

## ON POISSON EQUATIONS WITH A POTENTIAL IN THE WHOLE SPACE FOR “ERGODIC” GENERATORS

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ALEXANDER VERETENNIKOV

ABSTRACT. In [11, 12, 13, 19, 7] Poisson equation *in the whole space* was studied for so called ergodic generators  $L$  corresponding to homogeneous Markov diffusions  $(X_t, t \geq 0)$  in  $\mathbb{R}^d$ . Solving this equation is one of the main tools for *diffusion approximation* in the theory of stochastic averaging and homogenization. Here a similar equation *with a potential* is considered, firstly because it is natural for PDEs, and secondly with a hope that it may be also useful for some extensions related to homogenization and averaging.

### 1. INTRODUCTION

Let us consider a stochastic differential equation in  $\mathbb{R}^d$ ,

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt, \quad X_0 = x. \quad (1)$$

Assume the matrix function  $\sigma$  and the vector function  $b$  are Borel bounded, and

$$(a^{ij}(x)) := \frac{1}{2} \sigma \sigma^*(x)$$

uniformly non-degenerate. Some further conditions will be assumed in the sequel; however, the above already suffices for the existence of (weak) solution, see [4]. Denote

$$L = \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d b^i(x) \frac{\partial}{\partial x^i}.$$

The Poisson equation

$$Lu(x) = -f(x), \quad x \in \mathbb{R}^d, \quad (2)$$

is one of the well-known equations in mathematical physics. Only relatively recently, in the last two decades it was understood how useful is this equation *without boundary conditions and in the whole space*  $\mathbb{R}^d$ : namely, it is a powerful tool in the theory of averaging and homogenization, see [1, 2, 9], et al. This understanding was the reason for the series of papers [11, 12, 13] and for some further versions and extensions in [7, 19]. In all papers in the latter references such equations were *without zero order terms* (also known as potentials). On the other hand, equations *with* potentials are also very frequent in physics and even more popular than without them. In the cases where the author presented these results at the PDE seminars, the most frequent question was, why zero order terms are not included in the equation?

Hence, the goal of this paper is to transfer some of the recent advances about Poisson equation “in the whole space” without a potential to the equation

$$Lu - cu = -f, \quad (3)$$

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with a potential  $c$ . Note straight away that the case  $\inf_x c(x) > 0$  is the most simple one where – at least, for bounded functions  $f$  – convergence of the integral representing the solution

$$u(x) = \int_0^\infty \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt, \tag{4}$$

and the equation itself for this representation follow automatically. So, we will concentrate on the more interesting situations where either  $c$  is *not* separated away from zero yet remaining non-negative, or even if the function  $c$  may change its sign, with a hope that in the future it could be possibly useful, in particular, for controlled Markov processes and, perhaps, for a probabilistic interpretation of the Helmholtz equation. The problem of equations with parameters is not addressed here.

The paper consists of Introduction; Reminder of equations without potentials; Main results; Proofs; and the latter part is additionally split into several sections and subsections.

2. ASSUMPTIONS AND EQUATION WITHOUT POTENTIAL: QUICK REMINDER

Let us present the main assumptions. They will be used in the next section, too, except that the first one (A1) will be replaced by a stronger assumption (A5). Also note that the centering condition (A4) is needed only in this section, and also in one case out of three cases in the Main results.

(A1)

**Assumption 1** (polynomial recurrence).

$$\limsup_{|x| \rightarrow \infty} \langle b(x), x \rangle = -\infty. \tag{5}$$

(A2)

**Assumption 2** (boundedness and non-degeneracy).

*The coefficient  $\sigma\sigma^*$  is uniformly non-degenerate*

$$\inf_x \inf_{|\xi|=1} \xi^* \sigma\sigma^*(x) \xi > 0; \tag{6}$$

*the functions  $\sigma$ ,  $b$ ,  $f$  and  $c$  are Borel bounded.*

(A3)

