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BISPECTRUM AND A NONLINEAR MODEL FOR NON-GAUSSIAN HOMOGENOUS AND ISOTROPIC FIELD IN 3D

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Dedicated to Professor Nikolai N. Leonenko on the occasion of his 65th birthday

Abstract. The so-called bispectrum is a widely used construction for analyzing nonlinear time series. In this paper the generalized bispectrum of a homogenous and isotropic stochastic field in 3D is introduced. The isotropy is considered in third order, and we give some necessary and sufficient conditions for isotropy of homogenous random fields. The spatial three-point correlation function (bicovariance function) is given by the bispectrum in terms of a kernel function, which is a superposition of spherical Bessel-functions and Legendre-polynomials. In return, the same kernel function is used in expressing the bispectrum by the bicovariance function. As an example, we generalize a model for non-Gaussian fields, which is the sum of a Gaussian-field and its 2nd degree Hermite-polynomial. This model can be applied as an alternative to the Gaussian one used in Cosmology for non-Gaussian CMB temperature fluctuations.

1. Introduction

Homogenous and isotropic stochastic fields have got some growing attention recently in several fields of science, including Cosmology [31]. Data coming from the cosmic microwave radiation background – a courtesy of NASA (http://lambda.gsfc.nasa.gov/) – are available for statistical analysis. The data are placed into a particular pixel structure on the surface of a 2D sphere. Stochastic modeling of the data includes isotropic stochastic fields on spheres with small perturbations of gravitational potential fields on $\mathbb{R}^3$, according to Newtonian Cosmology, [30], [11], [31, p. 139].

The basic theory of homogenous and isotropic stochastic fields in frequency domain was developed by [32], and several interesting results has been published ever since, see e.g. [16], [17], [18], [7], and [15]. Another line of investigation is summarized in [3], and a general one in [33]. All these studies concern to either the Gaussian-case, or the one which is equivalent to second order structure (covariance function and spectrum) of the fields, see [20] for application in Geophysics. For instance, the problem of testing Gaussianity of cosmic microwave background temperature fluctuations on a sphere, involves higher order spectra, see [27] and the references therein.

In this paper the bispectrum for a homogenous and isotropic stochastic field in 3D is introduced as a generalization of the bispectrum widely used in time series analysis, [5], [25]. In this respect we introduce the isotropy in third order, and give some necessary and sufficient conditions for isotropy for a homogenous field. The spatial three-point correlation function (bicovariance function) will be given by the bispectrum in terms of a kernel function, which is a superposition of spherical Bessel-functions and Legendre-polynomials. In return, the same kernel function serves for expressing the bispectrum by the bicovariance function. We generalize the model which is the sum of a Gaussian-field

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and its 2nd degree Hermite-polynomial. This model is applied as an alternative to the Gaussian one used in Cosmology for non-Gaussian CMB temperature fluctuations for instance, see [14], [10], [2]; and also in signal processing context, see e.g. [22].

We note that the further generalization of the bispectrum to higher order spectra is not straightforward at all, at least the trispectrum is necessary for understanding the general pattern, see [26] and [28] for 2D fields.

2. Homogenous and isotropic field

We consider homogenous stochastic real-valued fields $X(\underline{x})$ on $\mathbb{R}^3$, let us suppose that $X(\underline{x})$ is continuous (in mean square sense) and apply Rayleigh plane wave expansion in 3D in terms of spherical harmonics Y_ℓ^m, see (B.1), (B.6), and spherical Bessel-function $j_\ell(z)$ of the first kind, (B.7),

$$\begin{aligned} X(\underline{x}) &= \int_{\mathbb{R}^3} e^{i\underline{x}\cdot\underline{\omega}} Z(d\underline{\omega}) \\ &= 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}})\, i^\ell \int_0^\infty j_\ell(\rho r) \int_{\mathbb{S}_2} Y_\ell^m(\widehat{\underline{\omega}})^* Z\left(\Omega(d\widehat{\underline{\omega}})\,\rho^2 d\rho\right) \\ &= 4\pi \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}}) \int_0^\infty j_\ell(\rho r)\, Z_\ell^m\left(\rho^2 d\rho\right), \end{aligned} \tag{2.1}$$

where $\underline{\omega}, \underline{x} \in \mathbb{R}^3$, $r = |\underline{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\rho = |\underline{\omega}|$, $\widehat{\underline{x}} = \underline{x}/|\underline{x}|$, $\widehat{\underline{\omega}} = \underline{\omega}/|\underline{\omega}|$, $\mathbb{S}_2$ denotes the unit sphere, $\Omega(d\widehat{\underline{\omega}}) = \sin\eta d\eta d\zeta$ is Lebesgue element of surface area on $\mathbb{S}_2$ and

$$Z_\ell^m\left(\rho^2 d\rho\right) = i^\ell \int_{\mathbb{S}_2} Y_\ell^m(\widehat{\underline{\omega}})^* Z\left(\Omega(d\widehat{\underline{\omega}})\,\rho^2 d\rho\right). \tag{2.2}$$

The second order isotropy of $X(\underline{x})$ is defined by $\operatorname{Cov}\left(X(\underline{x}), X(\underline{y})\right) = C\left(|\underline{x} - \underline{y}|\right)$, i.e. the covariance does not depend on direction but only the distance. This implies and implied by that the spectral measure $F(d\underline{\omega})/(2\pi)^3 = E|Z(d\underline{\omega})|^2 = E\left|Z\left(\Omega(d\widehat{\underline{\omega}})\,\rho^2 d\rho\right)\right|^2$ is separated $F\left(\Omega(d\widehat{\underline{\omega}})\,\rho^2 d\rho\right) = \Omega(d\widehat{\underline{\omega}})\, F\left(\rho^2 d\rho\right)$. There will be no confusion if the unit vector $\widehat{\underline{\omega}} = \widehat{\underline{\omega}}(\eta, \zeta)$, in frequency domain denotes also the Euler-angles (η, ζ), similarly (ϑ, φ) corresponds to the space unit vector $\widehat{\underline{x}}$ where $\eta, \vartheta \in [0, \pi]$ are co-latitudes and $\zeta, \varphi \in [0, 2\pi]$ are longitudes. In this way the isotropic random field $X(\underline{x})$ can be decomposed into a countable number of mutually uncorrelated stationary processes with a one dimensional parameter, since

$$\operatorname{Cov}\left(Z_{\ell_1}^k\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^m\left(\rho_2^2 d\rho_2\right)\right) = \delta_{\ell_1-\ell_2}\delta_{k-m}\delta(\rho_1 - \rho_2)\, F\left(\rho^2 d\rho\right)/(2\pi)^3,$$

where δ_{k-m} and $\delta(\rho_1 - \rho_2)$ denote the Kronecker and Dirac-delta respectively.

An important characterization of the isotropy (in Gaussian-case it concerns to the covariance function) is the *invariance under rotation.* Let us consider a rotation $g \in SO(3)$, it is known that the spherical harmonics Y_ℓ^m at the rotated location are given in terms of the Wigner D-matrix, more precisely

$$\Lambda(g)\, Y_\ell^m(\widehat{\underline{x}}) = \sum_{k=-\ell}^{\ell} D_{k,m}^{(\ell)}(g)\, Y_\ell^k(\widehat{\underline{x}}),$$

where $\Lambda(g)$ denotes the operator according to the rotation g, $\Lambda(g)\, Y_\ell^k(\underline{\widehat{x}}) = Y_\ell^k\left(g^{-1}\underline{\widehat{x}}\right)$. Hence the rotated field has the following form

$$\begin{aligned}\Lambda(g)\, X(\underline{x}) &= 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m\left(g^{-1}\underline{\widehat{x}}\right) \int_0^\infty j_\ell(\rho r)\, Z_\ell^m\left(\rho^2 d\rho\right) \\ &= 4\pi \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} Y_\ell^k(\underline{\widehat{x}}) \int_0^\infty j_\ell(\rho r) \sum_{m=-\ell}^{\ell} D_{k,m}^{(\ell)}(g)\, Z_\ell^m\left(\rho^2 d\rho\right) \\ &= 4\pi \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} Y_\ell^k(\underline{\widehat{x}}) \int_0^\infty j_\ell(\rho r)\, \mathcal{Z}_\ell^k\left(\rho^2 d\rho\right). \qquad (2.3)\end{aligned}$$

Definition 1. *A homogenous field $X(\underline{x})$ called strictly isotropic if $X(\underline{x})$ equals to $\Lambda(g)\, X(\underline{x})$ in distribution for each rotation $g \in SO(3)$.*

The assumption of strict isotropy is equivalent to that of the distribution of the rotated array

$$\mathcal{Z}_\ell^k\left(\rho^2 d\rho\right) = \sum_{m=-\ell}^{\ell} D_{k,m}^{(\ell)}(g)\, Z_\ell^m\left(\rho^2 d\rho\right), \qquad (2.4)$$

equals to the distribution of the array $Z_\ell^m\left(\rho^2 d\rho\right)$ for each $g \in SO(3)$. In Gaussian-case, isotropy (strictly) follows directly from the orthogonality of Wigner D-matrices, see Appendix (B.13), (B.14).

