representation takes place for the Laplace transform of the random variables $\xi_\varepsilon(1)$, for $s\geq 0$,

$$\mathsf{E}e^{-s\xi_\varepsilon(1)} = \sum_{i\in\mathbb{X}} q_{\varepsilon,i} \sum_{n=1}^{\infty} \sum_{i=i_0,i_1,\ldots,i_n\in\mathbb{X}} \prod_{k=1}^{n-1} \varphi_{\varepsilon,i_{k-1}i_k}(0,s)\varphi_{\varepsilon,i_{n-1}i_n}(1,s)$$
$$= \sum_{i\in\mathbb{X}} q_{\varepsilon,i} \sum_{n=1}^{\infty} \sum_{i=i_0,i_1,\ldots,i_{n-1}\in\mathbb{X}} \prod_{k=1}^{n-1} p_{\varepsilon,s,i_{k-1}i_k}$$
$$\times (1-p_{\varepsilon,i_{k-1}})\phi_{\varepsilon,i_{k-1}}(0,s)p_{\varepsilon,i_{n-1}}\phi_{\varepsilon,i_{n-1}}(1,s)$$
$$= \mathsf{E}\exp\{-\sum_{k=1}^{\nu_{\varepsilon,s}} -\ln\phi_{\varepsilon,\eta_{\varepsilon,s,k-1}}(0,s)$$
$$-\ln\phi_{\varepsilon,\eta_{\varepsilon,s,\nu_{\varepsilon,s}-1}}(0,s) + \ln\phi_{\varepsilon,\eta_{\varepsilon,s,\nu_{\varepsilon,s}-1}}(1,s)\}. \tag{22}$$

The above asymptotic relations for conditional Laplace transforms $\phi_{\varepsilon,i}(0,s)$ and $\phi_{\varepsilon,i}(1,s)$ obviously imply that $|\ln\phi_{\varepsilon,\eta_{\varepsilon,s,\nu_{\varepsilon,s}-1}}(0,s)| + |\ln\phi_{\varepsilon,\eta_{\varepsilon,s,\nu_{\varepsilon,s}-1}}(1,s)| \xrightarrow{\mathsf{P}} 0$ as $\varepsilon\to 0$, for $s\geq 0$. This relation and representation (22) imply the following relation,

$$\mathsf{E}e^{-s\xi_\varepsilon} \sim \mathsf{E}e^{-\tilde{\nu}_{\varepsilon,s}} \text{ as } \varepsilon\to 0, \text{ for } s>0. \tag{23}$$

where

$$\tilde{\nu}_{\varepsilon,s} = \sum_{n=1}^{\nu_{\varepsilon,s}} -\ln\phi_{\varepsilon,\eta_{\varepsilon,s,k-1}}(0,s). \tag{24}$$

Let us assume that condition **D** holds additionally to conditions **A** – **C**.

Condition **D** is equivalent to condition $\mathbf{D}_1$, and, thus, due to the above asymptotic relations for conditional Laplace transforms $\phi_{\varepsilon,i}(0,s)$ and $\phi_{\varepsilon,i}(1,s)$, condition **A** and proposition **(i)** of Lemma 1, to the following relation,

$$v_\varepsilon(1-\varphi_\varepsilon(s)) = v_\varepsilon\sum_{i\in\mathbb{X}}\pi_{\varepsilon,i}(1-\varphi_{\varepsilon,i}(s))$$
$$= v_\varepsilon\sum_{i\in\mathbb{X}}\pi_{\varepsilon,i}((1-p_{\varepsilon,i})(1-\phi_{\varepsilon,i}(0,s)) + p_{\varepsilon,i}(1-\phi_{\varepsilon,i}(1,s))$$
$$\sim v_\varepsilon\sum_{i\in\mathbb{X}}\pi_{\varepsilon,i}(1-\phi_{\varepsilon,i}(0,s)) \to A(s) \text{ as } \varepsilon\to 0, \text{ for } s>0, \tag{25}$$

where $A(s)>0$, for $s>0$ and $A(s)\to 0$ as $s\to 0$.

Relations (20) and (25) imply that, in this case,

$$A_\varepsilon(s) = -v_\varepsilon\sum_{i\in\mathbb{X}}\pi_{\varepsilon,s,i}\ln\phi_{\varepsilon,i}(0,s) \to A(s) \text{ as } \varepsilon\to 0, \text{ for } s>0. \tag{26}$$

Now, we can, for every $s>0$, apply the sufficiency statement of proposition **(iv)** of Lemma 5 to random variables $\tilde{\nu}_{\varepsilon,s}$. This yields, the following relation,

$$\tilde{\nu}_{\varepsilon,s} \xrightarrow{d} A(s)\nu_0 \text{ as } \varepsilon\to 0, \text{ for } s>0, \tag{27}$$

where $\nu_0$ is exponentially distributed random variable with parameter 1.

This relation implies, by continuity theorem for Laplace transforms, the following relation,

$$\mathsf{E}e^{-s\xi_\varepsilon} \sim \mathsf{E}e^{-\tilde{\nu}_{\varepsilon,s}} \to \mathsf{E}e^{-A(s)\nu_0} = \frac{1}{1 + A(s)} \text{ as } \varepsilon \to 0, \text{ for } s > 0. \tag{28}$$

Relation (28) proves sufficiency statements of propositions **(i)** and **(ii)** of Theorem 1.

Let now assume that conditions **A** – **C** plus the asymptotic relation penetrating proposition **(i)** of Theorem 1 hold.