**Assumption 3** (regularity).

*The coefficient  $a = \sigma\sigma^*$  is uniformly continuous.*

(A4)

**Assumption 4** ( $f$ -centering).

$$\int f(x) \mu(dx) = 0. \tag{7}$$

(A5)

**Assumption 5** ( exponential recurrence).

$$\limsup_{|x| \rightarrow \infty} \langle b(x), x/|x| \rangle = -r < 0. \tag{8}$$

**Remark 1.** *The assumption (A3) may be totally dropped in one-dimensional case. In the multi-variate case it is assumed so as to guarantee existence, weak uniqueness of solutions of the equation (1) and a strong Markov property. The assumption (A5) may be strengthened to  $\limsup_x \langle b(x), x/|x| \rangle = -\infty$ , in which case some other references on large deviation results would be needed but the results would be established not just for small values of  $\epsilon$  only; however, this would also require new references about convergence*

rates and mixing because for this assumption to be non-empty, the drift may not be globally bounded. The boundedness of other coefficients may also be relaxed a little bit, but we do not pursue this goal here,

Here is the main result from [11] about the equation (2); we do not show further advances already established, which relate to the more involved case of equations with parameters. Note that under the assumptions (A1) – (A3), the equation (1) has a weakly unique Markov and strong Markov solution  $(X_t)$  with a unique stationary measure  $\mu$  (cf. [4, 18]).

**Proposition 1** ([11]). *Under the assumptions (A1) – (A4), the equation (2) has a solution  $u$  in  $\bigcap_{p>1} W_{p,loc}^2$ . This solution is itself centered, it has a moderate growth (i.e., not faster than some polynomial), and this solution is unique in this class of functions. The representation (4) with  $c \equiv 0$  holds true for this solution.*

Recall that the assumption (A5) is not needed in this Proposition. On the other hand, where it will be used (in the next sections), it clearly replaces the weaker assumption (A1).

### 3. EQUATION WITH A POTENTIAL

Now we turn to the main goal of the paper, the equation (3) with a potential  $c$ . As it was already mentioned, a natural candidate for the solution is the function

$$u(x) = \int_0^\infty \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt,$$

see (4), of course, provided that this expression is well-defined. Recall that in the sequel it will be assumed that both  $f$  and  $c$  are bounded. Beside the most simple case  $\inf_x c(x) > 0$ , we are able to tackle three different situations. In all of these cases we will assume the assumption (A5), which, actually, replaces the weaker one (A1), see below.

**Case 1:**  $c(x) = \epsilon c_1(x)$ , where  $\epsilon > 0$  is *small enough*, and  $\int c_1 d\mu > 0$ .

**Case 2:**  $c(x) \geq 0$ ,  $\int c d\mu > 0$ .

**Case 3:**  $c(x) = -\epsilon < 0$  with  $\epsilon$  – which is a constant – small enough.

Note that all three cases do not include each other, although the case 1 and the case 2 do intersect. In all three cases we assume (A2) – (A5). In particular, the condition (A2) in terms of the function  $c_1$  necessarily assumes that  $c_1$  is bounded, likewise to  $c$ . The question about the case 2 was suggested by A. Piunovskiy; hopefully, it might be useful in the theory of controlled Markov processes (cf. with [10]).

In the cases 1 and 2, denote

$$\bar{c} = \int c d\mu,$$

and in the case 1 also

$$\bar{c}_1 = \int c_1 d\mu.$$

Clearly,  $\bar{c} = \epsilon \bar{c}_1$ .