It is well known, see [32] for instance, that the covariance function

$$C_2(r) = \operatorname{Cov}\left(X(\underline{x}), X(\underline{y})\right) = C\left(\left|\underline{x} - \underline{y}\right|\right)$$

of a homogenous and isotropic field $X(\underline{x})$ is expressed by the spectrum

$$C_2(r) = \frac{1}{2\pi^2} \int_0^\infty j_0(\rho r)\, F\left(\rho^2 d\rho\right),$$

in terms of spherical Bessel-function $j_\ell(z)$, actually $j_0(\rho) = \sin\rho/\rho$. In turn, when $F\left(\rho^2 d\rho\right) = S_2(\rho)\,\rho^2 d\rho$ we also have the inversion

$$S_2(\rho) = 4\pi \int_0^\infty j_0(\rho r)\, C_2(r)\, r^2 dr.$$

3. Bispectrum

A Gaussian homogenous and isotropic field $X(\underline{x})$ is invariant under translations and rotations, moreover, distributional properties are equivalent to the same properties of covariance function. It follows from (2.3) that a homogenous field $X(\underline{x})$ is strictly isotropic if and only if the distribution of rotated stochastic measures $\mathcal{Z}_\ell^m$, see (2.4) equal in distribution to Z_ℓ^m. From now on we *assume the existence of third order moments, at least.* Similarly to the second order case, we refer to the third order cumulants as spatial three-point covariance functions, or simply bicovariances. Strict isotropy implies the invariance under rotations of the bicovariances as well. If the bicovariances of $X(\underline{x})$ are invariant under rotations, then $X(\underline{x})$ will be called *isotropy in third order.*

Lemma 1. *Let us assume the absolute continuity of third order cumulants*

$$\operatorname{Cum}\left(Z_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{m_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{m_3}\left(\rho_2^2 d\rho_2\right)\right)$$

of stochastic measures Z_ℓ^m. The necessary and sufficient condition of isotropy in third order of a homogenous field $X(\underline{x})$ is that the triangle array Z_ℓ^m of the spectral measures *fulfil equation*

$$\begin{aligned}&\operatorname{Cum}\left(Z_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{m_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{m_3}\left(\rho_3^2 d\rho_3\right)\right)\\ &\quad=\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} B_{\ell_1,\ell_2,\ell_3}\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\rho_k^2 d\rho_k,\end{aligned}\tag{3.1}$$

such that

$$\begin{aligned}&B_{\ell_1,\ell_2,\ell_3}\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\rho_k^2 d\rho_k\\ &\quad=\sum_{p,q,r}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ p & q & r\end{pmatrix}\operatorname{Cum}\left(Z_{\ell_1}^{p}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{q}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{r}\left(\rho_3^2 d\rho_3\right)\right),\end{aligned}$$

where $\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}$ *denotes the Wigner* $3j$*-symbols (see Appendix B, 5).*

See Appendix A.1 for the proof.

Now, we return to the bicovariance, assume absolute continuity of bispectral measure

$$\operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right)\right)=\iiint_{\mathbb{R}^{3\times3}} e^{i\left(\Sigma_1^3\underline{x}_k\cdot\underline{\omega}_k\right)} S_3^0\left(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3\right)\delta\left(\Sigma_1^3\underline{\omega}_k\right)\prod_{k=1}^{3} d\underline{\omega}_k,$$

in this way we define the bispectrum S_3^0 for a homogenous process. It is translation invariant, hence it writes

$$\operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(\underline{0}\right)\right)=\iint_{\mathbb{R}^{2\times3}} e^{i\left(\underline{x}_1\cdot\underline{\omega}_1+\underline{x}_2\cdot\underline{\omega}_2\right)} S_3^0\left(\underline{\omega}_1,\underline{\omega}_2,-\underline{\omega}_1-\underline{\omega}_2\right)\prod_{k=1}^{2} d\underline{\omega}_k.$$

Let us assume isotropy in third order, we rotate first $\underline{x}_2$ into the North Pole $N=(0,0,1)$, it becomes r_2N, then $\underline{x}_1$ into the plane $\varphi=0$. The result is that the bicovariances are defined by the triplet (r_1,r_2,ϑ), $r_1,r_2\geq 0$, $\vartheta\in[0,\pi]$, hence $\mathcal{C}_3\left(r_1,r_2,\vartheta\right)=\operatorname{Cum}\left(X\left(r_1\widehat{\underline{x}}\right), X\left(r_2N\right), X(0)\right)$, where $\widehat{\underline{x}}=(\sin\vartheta,0,\cos\vartheta)$. The consequence for the bispectrum is that for each $g\in SO(3)$

$$\begin{aligned}&\operatorname{Cum}\left(X\left(g\underline{x}_1\right), X\left(g\underline{x}_2\right), X\left(g\underline{x}_3\right)\right)\\ &\quad=\iiint_{\mathbb{R}^{3\times3}} e^{i\left(\Sigma_1^3\underline{x}_k\cdot\underline{\omega}_k\right)} S_3^0\left(g\underline{\omega}_1,g\underline{\omega}_2,g\underline{\omega}_3\right)\delta\left(\Sigma_1^3\underline{\omega}_k\right)\prod_{k=1}^{3} d\underline{\omega}_k\\ &\quad=\operatorname{Cum}\left(X\left(\underline{x}_1\right), X\left(\underline{x}_2\right), X\left(\underline{x}_3\right)\right),\end{aligned}$$

hence $S_3^0\left(g\underline{\omega}_1,g\underline{\omega}_2,g\underline{\omega}_3\right)=S_3^0\left(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3\right)$, this yields that S_3 depends on ρ_1,ρ_2,ρ_3 and $\widehat{\underline{\omega}}_k\cdot\widehat{\underline{\omega}}_m$ only, more precisely it depends on the angles between the vectors, and all these angles belong into $[0,\pi]$. In addition $\underline{\omega}_1$, $\underline{\omega}_2$, $\underline{\omega}_3$ fulfill the equation $\Sigma_1^3\underline{\omega}_k=0$, i.e. $\underline{\omega}_1$, $\underline{\omega}_2$, $\underline{\omega}_3$ form a triangle. Now, a triangle is defined by its sides ρ_1, ρ_2, ρ_3 and is invariant under the movement of a rigid body, hence $S_3^0\left(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3\right)=S_3\left(\rho_1,\rho_2,\rho_3\right)/\left(4\pi\right)^3$. By the law of cosines a side, let's say ρ_3, is expressed by the two other sides ρ_1, ρ_2, and the angle η contained between these sides; $\rho_3=\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\eta}$, $\eta\in[0,\pi]$. Hence $S_3\left(\rho_1,\rho_2,\rho_3\right)=S_3\left(\rho_1,\rho_2,\eta\right)$, where $0<\rho_1,\rho_2$, and $\eta\in[0,\pi]$, we shall use both equivalent notations $S_3\left(\rho_1,\rho_2,\rho_3\right)$ and $S_3\left(\rho_1,\rho_2,\eta\right)$. Let us consider the cumulant of spectral measure, we have

$$\begin{aligned}\operatorname{Cum}\left(Z\left(d\underline{\omega}_1\right), Z\left(d\underline{\omega}_2\right), Z\left(d\underline{\omega}_3\right)\right)&=\delta\left(\Sigma_1^3\underline{\omega}_k\right)S_3\left(\underline{\omega}_1,\underline{\omega}_2,\underline{\omega}_3\right)d\underline{\omega}_1 d\underline{\omega}_2 d\underline{\omega}_3\\ &=\delta\left(\Sigma_1^3\rho_k\widehat{\underline{\omega}}_k\right)S_3\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\Omega\left(d\widehat{\underline{\omega}}_k\right)\frac{\rho_k^2 d\rho_k}{\left(2\pi\right)^2},\end{aligned}\tag{3.2}$$

again $\underline{\omega}_1 + \underline{\omega}_2 + \underline{\omega}_3 = 0$, hence the wave numbers ρ_1, ρ_2, and ρ_3 satisfy the triangle relation $|\rho_1 - \rho_2| \leq \rho_3 \leq \rho_1 + \rho_2$, (they should be able to form a triangle). The equation (3.2) implies that the stochastic measures Z_ℓ^m fulfill some particular connection with the bispectrum.

Lemma 2. *Let us assume the absolute continuity of third order cumulants* $\operatorname{Cum}\left(Z\left(d\underline{\omega}_1\right), Z\left(d\underline{\omega}_2\right), Z\left(d\underline{\omega}_3\right)\right)$. *Then*

$$\operatorname{Cum}\left(Z_{\ell_1}^{n_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{n_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{n_3}\left(\rho_3^2 d\rho_3\right)\right) = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \mathcal{J}_{\ell_1,\ell_2,\ell_3}\left(\rho_1,\rho_2,\rho_3\right) S_3\left(\rho_1,\rho_2,\rho_3\right) \prod_{k=1}^{3} \frac{\rho_k^2 d\rho_k}{(2\pi)^2}$$

where $\mathcal{J}_{\ell_1,\ell_2,\ell_3}$ *is defined by (A.4).*

See Appendix A.2 for the proof.

Remark 1. *We have some more specific form for the function* B_{ℓ_1,ℓ_2,ℓ_3} *in Lemma 1, more precisely the equation (A.3) shows that*

$$B_{\ell_1,\ell_2,\ell_3}\left(\rho_1,\rho_2,\rho_3\right) = \mathcal{J}_{\ell_1,\ell_2,\ell_3}\left(\rho_1,\rho_2,\rho_3\right) S_3\left(\rho_1,\rho_2,\rho_3\right).$$

We shall use the following particular case

$$\begin{aligned}
&\operatorname{Cum}\left(Z_{\ell_1}^{m}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{0}\left(\rho_2^2 d\rho_2\right), Z_{0}^{0}\left(\rho_3^2 d\rho_3\right)\right) \\
&= (-1)^{\Sigma\ell_k} \frac{\sqrt{\pi}\delta_{\rho\triangle}}{\rho_1\rho_2\rho_3} \begin{pmatrix} \ell_1 & \ell_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}^2 \sqrt{\prod_{k=1}^{2}(2\ell_k+1) S_3\left(\rho_1,\rho_2,\rho_3\right)} \prod_{k=1}^{3} \frac{\rho_k^2 d\rho_k}{4\pi} \\
&= \delta_{\ell_1-\ell_2}\delta_m \frac{\sqrt{\pi}\delta_{\rho\triangle}\left(2\ell_1+1\right)}{\rho_1\rho_2\rho_3} S_3\left(\rho_1,\rho_2,\rho_3\right) \prod_{k=1}^{3} \frac{\rho_k^2 d\rho_k}{(2\pi)^2},
\end{aligned}$$

where $\rho\triangle = 0$, if and only if the wave numbers ρ_1, ρ_2, and ρ_3 do not satisfy the triangle relation, and $\ell_1 = \ell_2$ since the triangular inequality $|\ell_1 - \ell_3| \leq \ell_2 \leq \ell_1 + \ell_3$ should be valid, see selection rules, Appendix B, 5.