The asymptotic relation (in proposition **(i)** of Theorem 1) expressed in terms of Laplace transforms takes the form of relation (which should be assumed to hold for some initial distributions $\bar{q}_\varepsilon$),

$$\mathsf{E}e^{-s\xi_\varepsilon} \to e^{-A_0(s)} \text{ as } \varepsilon \to 0, \text{ for } s > 0, \tag{29}$$

where $A_0(s) > 0$ for $s > 0$ and $A_0(s) \to 0$ as $s \to 0$.

Let us assume that conditions **A** – **C** hold but condition **D** does not hold.

The latter assumption means, due to relation (20), that either (a) $A_\varepsilon(s) \to A(s) \in (0, \infty)$ as $s \to 0$, for every $s > 0$, but $A(s) \not\to 0$ as $\varepsilon \to 0$, or (b) $A_\varepsilon(s^*) \not\to A(s^*) \in (0, \infty)$ as $\varepsilon \to 0$, for some $s^* > 0$. The latter relation holds if and only if there exist at least two subsequences $0 < \varepsilon'_n, \varepsilon''_n \to 0$ as $n \to \infty$ such that (b$_1$) $A_{\varepsilon'_n}(s^*) \to A'(s^*) \in [0, \infty]$ as $n \to \infty$, (b$_2$) $A_{\varepsilon''_n}(s^*) \to A''(s^*) \in [0, \infty]$ as $n \to \infty$ and (b$_3$) $A''(s^*) < A'(s^*)$.

In the case (a), we can repeat the part of the above proof presented in relations (20) – (28) and, taking into account relation (29), to get relation, $\mathsf{E}e^{-s\xi_\varepsilon} \sim \mathsf{E}e^{-\tilde{\nu}_{\varepsilon,s}} \to \frac{1}{1+A(s)} = e^{-A_0(s)}$ as $\varepsilon \to 0$, for $s > 0$. This relation implies that $A(s) \to 0$ as $\varepsilon \to 0$, i.e., the case (a) is impossible.

In the case (b), sub-case, $A'(s^*) = \infty$, is impossible. Indeed, by Lemma 5 applied to random variables $\tilde{\nu}_{\varepsilon,s^*}$, in this case, $\tilde{\nu}_{\varepsilon'_n,s^*} \xrightarrow{\mathsf{P}} \infty$ as $n \to \infty$, and, thus, $\mathsf{E}e^{-s^*\xi_{\varepsilon'_n}} \sim \mathsf{E}e^{-\tilde{\nu}_{\varepsilon'_n,s^*}} \to 0$ as $n \to \infty$. This relation contradicts to relation (29).

Sub-case, $A''(s^*) = 0$, is also impossible. Indeed, by Lemma 5 applied to random variables $\tilde{\nu}_{\varepsilon,s^*}$, in this case, $\tilde{\nu}_{\varepsilon''_n,s^*} \xrightarrow{\mathsf{P}} 0$ as $n \to \infty$, and, thus, $\mathsf{E}e^{-s^*\xi_{\varepsilon''_n}} \sim \mathsf{E}e^{-\tilde{\nu}_{\varepsilon''_n,s^*}} \to 1$ as $n \to \infty$. This relation also contradicts to relation (29).

Finally, the remaining sub-case, $0 < A''(s^*) < A'(s^*) < \infty$, is also impossible. Indeed, the sufficiency statement of Lemma 5, applied to random variables $\tilde{\nu}_{\varepsilon,s^*}$, yields, in this case, two relations $\tilde{\nu}_{\varepsilon'_n,s^*} \xrightarrow{d} A'(s^*)\nu_0$ as $n \to \infty$ and $\tilde{\nu}_{\varepsilon''_n,s^*} \xrightarrow{d} A''(s^*)\nu_0$ as $n \to \infty$, where $\nu_0$ is exponentially distributed random variable with parameter 1. These relations imply that $\mathsf{E}e^{-s^*\xi_{\varepsilon'_n}} \sim \mathsf{E}e^{-\tilde{\nu}_{\varepsilon'_n,s^*}} \to \frac{1}{1+A'(s^*)}$ as $n \to \infty$ and $\mathsf{E}e^{-s^*\xi_{\varepsilon''_n}} \sim \mathsf{E}e^{-\tilde{\nu}_{\varepsilon''_n,s^*}} \to \frac{1}{1+A''(s^*)}$ as $n \to \infty$. These relations contradict to relation (29), since $\frac{1}{1+A'(s^*)} \neq \frac{1}{1+A''(s^*)}$.

Therefore, condition **D** should hold. This complete the proof of propositions **(i)** and **(ii)** of Theorem 1. □

## 5. Proposition (iii) of Theorem 1

Let us consider step-sum reward stochastic processes,

$$\kappa_\varepsilon(t) = \sum_{n=1}^{[tv_\varepsilon]} \kappa_{\varepsilon,n}, t \geq 0. \tag{30}$$

**Lemma 7**. *Let condition* **B** *holds. Then,* **(i)** *condition* **D** *is necessary and sufficient condition for holding* (*for some or any initial distributions* $\bar{q}_\varepsilon$*, respectively, in statements of necessity and sufficiency*) *of the asymptotic relation,* $\kappa_\varepsilon(1) \xrightarrow{d} \theta_0$ *as* $\varepsilon \to 0$*, where* $\theta_0$ *is a non-negative random variable with distribution not concentrated in zero. In this case,* **(ii)** *the random variable* $\theta_0$ *has the infinitely divisible distribution with the Laplace* *transform* $\mathsf{E}e^{-s\theta_0} = e^{-A(s)}$, $s \geq 0$ *with the cumulant* $A(s)$ *defined in condition* **D**. *Moreover,* **(iii)** *stochastic processes* $\kappa_\varepsilon(t)$, $t \geq 0 \stackrel{\mathsf{J}}{\longrightarrow} \theta_0(t)$, $t \geq 0$ *as* $\varepsilon \to 0$, *where* $\theta_0(t)$, $t \geq 0$ *is a nonnegative càdlàg Lévy process with the Laplace transforms* $\mathsf{E}e^{-s\theta_0(t)} = e^{-tA(s)}$, $s, t \geq 0$.