**Theorem 1.** *In the cases 1 – 2, under the assumptions (A2)–(A3) and (A5), the function  $(u(x), x \in \mathbb{R}^d)$  given by the representation (4) is a continuous solution of the Poisson equation (3) in the Sobolev classes  $W_{p,loc}^2$  for each  $p > 0$ . This solution admits the bounds*

$$|u(x)| \leq C \exp(\gamma|x|), \tag{9}$$

*with any  $\gamma > 0$  and a corresponding  $C = C(\gamma)$ . In the case 3 the same assertions hold true under the assumptions (A2) – (A5), and, in addition, the function given by the formula (4) is centered. Recall that in the cases 1 and 3 the value of  $\epsilon > 0$  should be small enough, that is, there exists  $\epsilon_0 > 0$  such that all assertions hold true for any  $\epsilon < \epsilon_0$ .*

Note that in the cases 1–2 there is no need for the centering assumption (A4). The proof is split into the next three sections devoted to convergence (the section 4), existence of derivatives and verification of the equation (the section 5); uniqueness (the section 6).

4. PROOF: CONVERGENCE

In all cases we will use the bound from [15]

$$\|Q_t(x, dy) - \mu(dy)\|_{TV} \leq C \exp(\gamma|x|) \exp(-\lambda t), \quad t \geq 0, \tag{10}$$

where  $Q_t(x, dy)$  is the transition kernel of the process  $X_t$ , and  $\mu$  is its unique invariant measure, and “TV” is the total variation distance for two measures. Note that the inequality (10) may be read as follows: there exists  $\lambda_0 > 0$  such that for any  $\lambda \in (0, \lambda_0)$ , there exists  $\gamma > 0$  such that (10) holds; yet, it may be also read as follows: there exists  $\gamma_0 > 0$  such that for any  $\gamma \in (0, \gamma_0)$ , there exists  $\lambda > 0$  such that (10) holds true. Several close but a little different corollaries from this inequality will be used in the sequel.

Recall for the convenience of the reader that the bound (10) is a standard corollary of an exponential moment bound

$$\mathbb{E}_x \exp(\alpha \tau_K) \leq C \exp(\epsilon|x|), \tag{11}$$

for the stopping time  $\tau_K := \inf(t \geq 0 : |X_t| \leq K)$ ; in turn, such an exponential moment bound follows straightforwardly from Ito’s formula applied to the stochastic Lyapunov function  $\exp(\epsilon|X_t| + \alpha t)$  for  $t \leq \tau_K$ , with appropriately chosen (small enough)  $\epsilon > 0$  and  $\alpha > 0$ .

Another bound from [15] reads: there exists  $\gamma_0 > 0$  such that for any  $\gamma \in (0, \gamma_0)$  there exists  $C > 0$  such that

$$\sup_{t \geq 0} \mathbb{E}_x \exp(\gamma|X_t|) \leq C \exp(\gamma|x|). \tag{12}$$

Recall that this bound is derived also from Itô’s formula applied to the expression  $\exp(\epsilon|X_t|)$  for all values of  $t \geq 0$ : the equation naturally involves a certain local time, and the bound (12) follows from the comparison theorem applied to the process  $|X_t|$  and a simplified one-dimensional diffusion with a non-sticky reflection, for which simplified one-dimensional diffusion the bound like (12) is established by a direct calculation by using an explicit solution of a second order linear differential equation.

Also, the bound (12) implies that, at least, for  $\gamma > 0$  small enough,

$$\int \exp(\gamma|x|)\mu(dx) < \infty. \tag{13}$$

**4.1. Case 1, locally uniform convergence.** In this section we show convergence of the integral,

$$u(x) := \int_0^\infty \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt,$$

in the case with  $c(x) = \epsilon c_1(x)$ ,  $\bar{c}_1 = \int c_1 d\mu > 0$ , with  $\epsilon > 0$  small enough. Recall that the function  $c_1$  is bounded. Denote

$$H_T(\beta, x) := T^{-1} \ln \mathbb{E}_x \exp\left(\beta \int_0^T c_1(X_s) ds\right),$$

or, equivalently,

$$\mathbb{E}_x \exp\left(\beta \int_0^T c_1(X_s) ds\right) = \mathbb{E}_x \exp(T H_T(\beta, x)).$$