The bicovariances for a homogenous and isotropic field

$$\mathcal{C}_3\left(r_1, r_2, \vartheta\right) = \operatorname{Cum}(X(r_1\underline{\widehat{x}}), X(r_2 N), X(0)).$$

For deriving the bispectrum we shall use the series representation of the field (2.1) in the following locations

$$X\left(rN\right) = 4\pi \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} \int_0^\infty j_\ell\left(\rho r\right) Z_\ell^0\left(\rho^2 d\rho\right), \tag{3.3}$$

$$X\left(\underline{0}\right) = \int_{\mathbb{R}^3} Z\left(d\underline{\omega}\right) = \sqrt{4\pi} \int_0^\infty Z_0^0\left(\rho^2 d\rho\right), \tag{3.4}$$

where we applied the specific values $Y_\ell^m\left(N\right) = \delta_m \sqrt{\frac{\ell+1}{4\pi}}$, $Y_0^0\left(\underline{\widehat{x}}\right) = \sqrt{\frac{1}{4\pi}}$, of the spherical harmonics.

The main result of this paper is the construction of kernel functions

$$\mathcal{T}\left(r_1, r_2, \vartheta | \rho_1, \rho_2\right) = \frac{1}{8\pi^4} \sum_{\ell=0}^{\infty} (2\ell+1)^2 P_\ell\left(\cos\vartheta\right) j_\ell\left(\rho_1 r_1\right) j_\ell\left(\rho_2 r_2\right) \tag{3.5}$$

$$\mathcal{T}\left(r_1, r_2 | \rho_1, \rho_{2,}\eta\right) = 32\pi^2 \sum_{\ell=0}^{\infty} (2\ell+1)^{-1} P_\ell\left(\cos\eta\right) j_\ell\left(\rho_1 r_1\right) j_\ell\left(\rho_2 r_2\right), \tag{3.6}$$

such that it provides correspondence between bicovarince and bispectrum, and vica versa.

Theorem 1. *Assume isotropy in third order and the integrals below exist then*

$$\mathcal{C}_3(r_1, r_2, \vartheta) = \iint_0^\infty \int_0^\pi \mathcal{T}(r_1, r_2, \vartheta | \rho_1, \rho_2) S_3(\rho_1, \rho_2, \eta) \rho_1^2 d\rho_1 \rho_2^2 d\rho_2 \sin\eta d\eta, \tag{3.7}$$

and conversely

$$S_3(\rho_1, \rho_2, \eta) = \iint_0^\infty \int_0^\pi \mathcal{T}(r_1, r_2 | \rho_1, \rho_{2,}\eta) \mathcal{C}_3(r_1, r_2, \vartheta) r_1^2 dr_1 r_2^2 dr_2 \sin\vartheta d\vartheta.$$

See Appendix A.3 for the proof.

Remark 2. *The kernel (3.5) provides an orthogonal series expansion in terms of Legendre-polynomials for both bicovariance $\mathcal{C}_3$ and for bispectrum S_3.*

4. Laplacian-fields

Consider a homogeneous isotropic field X on $\mathbb{R}^3$ which fulfills the equation

$$\left(\nabla^2 - c^2\right) X = \partial W, \tag{4.1}$$

where ∇^2 denotes the Laplace-operator on $\mathbb{R}^3$, and ∂W is white noise, possible non-Gaussian. The stochastic equation (4.1) is meant by L_2 sense, see [32] p.16.

The Laplacian in spherical coordinates acts on $X(\underline{x})$, as

$$\begin{aligned}\nabla^2 X(\underline{x}) &= \left(\frac{1}{r^2}\triangle_B + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) X(\underline{x}) \\ &= -4\pi \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} \frac{\ell(\ell+1)}{r^2} Y_\ell^m(\widehat{\underline{x}}) \int_0^\infty j_\ell(\rho r) Z_\ell^m\left(\rho^2 d\rho\right) \\ &\quad + 4\pi \frac{\ell(\ell+1)}{r^2} X(\underline{x}) - 4\pi \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}}) \int_0^\infty j_\ell(\rho r) \rho^2 Z_\ell^m\left(\rho^2 d\rho\right),\end{aligned}$$

where $\triangle_B$ denotes the Lapalce–Beltrami-operator. Such that the solution of (4.1) is

$$X(\underline{x}) = -4\pi \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}}) \int_0^\infty j_\ell(\rho r) \frac{1}{\rho^2 + c^2} W_\ell^m\left(\rho^2 d\rho\right),$$

with $E\left|W_\ell^m\left(\rho^2 d\rho\right)\right|^2 = \rho^2 d\rho$. Its spectral density (according to measure $\rho^2 d\rho$), see [32], Example 6, p. 24, is

$$S(\rho) = \frac{1}{\left(\rho^2 + c^2\right)^2}, \quad \rho^2 = \|\underline{\omega}\|^2,$$

with covariance function of Matérn Class

$$\mathcal{C}(r) = \frac{1}{(2\pi)^2} \frac{(cr)^{1/2} K_{1/2}(cr)}{c},$$

where $K_{1/2}$ is the modified Bessel (Hankel) function, see [1]. A differential operator $\left(\nabla^2 - c^2\right)^\nu$ with $\nu > 0$ can also be considered In this case the covariance function is proportional to $(cr)^{1/2-\nu/2} K_{1/2-\nu/2}(cr)$, see [23] p. 179, 2.12.4.28 This can be generalized further by similar methods of the paper [13].

Bispectrum with measure $\prod_{k=1}^3 \left(\rho_k^2 d\rho_k / 4\pi\right)$ is

$$S_3(\rho_1, \rho_2, \rho_3) = \prod_{k=1}^3 \frac{1}{\rho_k^2 + c^2}, \; \rho_3^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\eta,$$

Theorem 1 gives the series representation

$$\begin{aligned}\mathcal{C}_3(r_1,r_2,\vartheta) &= \iint_0^\infty \int_0^\pi \mathcal{T}(r_1,r_2,\vartheta|\rho_1,\rho_2)\, S_3(\rho_1,\rho_2,\eta)\,\rho_1^2 d\rho_1 \rho_2^2 d\rho_2 \sin\eta d\eta \\ &= \frac{1}{8\pi^4}\sum_{\ell=0}^\infty (2\ell+1)^2 P_\ell(\cos\vartheta) \\ &\quad\times \iint_0^\infty \frac{j_\ell(\rho_1 r_1)}{\rho_1^2+c^2}\frac{j_\ell(\rho_2 r_2)}{\rho_2^2+c^2}\ln\frac{(\rho_1-\rho_2)^2+c^2}{(\rho_1+\rho_2)^2+c^2}\rho_1 d\rho_1\rho_2 d\rho_2,\end{aligned}$$

for the bicovariance function of the Laplacian-field (4.1), since

$$\int_0^\pi \frac{1}{\rho_3^2+c^2}\sin\eta d\eta = \frac{1}{2\rho_1\rho_2}\ln\frac{(\rho_1-\rho_2)^2+c^2}{(\rho_1+\rho_2)^2+c^2}.$$

5. A nonlinear model: Homogenous isotropic field with Hermite rank 2

We consider a Gaussian-field

$$\begin{aligned}X(\underline{x}) &= \int_{\mathbb{R}^3} e^{i\underline{x}\cdot\underline{\omega}} a(\rho)\, W(d\underline{\omega}) \\ &= 4\pi\sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}})\int_0^\infty j_\ell(\rho r)\, a(\rho)\, W_\ell^m\left(\rho^2 d\rho\right),\end{aligned}$$

where $W(d\underline{\omega})$ is Gaussian, $\mathsf{E}\,|W(d\underline{\omega})|^2 = \frac{1}{(2\pi)^3}d\underline{\omega}$. A model which is non-Gaussian is

$$H(\underline{x}) = X(\underline{x}) + f_{NL}\left(X^2(\underline{x}) - EX^2(\underline{x})\right),$$

see [14]. The coefficient f_{NL} is measuring the nonlinearity of the CMB observations for instance. Notice that $X^2(\underline{x}) - EX^2(\underline{x})$ is Hermite polynomial of degree 2 of the Gaussian random variable $X(\underline{x})$, and $H(\underline{x})$ is an elementary, very simple case of a chaotic field

$$H(\underline{x}) = \sum_{k=0}^\infty \int_{\mathbb{R}^{3\times k}} \exp(it\Sigma\underline{\omega}_{1:k}) f_k(\underline{\omega}_{1:k}) W(d\underline{\omega}_{1:k}), \tag{5.1}$$

where $\underline{\omega}_{1:k} = [\underline{\omega}_1,\underline{\omega}_2,\dots,\underline{\omega}_k]$, $W(d\underline{\omega}_{1:k})$ is the multiple Wiener-Itô stochastic spectral measure, see [6], [21]. $H(\underline{x})$ is subordinated to the complex Gaussian white noise spectral measure.

Consider a quadratic field

$$H_2(X(\underline{x})) = X^2(\underline{x}) - EX^2(\underline{x}),$$

and use the bipolar spherical harmonics $Y_{\ell_1,\ell_2}^{\ell,m}(\widehat{\underline{\omega}}_{1:2}) = \left[\underline{Y}_{\ell_1}(\widehat{\underline{\omega}}_1)\otimes\underline{Y}_{\ell_2}(\widehat{\underline{\omega}}_2)\right]_{\ell,m}$, see B, 7, hence we can rewrite

$$\begin{aligned}H_2(X(\underline{x})) &= (4\pi)^2\sum_{\ell,m} Y_\ell^m(\widehat{\underline{x}})\sum_{\ell_{1:2}=0}^\infty\sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}}C_{\ell_1,0;\ell_2,0}^{\ell,0} \\ &\quad\times\iint_0^\infty\prod_{q=1}^2 j_{\ell_q}(\rho_q r)\, a(\rho_q)\, W_{\ell_1,\ell_2}^{\ell,m}\left(\prod_{q=1}^2\rho_q^2 d\rho_q\right)\end{aligned} \tag{5.2}$$

where

$$W_{\ell_1,\ell_2}^{\ell,m}\left(\prod_{q=1}^2\rho_q^2 d\rho_q\right) = \int_{\mathbb{S}_2^2} Y_{\ell_1,\ell_2}^{\ell,m}(\widehat{\underline{\omega}}_{1:2})^* W\left(\prod_{q=1}^2\rho_q^2 d\rho_q \Omega(d\widehat{\underline{\omega}}_q)\right),$$

see B, A.4 for details. Here the stochastic integral would not change if we replace $Y_{\ell_1,\ell_2}^{\ell,m}$ with its $\widetilde{Y}_{\ell_1,\ell_2}^{\ell,m}(\widehat{\underline{\omega}}_{1:2})$ symmetrized version (according to $\widehat{\underline{\omega}}_1$ and $\widehat{\underline{\omega}}_2$).