**Proof**. Let us, first, prove that condition **D** implies holding of the asymptotic relations given in proposition **(iii)** of Lemma 7.

Let $\hat{\eta}_{\varepsilon,n}$, $n = 1, 2, \ldots$ be, for every $\varepsilon \in (0, \varepsilon'_0]$, a sequence of random variables such that: (a) it is independent of the Markov chain $(\eta_{\varepsilon,n}, \kappa_{\varepsilon,n})$, $n = 0, 1, \ldots$ and (b) it is a sequence of i.i.d. random variables taking value $i$ with probability $\pi_{\varepsilon,i}$, for $i \in \mathbb{X}$. In this case, the sequence of random variables $\hat{\eta}_{\varepsilon,n}$, $n = 1, 2, \ldots$ is also independent of the families of random variables $\langle \mu_{\varepsilon,i}(n), n = 0, 1, \ldots, i \in \mathbb{X} \rangle$ and $\langle \kappa_{\varepsilon,i,n}, n = 1, 2, \ldots, i \in \mathbb{X} \rangle$.

Let us consider the sequence of random variables $\theta_{\varepsilon,n} = \kappa_{\varepsilon,\hat{\eta}_{\varepsilon,n},n}$, $n = 1, 2, \ldots$. This is the sequence of i.i.d. random variables that follows from the above definition of the sequence of random variables $\hat{\eta}_{\varepsilon,n}$, $n = 1, 2, \ldots$ and the family of random variables $\kappa_{\varepsilon,i,n}$, $n = 1, 2, \ldots$, $i \in \mathbb{X}$. Also, $\mathsf{P}\{\theta_{\varepsilon,1} \leq t\} = \sum_{i \in \mathbb{X}} \pi_{\varepsilon,i} G_{\varepsilon,i}(t) = G_\varepsilon(t)$, $t \geq 0$. Let us also define the homogeneous step-sum processes with independent increments, $\theta_\varepsilon(t) = \sum_{n=1}^{[tv_\varepsilon]} \theta_{\varepsilon,n}$, $t \geq 0$. As well known, condition **D** is equivalent to the following relation, $\theta_\varepsilon(t)$, $t \geq 0 \stackrel{d}{\longrightarrow} \theta_0(t)$, $t \geq 0$ as $\varepsilon \to 0$.

Let us define random variables, $\hat{\mu}_{\varepsilon,i}(n) = \sum_{k=1}^{n} I(\hat{\eta}_{\varepsilon,n} = i)$, $n = 0, 1, \ldots$, $i \in \mathbb{X}$ and stochastic processes, $\hat{\kappa}_\varepsilon(t) = \sum_{i \in \mathbb{X}} \sum_{n=1}^{\hat{\mu}_{\varepsilon,i}([tv_\varepsilon])} \kappa_{\varepsilon,i,n}$, $t \geq 0$.

By the definition of the sequence of random variables $\langle \hat{\eta}_{\varepsilon,n}, n = 1, 2, \ldots \rangle$ and the family of random variables $\langle \kappa_{\varepsilon,i,n}, n = 1, 2, \ldots, i \in \mathbb{X} \rangle$, in particular, due to independence of the above sequence and family, the following relation holds, $\hat{\kappa}_\varepsilon(t)$, $t \geq 0 \stackrel{d}{=} \theta_\varepsilon(t)$, $t \geq 0$. Thus, $\hat{\kappa}_\varepsilon(t)$, $t \geq 0$ also is a homogeneous step-sum process with independent increments and condition **D** is equivalent to the following relation,

$$\hat{\kappa}_\varepsilon(t), t \geq 0 \stackrel{d}{\longrightarrow} \theta_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{31}$$

Random variables $I(\hat{\eta}_{\varepsilon,n} = i)$, $n = 1, 2, \ldots$ are, for every $i \in \mathbb{X}$, i.i.d. random variables taking values 1 and 0 with probabilities, respectively, $\pi_{\varepsilon,i}$ and $1 - \pi_{\varepsilon,i}$. According proposition **(i)** of Lemma 1, $0 < \underline{\lim}_{\varepsilon \to 0} \pi_{\varepsilon,i} \leq \overline{\lim}_{\varepsilon \to 0} \pi_{\varepsilon,i} < 1$, for every $i \in \mathbb{X}$. Taking into account the above remarks, this is easy to prove using the corresponding results from Skorokhod (1964, 1986), that the following relation holds,

$$\hat{\mu}^*_{\varepsilon,i}(t) = \frac{\hat{\mu}_{\varepsilon,i}([tv_\varepsilon])}{\pi_{\varepsilon,i} v_\varepsilon}, t \geq 0 \stackrel{\mathsf{J}}{\longrightarrow} \mu_{0,i}(t) = t, t \geq 0 \text{ as } \varepsilon \to 0, \text{ for } i \in \mathbb{X}. \tag{32}$$

Let us also introduce stochastic processes with independent increments,

$$\tilde{\kappa}_{\varepsilon,i}(t) = \sum_{n=1}^{[t\pi_{\varepsilon,i} v_\varepsilon]} \kappa_{\varepsilon,i,n}, t \geq 0, i \in \mathbb{X} \text{ and } \tilde{\kappa}_\varepsilon(t) = \sum_{i \in \mathbb{X}} \sum_{n=1}^{[t\pi_{\varepsilon,i} v_\varepsilon]} \kappa_{\varepsilon,i,n}, t \geq 0.$$

Note that, for every $\varepsilon \in (0, \varepsilon'_0]$, processes $\langle \tilde{\kappa}_{\varepsilon,i}(t), t \geq 0 \rangle$, $i \in \mathbb{X}$ are independent.