Let

$$H(\beta) := \lim_{T \rightarrow \infty} H_T(\beta, x), \quad \beta \in R^d. \tag{14}$$

Under the condition (A5), the limit in (14) does exist, at least, for all values of  $\beta$ , which do not exceed by absolute value some positive constant, locally uniformly with respect to  $\beta$  and  $x$ , see, e.g., [17, Theorem 1]. Since  $H_T(\beta, x)$  is convex with respect to  $\beta$ , and due to this locally uniform convergence, the limiting function  $H(\beta)$  is also convex in the domain where it is well-defined. Since  $H_T$  is clearly bounded for any value of  $|\beta| \leq K$  for any  $K$  uniformly in  $T$ , the function  $H$  is also finite and locally bounded in its domain, and, hence, it is also continuous as any finite convex function. It follows that  $H$  is differentiable at the origin  $\beta = 0$ , see, e.g., [16]. Because the reference may be not very well accessible, we recall now briefly the idea of this reasoning which is based on exponential bounds that precede the large deviation asymptotics in [16, 17].

To show the differentiability of the function  $H$  at zero, it is convenient to perform the following transformation,

$$H_T^1(\beta, x) := T^{-1} \ln \mathbb{E}_x \exp \left( \beta \int_0^T (c_1(X_s) - \bar{c}_1) ds \right) \equiv H_T(\beta, x) - \beta \bar{c}_1.$$

Further, due to the Law of Large Numbers – by virtue of a good mixing for  $X$ , see, e.g., [18] – we have,

$$(\mathbb{P}_x) \lim_{T \rightarrow \infty} T^{-1} \int_0^T (c_1(X_s) - \bar{c}_1) ds = 0,$$

and also  $\lim_{T \rightarrow \infty} T^{-1} \mathbb{E}_x \int_0^T c_1(X_s) ds = \bar{c}_1$ . Moreover, under the assumption (A5), for any  $\epsilon > 0$  there exist  $C = C(x), \lambda > 0$ , such that an exponential inequality holds,

$$\mathbb{P}_x \left( \left| T^{-1} \int_0^T (c_1(X_s) - \bar{c}_1) ds \right| \geq \epsilon \right) \leq C(x) \exp(-\lambda T), \tag{15}$$

with some  $C(x) = \exp(\gamma|x|), \gamma > 0$ ; here  $\lambda > 0$  does not depend on  $x$ . The estimate (15), actually, follows from the calculus in [17], although, it was not stated there explicitly. In order to show how (15) follows from the cited paper, let us use notations from [17]: for  $m > 0$  denote  $\tau_m^0 = 0, \tau_m^{k+1} := \inf(t \geq \tau_m^k + 1 : |X_{t-1}| \leq m, |X_t| \leq m), \tau_m^t := \tau_m^{\lfloor t \rfloor}$ , and  $\nu_m^{-1} = \mathbb{E}_{inv} \tau_m^1 \equiv \mathbb{E}_{inv} (\tau_m^{k+1} - \tau_m^k)$ , where  $\mathbb{E}_{inv}$  means the integration with respect to the invariant measure  $\mu$ . Also, let  $\tilde{c}_1(x) := c_1(x) - \bar{c}_1$ . Note that due to the Harris–Khasminsky principle  $\nu_m \mathbb{E}_{inv} \int_0^{\tau_m^1} c_1(X_s) ds = \bar{c}_1$ . Now, step XII of the section 3 “Proofs” in [17] includes the bound: there exists a constant  $K > 0$  such that for any  $b > 0, |\beta| \leq b$ , for any  $\delta > 0$  and for any  $t > 0$ ,

$$\begin{aligned} & \mathbb{E}_x \exp \left( \beta \int_0^t \tilde{c}_1(X_s) ds \right) \\ & \leq \mathbb{E}_x \exp \left( \beta \int_0^{\tau_m^{t\nu_m}} \tilde{c}_1(X_s) ds + \delta b \|\tilde{c}_1\|_{Bt} \right) + 2 \exp(b \|\tilde{c}_1\|_{Bt} - Kt). \end{aligned} \tag{16}$$