The nonlinear model we shall consider is

$$\begin{aligned} H(\underline{x}) &= X_1(\underline{x}) + X_2(\underline{x}) \\ &= \int_{\mathbb{R}^3} e^{i\underline{x}\cdot\underline{\omega}} a(\rho) W(d\underline{\omega}) + \int_{\mathbb{R}^{3\times 2}} e^{i\underline{x}\cdot(\underline{\omega}_1+\underline{\omega}_2)} a_2(\rho_{1:2}) W(d\underline{\omega}_{1:2}), \end{aligned}$$

where the quadratic transfer function $a_2(\rho_{1:2})$ is symmetric function of its variables. We put $H(\underline{x})$ in series expansion according to

$$X_1(\underline{x}) = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}}) \int_0^\infty j_\ell(\rho r) a_1(\rho) W_\ell^m(\rho^2 d\rho), \tag{5.3}$$

and

$$\begin{aligned} X_2(\underline{x}) = (4\pi)^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(\widehat{\underline{x}}) \sum_{\ell_{1:2}=0}^{\infty} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} C_{\ell_1,0;\ell_2,0}^{\ell,0} \\ \times \iint_0^\infty \prod_{q=1}^{2} j_{\ell_q}(\rho_q r) a_2(\rho_{1:2}) W_{\ell_1,\ell_2}^{\ell,m}\left(\prod_{q=1}^{2} \rho_q^2 d\rho_q\right), \end{aligned} \tag{5.4}$$

see A.4.

Remark 3. *The linear transfer function a_1 could be a function of $\underline{\omega}$, in this case as we have seen above, the isotropy requires that $|a_1(\underline{\omega})|^2$ should only depend on ρ.*

Remark 4. *One can show that if the quadratic transfer function a_2 is a function of $(\rho_{1:2}, \widehat{\underline{\omega}}_1 \cdot \widehat{\underline{\omega}}_2)$ ($\widehat{\underline{\omega}}_1 \cdot \widehat{\underline{\omega}}_2$ is the cosine of angle contained between $\widehat{\underline{\omega}}_1$ and $\widehat{\underline{\omega}}_2$) then the field X_2 is still isotropic. In this case we do not have the orthogonal series representation (5.4).*

Let us consider the rotation of the field $\Lambda(g) X_2(\underline{x})$

$$\begin{aligned} \Lambda(g) X_2(\underline{x}) &= (4\pi)^2 \sum_{\ell,m} Y_\ell^m(g^{-1}\widehat{\underline{x}}) \sum_{\ell_{1:2}=0}^{\infty} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} C_{\ell_1,0;\ell_2,0}^{\ell,0} \\ &\quad \times \iint_0^\infty \prod_{q=1}^{2} j_{\ell_q}(\rho_q r) a(\rho_q) \int_{\mathbb{S}_2^2} a_2(\rho_{1:2}) Y_{\ell_1,\ell_2}^{\ell,m}(\widehat{\underline{\omega}}_{1:2})^* dW \\ &\qquad \times \left(\prod_{q=1}^{2} \rho_q^2 d\rho_q \Omega(d\widehat{\underline{\omega}}_k)\right) \\ &= (4\pi)^2 \sum_{\ell,n} Y_\ell^{n*}(\widehat{\underline{x}}) \sum_{\ell_{1:2}=0}^{\infty} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi(2\ell+1)}} C_{\ell_1,0;\ell_2,0}^{\ell,0} \\ &\quad \times \iint_0^\infty \prod_{q=1}^{2} j_{\ell_q}(\rho_q r) a(\rho_q) \\ &\quad \times \int_{\mathbb{S}_2^2} a_2(\rho_{1:2}) \sum_{m=-\ell}^{\ell} D_{n,m}^{(\ell)}(g) Y_{\ell_1,\ell_2}^{\ell,m}(\widehat{\underline{\omega}}_{1:2})^* dW \left(\prod_{q=1}^{2} \rho_q^2 d\rho_q \Omega(d\widehat{\underline{\omega}}_k)\right), \end{aligned}$$

The rotated spherical harmonics $\sum_{m=-\ell}^{\ell} D_{n,m}^{(\ell)}(g) Y_{\ell_1,\ell_2}^{\ell,m}(\widehat{\underline{\omega}}_{1:2})^* = Y_{\ell_1,\ell_2}^{\ell,n}(g^{-1}\widehat{\underline{\omega}}_{1:2})^*$, and Wiener-Ito measure $W\left(\prod_{q=1}^{2} \rho_q^2 d\rho_q \Omega(d\widehat{\underline{\omega}}_k)\right)$ is rotational invariant therefore we obtain $\Lambda(g) X_2(\underline{x}) \stackrel{d}{=} X_2(\underline{x})$, hence the field is rotational invariant.

Once we have established that $X_2(\underline{x})$ is homogenous and isotropic, the covariance of $X_2(\underline{x})$ depends on distance r between two locations, and it can be calculated by $\mathcal{C}_{X_2}(r) = \operatorname{Cov}(X_2(rN), X_2(\underline{0}))$, these particular values are

$$X_2(\underline{0}) = 4\pi \iint_0^\infty a_2(\rho_{1:2}) W_{0,0}^{0,0} \left(\prod_{q=1}^2 \rho_q^2 d\rho_q\right), \tag{5.5}$$

and

$$X_2(rN) = 4\pi \sum_\ell \sum_{\ell_{1:2}=0}^\infty \sqrt{(2\ell_1+1)(2\ell_2+1)} C_{\ell_1,0;\ell_2,0}^{\ell,0} \times \iint_0^\infty \prod_{q=1}^2 j_{\ell_q}(\rho_q r) a_2(\rho_{1:2}) W_{\ell_1,\ell_2}^{\ell,0} \left(\prod_{q=1}^2 \rho_q^2 d\rho_q\right), \tag{5.6}$$

We have

$$C_{X_2}(r) = (4\pi)^2 \int_0^\infty \int_0^\infty j_0(\rho_1 r) j_0(\rho_2 r) |a_2(\rho_{1:2})|^2 \rho_2^2 d\rho_2 \rho_1^2 d\rho_1,$$

the inversion formula is used for the spectrum

$$\begin{aligned} S_{X_2}(\rho) &= 4\pi \int_0^\infty j_0(\rho r) C_2(r) r^2 dr \\ &= 64\pi^3 \int_0^\infty \int_0^\infty \int_0^\infty j_0(\rho_1 r) j_0(\rho_2 r) j_0(\rho r) r^2 dr |a_2(\rho_{1:2})|^2 \rho_2^2 d\rho_2 \rho_1^2 d\rho_1 \\ &= 16\pi^4 \int_\rho^\infty \frac{1}{\rho_1} \int_{\rho_1-\rho_2}^{\rho_1+\rho_2} |a_2(\rho_{1:2})|^2 \rho_1^2 d\rho_1 \rho_2^2 d\rho_2 \\ &= (2\pi)^4 \int_\rho^\infty \int_{\rho_1-\rho}^{\rho_1+\rho} |a_2(\rho_{1:2})|^2 \rho_1 d\rho_1 \rho_2^2 d\rho_2, \end{aligned}$$

see (B.9).

Now the spectrum of $H(\underline{x})$ is simple, it is the sum of spectra, since – because of the orthogonality of multiple Wiener-Itô integrals – we have

$$\mathcal{C}_2(r) = \frac{1}{2\pi^2} \int_0^\infty j_0(\rho r) \left(|a_1(\rho)|^2 + S_{X_2}(\rho)\right) \rho^2 d\rho,$$

therefore the spectrum is $\left(|a_1(\rho)|^2 + S_{X_2}(\rho)\right)/2\pi^2$.

5.1. **Bispectrum.** Since we have established homogeneity and isotropy of $H(\underline{x})$, the bispectrum is derived by finding an expression the bicovariance

$$\begin{aligned} &\operatorname{Cum}(H(r_1\underline{\widehat{x}}), H(r_2N), H(0)) \\ &\quad = \operatorname{Cum}(X_1(r_1\underline{\widehat{x}}) + X_2(r_1\underline{\widehat{x}}), X_1(r_2N) + X_2(r_2N), X_1(\underline{0}) + X_2(\underline{0})). \end{aligned}$$

The results for cumulants of Hermite-polynomials of Gaussian-processes applies here ([25]), and we get

$$\begin{aligned} \operatorname{Cum}(H(r_1\underline{\widehat{x}}), H(r_2N), H(0)) &= \operatorname{Cum}(X_1(r_1\underline{\widehat{x}}), X_1(r_2N), X_2(\underline{0})) \\ &+ \operatorname{Cum}(X_1(r_1\underline{\widehat{x}}), X_2(r_2N), X_1(\underline{0})) \\ &+ \operatorname{Cum}(X_2(r_1\underline{\widehat{x}}), X_1(r_2N), X_1(\underline{0})) \\ &+ \operatorname{Cum}(X_2(r_1\underline{\widehat{x}}), X_2(r_2N), X_2(\underline{0})). \end{aligned}$$