Let us choose some $0 < u < 1$. Since processes $\tilde{\kappa}_\varepsilon(t)$, $\hat{\kappa}_\varepsilon(t)$, and $\hat{\mu}^*_{\varepsilon,i}(t)$, $i \in \mathbb{X}$ are non-negative and non-decreasing, we get, for $x \geq 0$,

$$\begin{aligned} \mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x\} &\leq \mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x, \hat{\mu}^*_{\varepsilon,i}(1) > u, i \in \mathbb{X}\} \\ &\quad + \sum_{i \in \mathbb{X}} \mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x, \hat{\mu}^*_{\varepsilon,i}(1) \leq u\} \\ &\leq \mathsf{P}\{\hat{\kappa}_\varepsilon(1) > x\} + \sum_{i \in \mathbb{X}} \mathsf{P}\{\hat{\mu}^*_{\varepsilon,i}(1) \leq u\}. \end{aligned} \tag{33}$$

The first step is to prove that distributions of random variables $\tilde{\kappa}_{\varepsilon,i}(u)$, $i \in \mathbb{X}$ are relatively compact as $\varepsilon \to 0$, for some $u > 0$.

Relations (31)–(33) imply that distributions of random variables $\tilde{\kappa}_\varepsilon(u)$ are relatively compact as $\varepsilon \to 0$ that is,

$$\lim_{x\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x\} \leq \lim_{x\to\infty} \overline{\lim_{\varepsilon\to 0}} (\mathsf{P}\{\hat{\kappa}_\varepsilon(1) > x\} + \sum_{i\in\mathbb{X}} \mathsf{P}\{\hat{\mu}^*_{\varepsilon,i}(1) \leq u\})$$
$$= \lim_{x\to\infty} \mathsf{P}\{\theta_0(1) > x\} = 0.$$

Since, $\tilde{\kappa}_{\varepsilon,i}(u) \leq \tilde{\kappa}_\varepsilon(u)$, for $i \in \mathbb{X}$, the above relation implies that distributions of random variables $\tilde{\kappa}_{\varepsilon,i}(1)$, $i \in \mathbb{X}$ are also relatively compact as $\varepsilon \to 0$, i.e.,

$$\lim_{x\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\tilde{\kappa}_{\varepsilon,i}(u) > x\} \leq \lim_{x\to\infty} \overline{\lim_{\varepsilon\to 0}} \mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x\} = 0, \quad \text{for } i \in \mathbb{X}.$$

This implies that any sequence $0 < \varepsilon_n \to 0$ as $n \to \infty$ contains a subsequence $0 < \varepsilon_{n_k} \to 0$ as $k \to \infty$ such that random variables, $\tilde{\kappa}_{\varepsilon_{n_k},i}(u) \stackrel{d}{\longrightarrow} \theta_{0,i,u}$ as $k \to \infty$, for $i \in \mathbb{X}$, where $\theta_{0,i,u}$, $i \in \mathbb{X}$ are proper nonnegative random variables, with distributions possibly dependent of the choice of subsequence $\varepsilon_{n_k}$. Moreover, by the central criterion of convergence, the random variables $\theta_{0,i,u}$, $i \in \mathbb{X}$ have infinitely divisible distributions. Let $\mathsf{E}e^{-s\theta_{0,i,u}} = e^{-uA_i(s)}$, $s \geq 0$, $i \in \mathbb{X}$ be their Laplace transforms. As well known (see, for example, Skorokhod (1964, 1986)), the above relation of convergence in distribution implies that stochastic processes, $\tilde{\kappa}_{\varepsilon_{n_k},i}(t)$, $t \geq 0 \stackrel{\mathsf{J}}{\longrightarrow} \theta_{0,i}(t)$, $t \geq 0$ as $k \to \infty$, for $i \in \mathbb{X}$, where $\theta_{0,i}(t)$, $t \geq 0$, $i \in \mathbb{X}$ are non-negative càdlàg Lévy processes with Laplace transforms $\mathsf{E}e^{-s\theta_{0,i}(t)} = e^{-tA_i(s)}$, $s, t \geq 0$, $i \in \mathbb{X}$, possibly dependent of the choice of subsequence $\varepsilon_{n_k}$. Since processes $\tilde{\kappa}_{\varepsilon,i}(t)$, $t \geq 0$, $i \in \mathbb{X}$ are independent, vector processes $(\tilde{\kappa}_{\varepsilon_{n_k},1}(t), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(t))$, $t \geq 0 \stackrel{d}{\longrightarrow} (\theta_{0,1}(t), \ldots, \theta_{0,m}(t))$, $t \geq 0$ as $k \to \infty$, where $\theta_{0,i}(t)$, $t \geq 0$, $i \in \mathbb{X}$ are independent nonnegative càdlàg sLévy processes with Laplace transforms $\mathsf{E}e^{-s\theta_{0,i}(t)} = e^{-tA_i(s)}$, $s, t \geq 0$, $i \in \mathbb{X}$, possibly dependent of the choice of subsequence $\varepsilon_{n_k}$. By Theorem 3.8.1, in Silvestrov (2004), J-compactness of the vector processes $(\tilde{\kappa}_{\varepsilon_{n_k},1}(t), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(t))$ follows from J-compactness of their components $\tilde{\kappa}_{\varepsilon_{n_k},i}(t)$, $i \in \mathbb{X}$, since the corresponding limiting processes $\theta_{0,i}(t)$, $i \in \mathbb{X}$ are stochastically continuous and independent and, thus, they have not with probability 1 joint points of discontinuity. Thus, J-convergence of vector processes $(\tilde{\kappa}_{\varepsilon_{n_k},1}(t), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(t))$, $t \geq 0$ also takes place, i.e.,