Emphasize that here  $K$  does not depend on  $b$ . Let

$$T_m^\beta h(x) := \mathbb{E}_x \exp \left( \beta \int_0^{\tau_m^1} \tilde{c}_1(x) \right) h(X_{\tau_m^1})$$

for  $(x : |x| \leq m)$ ; this operator is well-defined, at least, for  $|\beta|$  small enough uniformly in  $m \rightarrow \infty$ . Let  $\rho_m(\beta)$  denote the spectral radius of the operator  $T_m^\beta$  in  $C(B_m)$ , where  $B_m = (x : |x| \leq m)$ , for those values of  $\beta$  for which it is well-defined. From the arguments in [17] based on the spectral gap theorem for compact positive operators [3] and from the estimate (11) it follows that

$$\mathbb{E}_x \exp \left( \beta \int_0^{\tau_{t\nu_m}} \tilde{c}_1(X_s) ds \right) \leq C \exp(\epsilon|x|) (\rho_m(\beta))^{[t\nu_m]-1}, \tag{17}$$

and it is shown in [17] that

$$\ln \rho_m(\beta) \rightarrow H(\beta), \quad m \rightarrow \infty.$$

Also,

$$\ln \rho_m(\beta) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \mathbb{E}_x \exp \left( \beta \int_0^{\tau_m^n} \tilde{c}_1(X_s) ds \right),$$

the function  $\rho_m(\beta)$  is differentiable with respect to  $\beta$  in the domain where it is well-defined (see [3]), and

$$(\ln \rho_m(\beta))'_{\beta=0} = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x \int_0^{\tau_m^n} \tilde{c}_1(X_s) ds,$$

by virtue of the properties of convex functions (see [14]). On the other hand, due to the ergodicity,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_x \int_0^{\tau_m^n} \tilde{c}_1(X_s) ds = \mathbb{E}_{\mu_m} \int_0^{\tau_m^1} \tilde{c}_1(X_s) ds,$$

where  $\mu_m$  is the invariant measure of the stopped process  $X_{\tau_m^n}$ ,  $n \geq 0$ . Also,

$$\nu_m \rightarrow 1, \quad \mu_m \implies \mu, \quad \tau_m^1 \rightarrow 1 \text{ (a.s.)}, \quad m \rightarrow \infty,$$

and eventually,

$$\mathbb{E}_{\mu_m} \int_0^{\tau_m^1} \tilde{c}_1(X_s) ds \rightarrow \mathbb{E}_{\mu} \int_0^1 \tilde{c}_1(X_s) ds = 0, \quad m \rightarrow \infty.$$

So, for  $|\beta| \leq b$  with some appropriate  $b > 0$ ,

$$\rho_m(\beta) = 1 + o(b) + o_m(1), \quad \text{as } b \rightarrow 0, \quad m \rightarrow \infty.$$

Hence, for  $b > 0$  small enough and  $m \rightarrow \infty$ , the desired bound (15) follows from exponential Chebyshev–Markov’s inequality and from (16)–(17).

Now, once the bound (15) holds true, let  $\epsilon > 0$  be small enough. We will be using the split of unity

$$1 = 1(|T^{-1} \int_0^T (c_1(X_s) - \bar{c}_1) ds| < \epsilon) + 1(|T^{-1} \int_0^T (c_1(X_s) - \bar{c}_1) ds| \geq \epsilon),$$

and the elementary inequality  $a + b \leq 2(a \vee b)$  and, hence,  $\ln(a + b) \leq \ln(2(a \vee b))$  (with  $a, b > 0$ ), we now compute for any  $|\beta| \leq b$ , say,