It turns out that

$$\operatorname{Cum}\left(W_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), W_{\ell_2}^{0}\left(\rho_2^2 d\rho_2\right), W_{0,0}^{0,0}\left(\prod_{q=1}^{2}\rho_q^2 d\rho_q\right)\right) = 2\delta_{\ell_1}\delta_{m_1}\delta_{\ell_2}\prod_{q=1}^{2}\rho_q^2 d\rho_q$$

$$\operatorname{Cum}\left(W_{\ell_3}^{m_3}\left(\rho_1^2 d\rho_1\right), W_{\ell_1,\ell_2}^{\ell,0}\left(\prod_{q=1}^{2}\rho_p^2 d\rho_p\right), W_0^{0}\left(\rho_2^2 d\rho_2\right)\right) = 2\delta_{\ell_1}\delta_{m_3}\delta_{\ell_2-\ell_3}\delta_{\ell-\ell_2}\prod_{q=1}^{2}\rho_q^2 d\rho_q$$

$$\operatorname{Cum}\left(W_{\ell_1,\ell_2}^{\ell,m}\left(\prod_{q=1}^{2}\rho_q^2 d\rho_q\right), W_{\ell_3}^{0}\left(\rho_2^2 d\rho_2\right), W_0^{0}\left(\rho_1^2 d\rho_1\right)\right) = 2\delta_{\ell_1}\delta_{m}\delta_{\ell_2-\ell_3}\delta_{\ell-\ell_2}\prod_{q=1}^{2}\rho_q^2 d\rho_q,$$

where we applied the identity $C_{0,0;\ell,0}^{\ell,0} = 1$. These cumulants imply readily

$$\begin{aligned}
&\operatorname{Cum}\left(X_1\left(r_1\widehat{\underline{x}}\right), X_1\left(r_2 N\right), X_2\left(\underline{0}\right)\right) + \operatorname{Cum}\left(X_1\left(r_1\widehat{\underline{x}}\right), X_2\left(r_2 N\right), X_1\left(\underline{0}\right)\right)\\
&\qquad + \operatorname{Cum}\left(X_2\left(r_1\widehat{\underline{x}}\right), X_1\left(r_2 N\right), X_1\left(\underline{0}\right)\right)\\
&\quad = 6\left(4\pi\right)\iint_0^{\infty} a_1\left(\rho_1\right) a_1\left(\rho_2\right) a_2\left(\rho_{1:2}\right) j_0\left(\rho_1 r_1\right) j_0\left(\rho_2 r_2\right)\prod_{q=1}^{2}\rho_q^2 d\rho_q.
\end{aligned}$$

Similar calculation can be found in [5] Example 4. The third order cumulants according to the field X_2 are given by

$$\begin{aligned}
&\operatorname{Cum}\left(W_{\ell_1,\ell_2}^{\ell,m}\left(\prod_{q=1}^{2}\rho_q^2 d\rho_q\right), W_{k_1,k_2}^{k,0}\left(\prod_{q=1}^{2}\rho_p^2 d\rho_p\right), W_{0,0}^{0,0}\left(\prod_{q=1}^{2}\rho_q^2 d\rho_q\right)\right)\\
&\quad = 8\int_{\mathbb{S}_2^{2\times 3}} Y_{\ell_1,\ell_2}^{\ell,m}\left(\widehat{\underline{\omega}}_{1:2}\right) Y_{k_1,k_2}^{k,0}\left(\widehat{\underline{\omega}}_{3:4}\right)\delta\left(\widehat{\underline{\omega}}_2+\widehat{\underline{\omega}}_3\right)\delta\left(\rho_2-\rho_3\right)\delta\left(\widehat{\underline{\omega}}_4+\widehat{\underline{\omega}}_5\right)\\
&\qquad\quad \times\delta\left(\rho_4-\rho_5\right)\delta\left(\widehat{\underline{\omega}}_6+\widehat{\underline{\omega}}_1\right)\delta\left(\rho_6-\rho_1\right)\prod_{q=1}^{6}\rho_p^2 d\rho_p\Omega\left(d\widehat{\underline{\omega}}_p\right)\\
&\quad = 8\left(4\pi\right)\delta_{\ell_1}\delta_{k_2}\delta_{\ell_2-k_1}\delta_{\ell_2-\ell}\delta_{k-\ell}\delta_m\delta\left(\rho_2-\rho_3\right)\delta\left(\rho_4-\rho_5\right)\delta\left(\rho_6-\rho_1\right)\prod_{q=1}^{6}\rho_p^2 d\rho_p,
\end{aligned}$$

since

$$\begin{aligned}
&\int_{\mathbb{S}_2^3} Y_{\ell_1,\ell_2}^{\ell,m}\left(\widehat{\underline{\omega}}_{1:2}\right) Y_{k_1,k_2}^{k,n}\left(-\widehat{\underline{\omega}}_2,\widehat{\underline{\omega}}_3\right)\prod_{q=1}^{3}\Omega\left(d\widehat{\underline{\omega}}_q\right)\\
&\quad = \sum_{m_{1:2}=-\ell_{1:2}}^{\ell_{1:2}}\sum_{n_{1:2}=-k_{1:2}}^{k_{1:2}} C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m} C_{k_1,n_1;k_2,n_2}^{k,n}\\
&\qquad \times\int_{\mathbb{S}_2^3} Y_{\ell_1}^{m_1}\left(\widehat{\underline{\omega}}_1\right) Y_{\ell_2}^{m_2}\left(\widehat{\underline{\omega}}_2\right) Y_{k_1}^{n_1}\left(-\widehat{\underline{\omega}}_2\right) Y_{k_2}^{n_2}\left(\widehat{\underline{\omega}}_3\right)\prod_{q=1}^{3}\Omega\left(d\widehat{\underline{\omega}}_q\right)\\
&\quad = 4\pi\sum_{m_{1:2}=-\ell_{1:2}}^{\ell_{1:2}}\delta_{\ell_1}\delta_{m_1}\delta_{\ell_2-k_1}\delta_{m_2+n_1}\delta_{k_2}\delta_{n_2} C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m} C_{k_1,n_1;k_2,n_2}^{k,n}\\
&\quad = 4\pi\left(C_{0,0;\ell,m}^{\ell,m}\right)^2\delta_{\ell_1}\delta_{k_2}\delta_{\ell_2-k_1}\delta_{\ell_2-\ell}\delta_{k-\ell}\delta_{n-m}.
\end{aligned}$$

It follows from the above

$$
\begin{aligned}
&\operatorname{Cum}\left(X_2\left(r_1\underline{\widehat{x}}\right), X_2\left(r_2 N\right), X_2\left(\underline{0}\right)\right)\\
&\quad= 8\left(4\pi\right)^4 \sum_{\ell}\left(2\ell+1\right) P_\ell\left(\cos\vartheta\right) \iiint_0^\infty j_0\left(\rho_1 r_1\right) j_0\left(\rho_2 r_2\right) j_\ell\left(\rho_2 r_1\right) j_\ell\left(\rho_3 r_2\right)\\
&\qquad\qquad a_2\left(\rho_{1:2}\right) a_2\left(\rho_2,\rho_3\right) a_2\left(\rho_3,\rho_1\right) \prod_{q=1}^{3} \rho_q^2 d\rho_q .
\end{aligned}
$$

The nonlinear model H which containes a linear Gaussian term X_1 and a nonlinear term X_2 has a non-zero bispectrum (as far as $X_2 \neq 0$). This bispectrum is showing up expressing the bicovariance function of H in terms of transfer functions a_1 and a_2.

Lemma 3.

$$
\begin{aligned}
&\operatorname{Cum}\left(H\left(r_1\underline{\widehat{x}}\right), H\left(r_2 N\right), H\left(\underline{0}\right)\right)\\
&\quad= 6\left(4\pi\right) \iint_0^\infty a_1\left(\rho_1\right) a_1\left(\rho_2\right) a_2\left(\rho_{1:2}\right) \prod_{q=1}^{2} j_0\left(\rho_q r\right) \prod_{q=1}^{2} \rho_q^2 d\rho_q\\
&\qquad+ 8\left(4\pi\right)^4 \sum_{\ell}\left(2\ell+1\right) P_\ell\left(\cos\vartheta\right)\\
&\qquad\times \iiint_0^\infty j_0\left(\rho_1 r_1\right) j_\ell\left(\rho_2 r_1\right) j_0\left(\rho_2 r_2\right) j_\ell\left(\rho_3 r_2\right)\\
&\qquad\qquad\times a_2\left(\rho_{1:2}\right) a_2\left(\rho_2,\rho_3\right) a_2\left(\rho_3,\rho_1\right) \prod_{q=1}^{3} \rho_q^2 d\rho_q .
\end{aligned}
$$

Appendix A. Proofs

A.1. **Proof for Lemma 1.**

Proof. It follows from (2.3) that a homogenous field $X\left(\underline{x}\right)$ is isotropic in third order iff the bicovariances of the rotated $\mathcal{Z}_\ell^m$ (2.4) equal to the bicovariances of Z_ℓ^m, i.e.

$$
\begin{aligned}
&\operatorname{Cum}\left(\mathcal{Z}_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), \mathcal{Z}_{\ell_2}^{m_2}\left(\rho_2^2 d\rho_2\right), \mathcal{Z}_{\ell_3}^{m_3}\left(\rho_2^2 d\rho_2\right)\right)\\
&\quad= \operatorname{Cum}\left(Z_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{m_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{m_3}\left(\rho_3^2 d\rho_3\right)\right).
\end{aligned}
\tag{A.1}
$$

The left side is written by (2.4)

$$
\begin{aligned}
&\operatorname{Cum}\left(\mathcal{Z}_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), \mathcal{Z}_{\ell_2}^{m_2}\left(\rho_2^2 d\rho_2\right), \mathcal{Z}_{\ell_3}^{m_3}\left(\rho_2^2 d\rho_2\right)\right)\\
&\quad= \sum_{p,q,r=-\ell_1,\ell_2,\ell_3}^{\ell_1,\ell_2,\ell_3} D_{m_1,p}^{(\ell_1)}\left(g\right) D_{m_2,q}^{(\ell_2)}\left(g\right) D_{m_3,r}^{(\ell_3)}\left(g\right)\\
&\qquad\qquad\times \operatorname{Cum}\left(Z_{\ell_1}^{p}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{q}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{r}\left(\rho_3^2 d\rho_3\right)\right).
\end{aligned}
$$