$$(\tilde{\kappa}_{\varepsilon_{n_k},1}(t), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(t)), t \geq 0$$
$$\stackrel{\mathsf{J}}{\longrightarrow} (\theta_{0,1}(t), \ldots, \theta_{0,m}(t)), t \geq 0 \text{ as } k \to \infty, \tag{34}$$

where $\theta_{0,i}(t)$, $t \geq 0$, $i \in \mathbb{X}$ are independent nonnegative càdlàg Lévy processes described above.

Since, the limiting processes in (6) and (32) are non-random functions, relations (6), (32) and (34) imply (see, for example, Subsection 1.2.4 in Silvestrov (2004)), by Slutsky theorem, that,

$$(\mu^*_{\varepsilon_{n_k},1}(t), \ldots, \mu^*_{\varepsilon_{n_k},m}(t), \tilde{\kappa}_{\varepsilon_{n_k},1}(t), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(t)),\ t \geq 0$$
$$\stackrel{d}{\longrightarrow} (\mu_{0,1}(t), \ldots, \mu_{0,m}(t), \theta_{0,1}(t), \ldots, \theta_{0,m}(t)),\ t \geq 0 \quad \text{as } k \to \infty,$$

and

$$(\hat{\mu}^*_{\varepsilon_{n_k},1}(t), \ldots, \hat{\mu}^*_{\varepsilon_{n_k},m}(t), \tilde{\kappa}_{\varepsilon_{n_k},1}(t), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(t)),\ t \geq 0$$
$$\stackrel{d}{\longrightarrow} (\mu_{0,1}(t), \ldots, \mu_{0,m}(t), \theta_{0,1}(t), \ldots, \theta_{0,m}(t)),\ t \geq 0 \quad \text{as } k \to \infty,$$

where $\mu_{0,i}(t) = t$, $t \geq 0$, $i \in \mathbb{X}$ and $\theta_{0,i}(t)$, $t \geq 0$, $i \in \mathbb{X}$ are independent nonnegative càdlàg Lévy processes defined above. We can now apply Theorem 3.8.2, from Silvestrov (2004), which give conditions of J-convergence for vector compositions of càdlàg stochastic processes, and get the asymptotic relations,

$$(\tilde{\kappa}_{\varepsilon_{n_k},1}(\mu^*_{\varepsilon_{n_k},1}(t)), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(\mu^*_{\varepsilon_{n_k},m}(t))),\ t \geq 0 \\ \stackrel{\mathsf{J}}{\longrightarrow} (\theta_{0,1}(\mu_{0,1}(t)), \ldots, \theta_{0,m}(\mu_{0,m}(t))) = (\theta_{0,1}(t), \ldots, \theta_{0,m}(t)),\ t \geq 0 \quad \text{as } k \to \infty,$$

and

$$(\tilde{\kappa}_{\varepsilon_{n_k},1}(\hat{\mu}^*_{\varepsilon_{n_k},1}(t)), \ldots, \tilde{\kappa}_{\varepsilon_{n_k},m}(\hat{\mu}^*_{\varepsilon_{n_k},m}(t))),\ t \geq 0 \\ \stackrel{\mathsf{J}}{\longrightarrow} (\theta_{0,1}(\mu_{0,1}(t)), \ldots, \theta_{0,m}(\mu_{0,m}(t))) = (\theta_{0,1}(t), \ldots, \theta_{0,m}(t)),\ t \geq 0 \quad \text{as } k \to \infty,$$

where $\theta_{0,i}(t)$, $t \geq 0$, $i \in \mathbb{X}$ are independent nonnegative càdlàg Lévy processes defined in relation above. The latter two asymptotic relations obviously imply J-convergence for sum of components of the processes in these relations, i.e. that, respectively, the following relations hold,

$$\kappa_{\varepsilon_{n_k}}(t) = \sum_{i \in \mathbb{X}} \tilde{\kappa}_{\varepsilon_{n_k},i}(\mu^*_{\varepsilon_{n_k},i}(t)), t \geq 0 \\ \stackrel{\mathsf{J}}{\longrightarrow} \theta'_0(t) = \sum_{i \in \mathbb{X}} \theta_{0,i}(t), t \geq 0 \text{ as } k \to \infty, \tag{35}$$

and

$$\hat{\kappa}_{\varepsilon_{n_k}}(t) = \sum_{i \in \mathbb{X}} \tilde{\kappa}_{\varepsilon_{n_k},i}(\hat{\mu}^*_{\varepsilon_{n_k},i}(t)), t \geq 0 \\ \stackrel{\mathsf{J}}{\longrightarrow} \theta'_0(t) = \sum_{i \in \mathbb{X}} \theta_{0,i}(t), t \geq 0 \text{ as } k \to \infty, \tag{36}$$

where $\theta_{0,i}(t)$, $t \geq 0$, $i \in \mathbb{X}$ are independent nonnegative càdlàg Lévy processes defined above.

Relation (31) implies that $\theta'_0(t)$, $t \geq 0 \stackrel{d}{=} \theta_0(t)$, $t \geq 0$. Thus, the limiting process $\theta'_0(t) = \sum_{i \in \mathbb{X}} \theta_{0,i}(t)$, $t \geq 0$ has the same finite dimensional distributions for all subsequences $\varepsilon_{n_k}$ described above. Moreover, the cumulant $A(s)$ of the limiting Lévy process $\theta_0(t)$ is connected with cumulants $A_i(s)$, $i \in \mathbb{X}$ of Lévy processes $\theta_{0,i}(t)$ by relation, $A(s) = \sum_{i \in \mathbb{X}} A_i(s)$, $s \geq 0$.

Therefore, relations (35) and (36) imply that, respectively, the following relations hold,

$$\kappa_\varepsilon(t) = \sum_{i \in \mathbb{X}} \tilde{\kappa}_{\varepsilon,i}(\mu^*_{\varepsilon,i}(t)), t \geq 0 \stackrel{\mathsf{J}}{\longrightarrow} \theta_0(t), t \geq 0 \text{ as } \varepsilon \to 0, \tag{37}$$

and

$$\hat{\kappa}_\varepsilon(t) = \sum_{i \in \mathbb{X}} \tilde{\kappa}_{\varepsilon,i}(\hat{\mu}^*_{\varepsilon,i}(t)), t \geq 0 \stackrel{\mathsf{J}}{\longrightarrow} \theta_0(t), t \geq 0 \text{ as } \varepsilon \to 0. \tag{38}$$

Let us now prove that the asymptotic relation given in proposition **(i)** of Lemma 7 implies that condition **D** holds.

Again, the first step is to prove that distributions of random variables $\tilde{\kappa}_{\varepsilon,i}(u)$, $i \in \mathbb{X}$ are relatively compact as $\varepsilon \to 0$, for some $u > 0$.

Let us choose some $0 < u < 1$. Since processes $\kappa_\varepsilon(t)$, $\tilde{\kappa}_\varepsilon(t)$, and $\mu^*_{\varepsilon,i}(t)$, $i \in \mathbb{X}$ are non-negative and non-decreasing, we get, for any $x \geq 0$,

$$\mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x\} \leq \mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x, \mu^*_{\varepsilon,i}(1) > u, i \in \mathbb{X}\} \\ + \sum_{i \in \mathbb{X}} \mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x, \mu^*_{\varepsilon,i}(1) \leq u\} \\ \leq \mathsf{P}\{\kappa_\varepsilon(1) > x\} + \sum_{i \in \mathbb{X}} \mathsf{P}\{\mu^*_{\varepsilon,i}(1) \leq u\}. \tag{39}$$

Relation (39) and the asymptotic relation given in proposition **(i)** of Lemma 7 imply that,

$$\lim_{x\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\mathsf{P}\{\tilde{\kappa}_\varepsilon(u) > x\} \leq \lim_{x\to\infty}\overline{\lim_{\varepsilon\to 0}}(\mathsf{P}\{\kappa_\varepsilon(1) > x\} + \sum_{i\in\mathbb{X}}\mathsf{P}\{\mu^*_{\varepsilon,i}(1) \leq u\})$$
$$= \lim_{x\to\infty}\mathsf{P}\{\theta_0 > x\} = 0.$$

Note that, in this necessity case, the asymptotic relation given in proposition **(i)** of Lemma 7 is required to hold only for at least one family initial distributions $\bar{q}_\varepsilon$, $\varepsilon \in (0,\varepsilon_0]$. Since, $\tilde{\kappa}_{\varepsilon,i}(u) \leq \tilde{\kappa}_\varepsilon(u)$, for $i \in \mathbb{X}$, the above relation implies that distributions of random variables $\tilde{\kappa}_{\varepsilon,i}(1)$, $i \in \mathbb{X}$ are relatively compact as $\varepsilon \to 0$.

Now, we can repeat the part of the above prove related to relations (34) – (36).

Relation (35) and the asymptotic relation given in proposition **(i)** of Lemma 7 imply that the random variables $\theta'(1)$ and $\theta_0$, which appears in these asymptotic relations, have the same distribution, i.e. $\theta'(1) \stackrel{d}{=} \theta_0$. Moreover, cumulant $A(s)$ of the limiting Lévy process $\theta'_0(t)$ coincides with the cumulant of the random variable $\theta_0$, which, therefore, has infinitely divisible distribution. Moreover, relation (36) implies that cumulant $A(s)$ is connected with cumulants $A_i(s)$, $i \in \mathbb{X}$ of Lévy processes $\theta'_{0,i}(t)$ by relation $A(s) = \sum_{i\in\mathbb{X}} A_i(s)$, $s \geq 0$. Thus, the limiting process $\theta'_0(t)$, $t \geq 0 = \sum_{i\in\mathbb{X}}\theta_{0,i}(t)$, $t \geq 0$ has the same finite dimensional distributions for all subsequences $\varepsilon_{n_k}$ described above.