Now from third order isotropy follows

$$
\begin{aligned}
&\operatorname{Cum}\left(Z_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{m_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{m_3}\left(\rho_3^2 d\rho_3\right)\right)\\
&\quad= \sum_{p,q,r=-\ell_1,\ell_2,\ell_3}^{\ell_1,\ell_2,\ell_3} D_{m_1,p}^{(\ell_1)}\left(g\right) D_{m_2,q}^{(\ell_2)}\left(g\right) D_{m_3,r}^{(\ell_3)}\left(g\right)\\
&\qquad\qquad\times \operatorname{Cum}\left(Z_{\ell_1}^{p}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{q}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{r}\left(\rho_3^2 d\rho_3\right)\right),
\end{aligned}
$$

integrate both sides over the sphere according to the invariant Haar-measure, and by Gaunt-integral (B.16) we have

$$\begin{aligned}&\operatorname{Cum}\left(Z_{\ell_1}^{m_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{m_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{m_3}\left(\rho_3^2 d\rho_3\right)\right)\\&\quad=\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix} B_{\ell_1,\ell_2,\ell_3}\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\rho_k^2 d\rho_k,\end{aligned}$$

where

$$\begin{aligned}&B_{\ell_1,\ell_2,\ell_3}\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\rho_k^2 d\rho_k\\&\quad=\sum_{p,q,r}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ p & q & r\end{pmatrix}\operatorname{Cum}\left(Z_{\ell_1}^{p}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{q}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{r}\left(\rho_3^2 d\rho_3\right)\right).\end{aligned}$$

In other way around from (3.1) using (B.15) follows (A.1). □

A.2. **Proof for Lemma 2.**

Proof. We apply the Rayleigh plane wave expansion (B.6) for $e^{i\underline{\lambda}\cdot\Sigma_1^3\underline{x}_k}$ and get

$$\begin{aligned}\delta\left(\Sigma_1^3\rho_k\widehat{\underline{\omega}}_k\right)&=\frac{1}{(2\pi)^3}\int_0^\infty\int_{\mathbb{S}_2} e^{i\underline{\lambda}\cdot\Sigma_1^3\underline{\omega}_k}\Omega\left(d\widehat{\underline{\lambda}}\right)\lambda^2 d\lambda\\&=2^3\int_0^\infty\int_{\mathbb{S}_2}\sum_{g_{1:3}=0}^{\infty}\sum_{m_{1:3}=-g_{1:3}}^{g_{1:3}}\prod_{a=1}^{3} i^{g_a} j_{g_a}\left(\rho_k\lambda\right) Y_{g_a}^{m_a}\left(\widehat{\underline{\omega}}_k\right) Y_{g_a}^{m_a}\left(\widehat{\underline{\lambda}}\right)\\&\qquad\times\Omega\left(d\widehat{\underline{\lambda}}\right)\lambda^2 d\lambda.\end{aligned}\tag{A.2}$$

The bicovariance of the stochastic measure

$$Z_\ell^m\left(\rho^2 d\rho\right)=i^\ell\int_{\mathbb{S}_2} Y_\ell^m\left(\widehat{\underline{\omega}}\right)^* Z\left(\Omega\left(d\widehat{\underline{\omega}}\right)\rho^2 d\rho\right),$$

according to (3.2) is

$$\begin{aligned}&\operatorname{Cum}\left(Z_{\ell_1}^{n_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{n_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{n_3}\left(\rho_3^2 d\rho_3\right)\right)\\&\quad=\iiint_{\mathbb{S}_2}\prod_{k=1}^{3} i^{\ell_k} Y_{\ell_k}^{n_k}\left(\widehat{\underline{\omega}}_k\right)^*\delta\left(\Sigma_1^3\rho_k\widehat{\underline{\omega}}_k\right)\prod_{k=1}^{3}\Omega\left(d\widehat{\underline{\omega}}_k\right) S_3\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\frac{\rho_k^2 d\rho_k}{(2\pi)^2}\\&\quad=2^3\int_0^\infty\int_{\mathbb{S}_2}\sum_{g_{1:3}=0}^{\infty}\sum_{m_{1:3}=-g_{1:3}}^{g_{1:3}}\prod_{a=1}^{3} i^{g_a} j_{g_a}\left(\rho_k\lambda\right) Y_{g_a}^{m_a}\left(\widehat{\underline{\lambda}}\right)\Omega\left(d\widehat{\underline{\lambda}}\right)\lambda^2 d\lambda\\&\quad\times\iiint_{\mathbb{S}_2} i^{\ell_k}\prod_{k=1}^{3} Y_{\ell_k}^{n_k}\left(\widehat{\underline{\omega}}_k\right)^*\prod_{a=1}^{3} Y_{g_a}^{m_a}\left(\widehat{\underline{\omega}}_k\right)\prod_{k=1}^{3}\Omega\left(d\widehat{\underline{\omega}}_k\right) S_3\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\frac{\rho_k^2 d\rho_k}{(2\pi)^2}\\&\quad=\frac{(-1)^{\Sigma\ell_k}}{\sqrt{4\pi}}\sqrt{\prod_{k=1}^{3}(2\ell_k+1)}\mathcal{G}_{n_1,n_2,n_3}^{\ell_1,\ell_2,\ell_3}\int_0^\infty\prod_{k=1}^{3} j_{\ell_k}\left(\rho_k\lambda\right)\lambda^2 d\lambda S_3\left(\rho_1,\rho_2,\rho_3\right)\prod_{k=1}^{3}\frac{\rho_k^2 d\rho_k}{(2\pi)^2},\end{aligned}$$

since the spherical harmonics are orthogonal, where $\mathcal{G}_{n_1,n_2,n_3}^{\ell_1,\ell_2,\ell_3}$ is the Gaunt integral $\mathcal{G}_{n_1,n_2,n_3,0,0,0}^{\ell_1,\ell_2,\ell_3}$, see (B.16), based on Condon and Shortley phase convention (B.12) which gives the connection between Wigner rotation $D_{k,m}^{(\ell)}(g)$ and spherical harmonics $Y_\ell^m\left(\widehat{\underline{\lambda}}\right)$, namely

$$\int_{\mathbb{S}_2} Y_{\ell_1}^{n_1}\left(\widehat{\underline{\lambda}}\right) Y_{\ell_2}^{n_2}\left(\widehat{\underline{\lambda}}\right) Y_{\ell_3}^{n_3}\left(\widehat{\underline{\lambda}}\right) \Omega\left(d\widehat{\underline{\lambda}}\right) = \frac{1}{\sqrt{4\pi}}\sqrt{\prod_{k=1}^{3}(2\ell_k+1)}\mathcal{G}_{n_1,n_2,n_3}^{\ell_1,\ell_2,\ell_3}.$$

The integral of the spherical Bessel-functions included in the bicovariance has a clear form, see [12] 6.578.8, p. 709. Introduce $\delta_{\rho\triangle} = \delta_{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\eta-\rho_3^2}$, equivalently the wave numbers ρ_1, ρ_2, and ρ_3 should satisfy the triangle relation: $|\rho_1-\rho_2| < \rho_3 < \rho_1+\rho_2$, $\rho_1, \rho_2 > 0$, $\rho_3^2 = \rho_1^2+\rho_2^2-2\rho_1\rho_2\cos\eta$, then

$$\int_0^\infty j_\ell(\rho_1\lambda) j_\ell(\rho_2\lambda) j_0(\rho_3\lambda)\lambda^2 d\lambda = \delta_{\rho\triangle}\frac{\pi}{4\rho_1\rho_2\rho_3},$$

see (B.10), replace this into the bicovariance, we have

$$\begin{aligned}&\operatorname{Cum}\left(Z_{\ell_1}^{n_1}\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^{n_2}\left(\rho_2^2 d\rho_2\right), Z_{\ell_3}^{n_3}\left(\rho_3^2 d\rho_3\right)\right) \\ &= \begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ n_1 & n_2 & n_3\end{pmatrix}\mathcal{J}_{\ell_1,\ell_2,\ell_3}(\rho_1,\rho_2,\rho_3) S_3(\rho_1,\rho_2,\rho_3)\prod_{k=1}^{3}\frac{\rho_k^2 d\rho_k}{(2\pi)^2}\end{aligned} \tag{A.3}$$

where

$$\mathcal{J}_{\ell_1,\ell_2,\ell_3}(\rho_1,\rho_2,\rho_3) = (-1)^{\Sigma\ell_k}\frac{\sqrt{\pi}\delta_{\rho\triangle}}{\rho_1\rho_2\rho_3}\sqrt{\prod_{k=1}^{3}(2\ell_k+1)}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3 \\ 0 & 0 & 0\end{pmatrix}. \tag{A.4}$$

This completes the proof. □

A.3. **Proof for Theorem 1.**

Proof.

$$\begin{aligned}&\operatorname{Cum}\left(X\left(r_1\widehat{\underline{x}}\right), X(r_2N), X(0)\right) \\ &= \sqrt{4\pi}(4\pi)^2\iiint_0^\infty\sum_{\ell_1=0}^{\infty}\sum_{m=-\ell_1}^{\ell_1} Y_{\ell_1}^m\left(\widehat{\underline{x}}\right)\sqrt{\frac{2\ell_2+1}{4\pi}}\sum_{\ell_2=0}^{\infty} j_{\ell_1}(\rho_1 r_1) j_{\ell_2}(\rho_2 r_2) \\ &\quad\times\operatorname{Cum}\left(Z_{\ell_1}^m\left(\rho_1^2 d\rho_1\right), Z_{\ell_2}^0\left(\rho_2^2 d\rho_2\right), Z_0^0\left(\rho_3^2 d\rho_3\right)\right) \\ &= \sum_{\ell_1=0}^{\infty} Y_{\ell_1}^0\left(\widehat{\underline{x}}\right)\sqrt{2\ell_1+1}(2\ell_1+1) \\ &\quad\times\iiint_0^\infty j_{\ell_1}(\rho_1 r_1) j_{\ell_1}(\rho_2 r_2)\frac{\sqrt{\pi}\delta_{\rho\triangle}}{4\pi^4\rho_1\rho_2\rho_3} S_3(\rho_1,\rho_2,\rho_3)\prod_{k=1}^{3}\rho_k^2 d\rho_k.\end{aligned}$$