This let us write down relations (37) – (38). Relation (38) proves, in this case, that condition **D** holds. Relation (37) proves proposition **(iii)** of Lemma 7. □

It is useful to note that the flag variables $\zeta_{\varepsilon,n}$ are not involved in the definition of the processes $\kappa_\varepsilon(t)$. This let one replace function $v_\varepsilon = p_\varepsilon^{-1}$ by an arbitrary function $0 < v_\varepsilon \to \infty$ as $\varepsilon \to 0$ in condition **D** and Lemma 7. In this case, Lemma 7 gives necessary and sufficient conditions for convergence in J-topology for the step-sum reward processes $\kappa_\varepsilon(t)$, $t \geq 0$. As I think, it is the result of some independent interest.

The following lemma brings together the asymptotic relations given in Lemma 4 and 7.

**Lemma 8**. *Let conditions* **A**, **B**, **C** *and* **D** *hold. Then, the following asymptotic relation holds,* $(\nu^*_\varepsilon, \kappa_\varepsilon(t))$, $t \geq 0 \stackrel{d}{\longrightarrow} (\nu_0, \theta_0(t))$, $t \geq 0$ *as* $\varepsilon \to 0$, **(a)** $\nu_0$ *is a random variable, which has the exponential distribution with parameter* 1, **(b)** $\theta_0(t)$, $t \geq 0$ *is a nonnegative càdlàg Lévy process with the Laplace transforms* $\mathsf{E}e^{-s\theta_0(t)} = e^{-tA(s)}$, $s, t \geq 0$, *with the cumulant* $A(s)$ *defined in condition* **D**, **(c)** *the random variable* $\nu_0$ *and the process* $\theta_0(t)$, $t \geq 0$ *are independent.*

**Proof**. The following representation takes place, for $s, t \geq 0$,

$$\mathsf{E}I(\nu^*_\varepsilon > t)e^{-s\kappa_\varepsilon(t)} = \sum_{i\in\mathbb{X}} q_{\varepsilon,i} \sum_{i=i_0,i_1,\ldots,i_{[tv_\varepsilon]}\in\mathbb{X}} \prod_{k=1}^{[tv_\varepsilon]} \varphi_{\varepsilon,i_{k-1}i_k}(0,s)$$
$$= \mathsf{E}\exp\{-\sum_{k=1}^{[tv_\varepsilon]}(-\ln(1-p_{\varepsilon,\tilde{\eta}_{\varepsilon,s,k-1}}) - \ln\phi_{\varepsilon,\tilde{\eta}_{\varepsilon,s,k-1}}(0,s))\}. \tag{40}$$

Using condition **A**, **B**, Lemma 1 and relation $\pi_{\varepsilon,s,i} - \pi_{\varepsilon,i} \to 0$ as $\varepsilon \to 0$, for $i \in \mathbb{X}$., we get relation, $f_{\varepsilon,s} = -v_\varepsilon\sum_{i\in\mathbb{X}}\tilde{\pi}_{\varepsilon,s,i}\ln(1-p_{\varepsilon,i}) \sim v_\varepsilon\sum_{i\in\mathbb{X}}\pi_{\varepsilon,i}p_{\varepsilon,i} = v_\varepsilon p_\varepsilon = 1$ as $\varepsilon \to 0$, for $s \geq 0$. Thus, Lemma 3 can, for every $s > 0$, be applied to the processes, $\kappa_{\varepsilon,s}(t) = \sum_{k=1}^{[tv_\varepsilon]}(-\ln(1-p_{\varepsilon,\tilde{\eta}_{\varepsilon,k-1}}) - \ln\phi_{\varepsilon,\tilde{\eta}_{\varepsilon,k-1}}(0,s))$, $t \geq 0$.

This yields that the following relation holds, for every $s > 0$,

$$\kappa_{\varepsilon,s}(t), t \geq 0 \stackrel{d}{\longrightarrow} t + A(s)t, t \geq 0 \text{ as } \varepsilon \to 0. \tag{41}$$

Let us denote,

$$\Psi_{\varepsilon,ij}(n,s) = \mathsf{E}_i I(\nu_\varepsilon > n, \eta_{\varepsilon,n} = j) e^{-s\sum_{k=1}^n \kappa_{\varepsilon,k}},$$
$$\Psi_{\varepsilon,i}(n,s) = \mathsf{E}_i I(\nu_\varepsilon > n) e^{-s\sum_{k=1}^n \kappa_{\varepsilon,k}} = \sum_{j\in\mathbb{X}} \Psi_{\varepsilon,ij}(n,s)$$

and

$$\psi_{\varepsilon,ij}(n,s) = \mathsf{E}_i I(\eta_{\varepsilon,n} = j) e^{-s\sum_{k=1}^n \kappa_{\varepsilon,k}}, \psi_{\varepsilon,i}(n,s) = \mathsf{E}_i e^{-s\sum_{k=1}^n \kappa_{\varepsilon,k}} = \sum_{j\in\mathbb{X}} \psi_{\varepsilon,ij}(n,s)$$

for $i,j \in \mathbb{X}$, $n = 0,1,\ldots$, $s \geq 0$.