We rewrite the integral

$$\begin{aligned}&\iiint_0^\infty j_\ell(\rho_1 r_1) j_\ell(\rho_2 r_2) S_3(\rho_1,\rho_2,\rho_3)\frac{\delta_{\rho\triangle}}{\rho_1\rho_2\rho_3}\prod_{k=1}^{3}\rho_k^2 d\rho_k \\ &= \iint_0^\infty\int_{|\rho_1-\rho_2|}^{\rho_1+\rho_2} j_\ell(\rho_1 r_1) j_\ell(\rho_2 r_2) S_3(\rho_1,\rho_2,\rho_3)\rho_1 d\rho_1\rho_2 d\rho_2\rho_3 d\rho_3 \\ &= \iint_0^\infty\int_0^\pi j_\ell(\rho_1 r_1) j_\ell(\rho_2 r_2) S_3(\rho_1,\rho_2,\rho_3)\sin\eta d\eta\rho_1^2 d\rho_1\rho_2^2 d\rho_2,\end{aligned}$$

where $\rho_3^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos\eta$, and $\rho_3 d\rho_3 = \rho_1\rho_2 \sin\eta d\eta$, therefore

$$\begin{aligned}\mathcal{C}_3(r_1, r_2, \vartheta) &= \operatorname{Cum}\left(X\left(r_1\underline{\widehat{x}}\right), X\left(r_2 N\right), X(0)\right)\\ &= \frac{1}{8\pi^4}\iint_0^\infty \int_0^\pi \sum_{\ell=0}^\infty (2\ell+1)^2 P_\ell(\cos\vartheta)\, j_\ell(\rho_1 r_1)\, j_\ell(\rho_2 r_2)\, S_3(\rho_1, \rho_2, \rho_3)\\ &\qquad\qquad \times \prod_{k=1}^3 \rho_k^2 d\rho_k\\ &= \iint_0^\infty \int_0^\pi \mathcal{T}(r_1, r_2, \vartheta|\rho_1, \rho_2)\, S_3(\rho_1, \rho_2, \rho_3)\, \rho_1^2 d\rho_1 \rho_2^2 d\rho_2 \sin\eta d\eta\end{aligned}$$

kernel $\mathcal{T}(r_1, r_2, \vartheta|\rho_1, \rho_2)$, (3.5), where $\underline{\widehat{x}} = (\sin\vartheta, 0, \cos\vartheta)$ and $\underline{\widehat{\omega}} = (\sin\eta, 0, \cos\eta)$, $\rho_3^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\eta$. Define

$$\mathcal{T}(r_1, r_2|\rho_1, \rho_{2,}\eta) = 32\pi^2 \sum_{\ell=0}^\infty (2\ell+1)^{-1} P_\ell(\cos\eta)\, j_\ell(\rho_1 r_1)\, j_\ell(\rho_2 r_2),$$

and consider

$$I(\rho_1, \rho_2, \eta|\kappa_1, \kappa_2, \beta) = \iint_0^\infty \mathcal{T}(r_1, r_2, \vartheta|\rho_1, \rho_2)\, \mathcal{T}(r_1, r_2|\kappa_1, \kappa_2, \beta)\, r_1^2 dr_1 r_2^2 dr_2,$$

when $\underline{\widehat{x}} = (\sin\vartheta, 0, \cos\vartheta)$, $Y_\ell^0(\underline{\widehat{x}}) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\underline{\widehat{x}} \cdot N) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\vartheta)$. Use the orthogonality of the spherical harmonics and the Bessel-functions ((B.8) and [29]. 5.6.1.1, p.141)

$$\begin{aligned}I(\rho_1, \rho_2, \eta|\kappa_1, \kappa_2, \beta) &= \frac{4}{\pi^2} \frac{\pi\delta(\rho_1 - \kappa_1)}{2\rho_1^2} \frac{\pi\delta(\rho_2 - \kappa_2)}{2\rho_2^2} \sum_{\ell=0}^\infty \frac{2\ell+1}{2} P_\ell(\cos\vartheta)\, P_\ell(\cos\eta)\\ &= \frac{\delta(\rho_1 - \kappa_1)}{\rho_1^2} \frac{\delta(\rho_2 - \kappa_2)}{\rho_2^2} \delta(\vartheta - \eta),\end{aligned}$$

here we have put $\underline{\widehat{\omega}} = (\sin\beta, 0, \cos\beta)$. Now

$$\begin{aligned}&\iint_0^\infty \int_0^\pi \mathcal{C}_3(r_1, r_2, \vartheta)\, \mathcal{T}(r_1, r_2|\kappa_1, \kappa_2, \beta)\, r_1^2 dr_1 r_2^2 dr_2 \sin\vartheta d\vartheta\\ &\quad = \iint_0^\infty \int_0^\pi \left(\iint_0^\infty \int_0^\pi \mathcal{T}(r_1, r_2, \vartheta|\rho_1, \rho_2)\, S_3(\rho_1, \rho_2, \rho_3)\, \rho_1^2 d\rho_1 \rho_2^2 d\rho_2 \sin\eta d\eta\right)\\ &\qquad\qquad \times \mathcal{T}(\kappa_1, \kappa_2, \beta|r_1, r_2)\, r_1^2 dr_1 r_2^2 dr_2 \sin\vartheta d\vartheta\\ &\qquad\qquad = \iint_0^\infty \frac{\delta(\rho_1 - \kappa_1)}{\rho_1^2} \frac{\delta(\rho_2 - \kappa_2)}{\rho_2^2}\\ &\quad \times \int_0^\pi \int_0^\pi \sum_{\ell=0}^\infty \frac{2\ell+1}{2} P_\ell(\cos\vartheta)\, P_\ell(\cos\eta)\, S_3(\rho_1, \rho_2, \vartheta) \sin\vartheta d\vartheta \sin\eta d\eta \rho_1^2 d\rho_1 \rho_2^2 d\rho_2\\ &\qquad\qquad = \int_{\mathbb{S}_2} S_3(\kappa_1, \kappa_2, \kappa_3)\, \delta(\underline{\widehat{x}} - \underline{\widehat{\omega}})\, \Omega(d\underline{\widehat{x}})/2\pi\\ &\qquad\qquad = S_3(\kappa_1, \kappa_2, \beta),\end{aligned}$$

which completes the proof. □

A.4. **Proof of (5.2).**

Proof. We have

$$H_2\left(X\left(\underline{x}\right)\right)=\left(4\pi\right)^2\sum_{\ell_{1:2}=0}^{\infty}\sum_{m_{1:2}=-\ell_{1:2}}^{\ell_{1:2}}Y_{\ell_1}^{m_1}\left(\widehat{\underline{x}}\right)Y_{\ell_2}^{m_2}\left(\widehat{\underline{x}}\right)$$
$$\times\iint_0^\infty\prod_{q=1}^{2}j_{\ell_q}\left(\rho_q r\right)a\left(\rho_q\right)\int_{\mathbb{S}_2^2}Y_{\ell_1}^{m_1}\left(\widehat{\underline{\omega}}_1\right)^*Y_{\ell_2}^{m_2}\left(\widehat{\underline{\omega}}_2\right)^*W\left(\prod_{q=1}^{2}\rho_q^2d\rho_q\Omega\left(d\widehat{\underline{\omega}}_q\right)\right),$$

now use Appendix B, 7,

$$H_2\left(X\left(\underline{x}\right)\right)=\left(4\pi\right)^2\sum_{\ell_{1:2}=0}^{\infty}\sum_{m_{1:2}=-\ell_{1:2}}^{\ell_{1:2}}\sum_{\ell,m}C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m}Y_{\ell_1,\ell_2}^{\ell,m}\left(\widehat{\underline{x}},\widehat{\underline{x}}\right)$$
$$\times\iint_0^\infty\prod_{q=1}^{2}j_{\ell_q}\left(\rho_q r\right)a\left(\rho_q\right)\int_{\mathbb{S}_2^2}\sum_{\ell,m}C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m}Y_{\ell_1,\ell_2}^{\ell,m}\left(\widehat{\underline{\omega}}_{1:2}\right)^*$$
$$\times W\left(\prod_{q=1}^{2}\rho_q^2d\rho_q\Omega\left(d\widehat{\underline{\omega}}_q\right)\right)$$
$$=\left(4\pi\right)^2\sum_{\ell_{1:2}=0}^{\infty}\sum_{\ell,m}Y_{\ell_1,\ell_2}^{\ell,m}\left(\widehat{\underline{x}},\widehat{\underline{x}}\right)\iint_0^\infty\prod_{q=1}^{2}j_{\ell_q}\left(\rho_q r\right)a\left(\rho_q\right)$$
$$\times\int_{\mathbb{S}_2^2}Y_{\ell_1,\ell_2}^{\ell,m}\left(\widehat{\underline{\omega}}_{1:2}\right)^*W\left(\prod_{q=1}^{2}\rho_q^2d\rho_q\Omega\left(d\widehat{\underline{\omega}}_q\right)\right),$$

where we applied

$$\sum_{m_{1:2}=-\ell_{1:2}}^{\ell_{1:2}}C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m}C_{\ell_1,m_1;\ell_2,m_2}^{\ell_0,m_0}=\delta_{\ell-\ell_0,}\delta_{m-m_0,},$$

and

$$Y_{\ell_1,\ell_2}^{\ell,m}\left(\widehat{\underline{x}},\widehat{\underline{x}}\right)=\sqrt{\frac{\left(2\ell_1+1\right)\left(2\ell_2+1\right)}{4\pi\left(2\ell+1\right)}}C_{\ell_1,0;\ell_2,0}^{\ell,0}Y_{\ell}^{m}\left(\widehat{\underline{x}}\right).\qquad\square$$

Appendix B. Formulae

(1) **Orthonormal spherical harmonics with complex values** $Y_\ell^m\left(\vartheta,\varphi\right)$, $\ell=0,1,2,\ldots$, $m=-\ell,-\ell+1,\ldots-1,0,1,\ldots,\ell-1,\ell$ of **degree** ℓ and **order** m (rank ℓ and projection m)

$$Y_\ell^m\left(\vartheta,\varphi\right)=\left(-1\right)^m\sqrt{\frac{2\ell+1}{4\pi}\frac{\left(\ell-m\right)!}{\left(\ell+m\right)!}}P_\ell^m\left(\cos\vartheta\right)e^{im\varphi},\ \varphi\in\left[0,2\pi\right],\ \vartheta\in\left[0,\pi\right],\tag{B.1}$$

where P_ℓ^m denotes *associated normalized* **Legendre function** *of the first kind*. The spherical harmonics are eigenfunctions of the square of the orbital angular momentum operator.

$$Y_\ell^0\left(\vartheta,\varphi\right)=\sqrt{\frac{2\ell+1}{4\pi}}P_\ell\left(\cos\vartheta\right),\tag{B.2}$$
$$Y_0^0\left(\vartheta,\varphi\right)=\sqrt{\frac{1}{4\pi}},$$

moreover

$$Y_\ell^m\left(N\right)=\delta_{m=0}\sqrt{\frac{2\ell+1}{4\pi}}.\tag{B.3}$$

Y_ℓ^m is fully normalized

$$\int_0^{2\pi}\int_0^{\pi}\left|Y_\ell^m\left(\vartheta,\varphi\right)\right|^2\sin\vartheta d\vartheta d\varphi=1.$$

Some detailed account of spherical harmonics Y_ℓ^m can be found in [29] and [24]. some authors do not apply $1/\sqrt{4\pi}$ in the definition of Y_ℓ^m, also for a sphere with radius R *spherical harmonics are normalized additionally* $Y_\ell^m\left(\vartheta,\varphi\right)/R$. It also follows

$$\begin{aligned}Y_\ell^{m*}\left(\vartheta,\varphi\right)&=Y_\ell^m\left(\vartheta,-\varphi\right)\\&=\left(-1\right)^m Y_\ell^{-m}\left(\vartheta,\varphi\right),\\Y_\ell^{-m}\left(\vartheta,\varphi\right)&=\left(-1\right)^m e^{-i2m\varphi}Y_\ell^m\left(\vartheta,\varphi\right).\end{aligned}$$

Addition formula (see [12], 8.814, [9], 11.4(8)),

$$\sum_{m=-\ell}^{\ell}Y_\ell^{m*}\left(\underline{\widetilde{x}}_1\right)Y_\ell^m\left(\underline{\widetilde{x}}_2\right)=\frac{2\ell+1}{4\pi}P_\ell\left(\cos\vartheta\right), \tag{B.4}$$

where $\cos\vartheta=\underline{\widetilde{x}}_1\cdot\underline{\widetilde{x}}_2$.

$$\sum_{m=-\ell}^{\ell}Y_\ell^{m*}\left(\underline{\widetilde{x}}\right)Y_\ell^m\left(\underline{\widetilde{x}}\right)=\frac{2\ell+1}{4\pi}, \tag{B.5}$$

(2) **Rayleigh plane wave expansion in 3D:**

$$\begin{aligned}e^{i\underline{\omega}\cdot\underline{x}}&=\sum_{\ell=0}^{\infty}i^\ell\left(2\ell+1\right)j_\ell\left(\rho r\right)P_\ell\left(\underline{\widehat{\omega}}\cdot\underline{\widehat{x}}\right)\\&=4\pi\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}i^\ell j_\ell\left(\rho r\right)Y_\ell^m\left(\underline{\widehat{\omega}}\right)^*Y_\ell^m\left(\underline{\widehat{x}}\right),\end{aligned} \tag{B.6}$$

[1, 10.1.47],

(3) **Spherical Bessel-function** j_ℓ of the first kind ([1] 10.1.1), is given by the is the Bessel-function of the first kind $J_{\ell+1/2}$,

$$j_\ell\left(z\right)=\sqrt{\frac{\pi}{2z}}J_{\ell+1/2}\left(z\right), \tag{B.7}$$

$$\frac{2a^2}{\pi}\int j_\ell\left(az\right)j_\ell\left(bz\right)z^2dz=\delta\left(a-b\right), \tag{B.8}$$

see [4, Sect 11, p.735.] and

$$\int_0^\infty j_0\left(\rho_1 r\right)j_0\left(\rho_2 r\right)j_0\left(\rho r\right)r^2dr=\begin{cases}0, & \rho_2<\rho_1-\rho, \quad \rho_2>\rho_1+\rho,\\ \frac{\pi}{8\rho_1}, & \rho_2=\rho_1-\rho, \quad \rho_2=\rho_1+\rho,\\ \frac{\pi}{4\rho_1}, & \rho_1-\rho<\rho_2<\rho_1+\rho, \quad \rho_1\geq\rho>0,\rho_2>0,\end{cases} \tag{B.9}$$

see [12, 3.763.2, p.438],

$$\int_0^\infty J_{\ell+1/2}\left(\rho_1\lambda\right)J_{\ell+1/2}\left(\rho_2\lambda\right)J_{g+1/2}\left(\rho_3\lambda\right)\lambda^{1/2-g}d\lambda=\delta_{\rho\triangle}\frac{\left(\rho_1\rho_2\right)^{g-1/2}}{\sqrt{2\pi}\rho_3^{g+1/2}}\sin^g\eta P_\ell^{-g}\left(\cos\eta\right), \tag{B.10}$$

by [12, 6.578.8, p.686.]

Introduce $\delta_{\rho\triangle} = \delta\left(\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\cos\eta - \rho_3^2\right)$, equivalently the wave numbers ρ_1, ρ_2, and ρ_3 should satisfy the triangle relation, we have

$$\begin{aligned}&\int_0^\infty j_\ell(\rho_1\lambda)\, j_\ell(\rho_2\lambda)\, j_0(\rho_3\lambda)\,\lambda^2 d\lambda\\ &\quad= \sqrt{\frac{\pi^3}{8\rho_1\rho_2\rho_3}}\int_0^\infty J_{\ell+1/2}(\rho_1\lambda)\, J_{\ell+1/2}(\rho_2\lambda)\, J_{1/2}(\rho_3\lambda)\,\lambda^{1/2} d\lambda \\ &\quad= \delta_{\rho\triangle}\frac{\pi}{4\rho_1\rho_2\rho_3}.\end{aligned} \tag{B.11}$$

(4) **Condon and Shortley phase convention**, [8], (4.3.3)

$$\begin{aligned}Y_\ell^m(\vartheta,\varphi) &= \sqrt{\frac{2\ell+1}{4\pi}} D_{0,-m}^{(\ell)}(\gamma,\vartheta,\varphi) \\ &= \sqrt{\frac{2\ell+1}{4\pi}} D_{m,0}^{(\ell)*}(\varphi,\vartheta,\gamma).\end{aligned} \tag{B.12}$$

(5) **Wigner $3j$-symbols** (see [19]), notation

$$\begin{pmatrix}\ell_{1:3}\\ m_{1:3}\end{pmatrix} = \begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}.$$

Selection rules: a Wigner 3j symbols vanishes unless

- $m_1 + m_2 + m_3 = 0$,
- Integer perimeter rule: $\mathcal{L} = \ell_1 + \ell_2 + \ell_3$ is an integer (if $m_1 = m_2 = m_3 = 0$, then $\mathcal{L}$ is even).
- Triangular inequality $|\ell_1 - \ell_2| \le \ell_3 \le \ell_1 + \ell_2$ is fulfilled.
- There is a one to one correspondence between Wigner $3j$-symbols and Clebsch–Gordan-coefficients

$$\frac{(-1)^{\ell_1-\ell_2+k}}{\sqrt{2\ell+1}} C_{\ell_1,k_1;\ell_2,k_2}^{\ell,k} = \begin{pmatrix}\ell_{1:2} & \ell\\ k_{1:2} & -k\end{pmatrix},$$

(6) **Wigner D-matrix** Let $\Lambda(g)\, Y_\ell^m(L) = Y_\ell^m\left(g^{-1}L\right)$,

$$\Lambda(g)\, Y_\ell^m(L) = \sum_{k=-\ell}^{\ell} D_{k,m}^{(\ell)}(g)\, Y_\ell^k(L), \tag{B.13}$$

if ℓ is fixed $D_{m,k}^{(\ell)}(g)$ is unitary

$$\sum_{k=-\ell}^{\ell} D_{m_1,k}^{(\ell)}(g)\, D_{m_2,k}^{(\ell)*}(g) = \delta_{m_1,m_2}, \tag{B.14}$$

see [29], pp79 for details, also

$$\sum_{m_1,m_2,m_3} D_{m_1,k_1}^{(\ell_1)} D_{m_2,k_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} \begin{pmatrix}\ell_{1:3}\\ m_{1:3}\end{pmatrix} = \begin{pmatrix}\ell_{1:3}\\ k_{1:3}\end{pmatrix}. \tag{B.15}$$

The **Gaunt-type integral**

$$\begin{aligned}\mathcal{G}_{k_1,k_2,k_3;m_1,m_2,m_3}^{\ell_1,\ell_2,\ell_3} &= \int_{SO(3)} D_{m_1,k_1}^{(\ell_1)} D_{m_2,k_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} dg \\ &= \begin{pmatrix}\ell_{1:3}\\ m_{1:3}\end{pmatrix}\begin{pmatrix}\ell_{1:3}\\ k_{1:3}\end{pmatrix},\end{aligned} \tag{B.16}$$

where $dg = \sin\vartheta d\vartheta d\varphi d\gamma/8\pi^2$ is the Haar-measure: (see [29].

(7) **Bipolar spherical harmonics,** see [29] 5.16.1, p.160,

$$Y_{\ell_1,\ell_2}^{\ell,m}\left(\underline{\widehat{\omega}}_{1:2}\right) = \left[\underline{Y}_{\ell_1}\left(\underline{\widehat{\omega}}_1\right) \otimes \underline{Y}_{\ell_2}\left(\underline{\widehat{\omega}}_2\right)\right]_{\ell,m}$$

$$= \sum_{m_{1:2}=-\ell_{1:2}}^{\ell_{1:2}} C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m} Y_{\ell_1}^{m_1}\left(\underline{\widehat{\omega}}_1\right) Y_{\ell_2}^{m_2}\left(\underline{\widehat{\omega}}_2\right),$$

$$\sum_{\ell,m} C_{\ell_1,m_1;\ell_2,m_2}^{\ell,m}\left[\underline{Y}_{\ell_1}\left(\underline{\widehat{x}}_1\right) \otimes \underline{Y}_{\ell_2}\left(\underline{\widehat{x}}_2\right)\right]_{\ell,m} = Y_{\ell_1}^{m_1}\left(\underline{\widehat{\omega}}_1\right) Y_{\ell_2}^{m_2}\left(\underline{\widehat{\omega}}_2\right).$$

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