The following representation for multivariate joint distributions of random variable $\nu_\varepsilon^*$ and increments of stochastic process $\kappa_\varepsilon(t)$ takes place, for $0 = t_0 \leq t_1 < \cdots t_k = t \leq t_{k+1} \leq \cdots \leq t_n < \infty$, $1 \leq k < n < \infty$ and $s_1,\ldots,s_n \geq 0$,

$$\begin{aligned}
&\mathsf{E}I(\nu_\varepsilon^* > t_k)\exp\{-\sum_{r=1}^n s_r(\kappa_\varepsilon(t_r) - \kappa_\varepsilon(t_{r-1})\} \\
&\quad = \sum_{i_0\in\mathbb{X}} q_{\varepsilon,i_0}\prod_{l=1}^k\sum_{i_l\in\mathbb{X}}\Psi_{\varepsilon,i_{l-1}i_l}([t_l v_\varepsilon] - [t_{l-1}v_\varepsilon], s_l) \\
&\quad\quad \times \prod_{l=k+1}^n\sum_{i_l\in\mathbb{X}}\psi_{\varepsilon,i_{l-1}i_l}([t_l v_\varepsilon] - [t_{l-1}v_\varepsilon], s_l).
\end{aligned} \tag{42}$$

Relation (41) readily implies asymptotic relation, $\Psi_{\varepsilon,i}([t'v_\varepsilon] - [t''v_\varepsilon], s) \sim \Psi_{\varepsilon,i}([(t' - t'')v_\varepsilon], s) \to e^{-(t'-t'')}e^{-A(s)(t'-t'')}$ as $\varepsilon \to 0$, for $s > 0$, $0 \leq t'' \leq t' < \infty$. Also the proposition **(iii)** of Lemma 7 implies asymptotic relation, $\psi_{\varepsilon,i}([t'v_\varepsilon] - [t''v_\varepsilon], s) \sim \psi_{\varepsilon,i}([(t' - t'')v_\varepsilon], s) \to e^{-A(s)(t'-t'')}$ as $\varepsilon \to 0$, for $s > 0$ and $0 \leq t'' \leq t' < \infty$. Using these asymptotic relations and representation (42) we get recurrently, for $0 = t_0 \leq t_1 < \cdots t_k = t \leq t_{k+1} \leq \cdots \leq t_n < \infty$, $1 \leq k < n < \infty$ and $s_1,\ldots,s_n > 0$,

$$\begin{aligned}
&\mathsf{E}I(\nu_\varepsilon^* > t_k)\exp\{-\sum_{r=1}^n s_r(\kappa_\varepsilon(t_r) - \kappa_\varepsilon(t_{r-1}))\} \\
&\sim \mathsf{E}I(\nu_\varepsilon^* > t_k)\exp\{-\sum_{r=1}^{n-1} s_r(\kappa_\varepsilon(t_r) - \kappa_\varepsilon(t_{r-1}))\}e^{-A(s_n)(t_n - t_{n-1})} \\
&\cdots \sim \mathsf{E}I(\nu_\varepsilon^* > t_k)\exp\{-\sum_{r=1}^k s_r(\kappa_\varepsilon(t_r) - \kappa_\varepsilon(t_{r-1}))\} \\
&\quad \times \exp\{\sum_{r=k+1}^n -A(s_r)(t_r - t_{r-1})\} \\
&\sim \mathsf{E}I(\nu_\varepsilon^* > t_{k-1})\exp\{-\sum_{r=1}^{k-1} s_r(\kappa_\varepsilon(t_r) - \kappa_\varepsilon(t_{r-1}))\} \\
&\quad \times \exp\{-(t_k - t_{k-1})\}\exp\{\sum_{r=k}^n -A(s_r)(t_r - t_{r-1})\} \\
&\cdots \sim \exp\{-\sum_{r=1}^k (t_r - t_{r-1})\}\exp\{\sum_{r=1}^n -A(s_r)(t_r - t_{r-1})\} \\
&= \exp\{-t\}\exp\{\sum_{r=1}^n -A(s_r)(t_r - t_{r-1})\} \text{ as } \varepsilon \to 0.
\end{aligned} \tag{43}$$

This relation is equivalent an form of the asymptotic relation given in Lemma 8. □

Now, we can complete the proof of proposition **(iii)** of Theorem 1.

The asymptotic relation given in Lemma 8 can, obviously, be rewritten in the following equivalent form,

$$(t\nu_\varepsilon^*, \kappa_\varepsilon(t)), t \geq 0 \xrightarrow{d} (t\nu_0, \theta_0(t)), t \geq 0 \quad \text{as } \varepsilon \to 0,$$

where the random variable $\nu_0$ and the stochastic process $\theta_0(t)$, $t \geq 0$ are described in Lemma 8. This asymptotic relation and the asymptotic relation given in proposition **(iii)** of Lemma 7 let us apply Theorem 3.4.1 from Silvestrov (2004) to the compositions of stochastic processes $\kappa_\varepsilon(t)$, $t \geq 0$ and $t\nu_\varepsilon^*$, $t \geq 0$ that yield the following relation, $\xi_\varepsilon(t) = \kappa_\varepsilon(t\nu_\varepsilon^*)$, $t \geq 0 \xrightarrow{\mathrm{J}} \theta_0(t\nu_0)$, $t \geq 0$ as $\varepsilon \to 0$. □

As a concluding remark, I would like to say that getting of necessary and sufficient conditions of convergence without a gap between their necessary and sufficient parts is usually a challenging and difficult problem. Here, I would like to mention some prospectives directions for further studies of such conditions of convergence for first-rare-event-type functionals. It would be interesting to try to weaken the model ergodicity condition **B**, i.e., to expand studies to singularly perturbed models. Another model condition **C**, seems, also can be weaken. This can cause appearance of additional non-trivial components for limiting first-rare-event processes. It would also be interesting to generalize the above results to the model of first-rare-event reward functionals and processes with real-valued random summands in defining relations. A generalization of the above results to the model with countable and general phase spaces $\mathbb{X}$ is another open problem. A conjecture is that some additional compactness conditions for averaging stationary distributions should be involved in this case. Applications of the above results to queuing and reliability models, communication networks, models of population dynamics, insurance models, etc. is a prospective and unbounded direction for future studies.

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