$$\begin{aligned} H_T^1(\beta, x) &= T^{-1} \ln \left( \mathbb{E}_x \exp(\beta \int_0^T (c_1(X_s) - \bar{c}_1) ds) 1(|T^{-1} \int_0^T (c_1(X_s) - \bar{c}_1) ds| < \epsilon) \right. \\ &\quad \left. + \mathbb{E}_x \exp \left( \beta \int_0^T (c_1(X_s) - \bar{c}_1) ds \right) 1(|T^{-1} \int_0^T (c_1(X_s) - \bar{c}_1) ds| \geq \epsilon) \right) \end{aligned}$$

$$\leq T^{-1} \ln (2(\exp(|\beta| T \epsilon) \vee (\exp(|\beta| T \|c_1 - \bar{c}_1\|) C(x) \exp(-\lambda T)))) \leq 2b\epsilon.$$

Here  $b > 0$  may be taken small enough in comparison to  $\lambda$  (since the latter does not depend on  $\beta$ ). This implies that for a fixed  $x$  – and, actually, locally uniformly with respect to  $x$  – the function  $H_T^1(\beta, x) = o(|\beta|)$  uniformly in  $T \rightarrow \infty$ . Thus,  $H^1(\beta) := H(\beta) - \bar{c}_1\beta = o(|\beta|)$ , which, clearly, means that  $H(\beta)$  is differentiable at zero and that  $H'(0) = \bar{c}_1$ . Note that this is also in accordance with the fact that  $H_T(\beta, x) \rightarrow H(\beta)$  and since both functions are convex in  $\beta$ , we also have  $H_T'(0, x) \rightarrow H'(0)$  (see, e.g., [14]). For any  $\delta > 0$  we may assume that

$$|H_T(\beta, x) - H(\beta)| \leq \delta + o_T(1), \quad \text{as } T \rightarrow \infty.$$

In any case, since  $H'(0) = \bar{c}_1 > 0$ , in some neighbourhood of zero we have,

$$H(\beta) > 0, \quad \beta > 0, \quad \& \quad H(\beta) < 0, \quad \beta < 0. \tag{18}$$

Therefore, convergence of the integral in the definition of  $u$  follows from (18). Indeed, we estimate, for  $\epsilon > 0$  small and independent on  $x$ , and with any  $\delta > 0$  and taking  $\beta = -\epsilon$  we have,

$$\begin{aligned} |u(x)| &\leq \|f\|_B \int_0^\infty \mathbb{E}_x \exp\left(-\epsilon \int_0^t c_1(X_s) ds\right) dt = \|f\|_B \int_0^\infty \exp(tH_t(-\epsilon, x)) dt \\ &\leq \|f\|_B \int_0^\infty \exp(tH(-\epsilon) + \delta + o_t(1) + \gamma|x|) dt < \infty. \end{aligned}$$

Also note that here  $\gamma > 0$  may be chosen arbitrarily small, which means that the rate of growth of the function  $u$  is slower than any exponential of  $|x|$ . (In fact, some better polynomial growth bound on  $u(x)$  holds true, too.) More precisely, for any  $\gamma > 0$  small enough (and, hence, actually, for any  $\gamma > 0$ ) there exists  $C > 0$  such that

$$|u(x)| \leq C \exp(\gamma|x|). \quad (19)$$

**4.2. Case 2, locally uniform convergence.** Recall that we wish to show convergence of the integral,

$$u(x) := \int_0^\infty \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt,$$

in the case where

$$c(x) \geq 0, \quad \& \quad \int c d\mu > 0.$$

We will use an exponential estimate similar to (15),

$$P_x\left(\int_0^t c(X_s) ds < (\bar{c} - \delta)t\right) \leq C \exp(\nu|x|) \exp(-\lambda t), \quad (20)$$

along with a split of unity into two indicators. It follows that again the expression for  $u$  is well defined. Indeed,

$$\begin{aligned} &\int_0^\infty \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt \\ &= \int_0^\infty \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) 1_{\left(\int_0^t c(X_s) ds < (\bar{c} - \delta)t\right)} dt \\ &\quad + \int_0^\infty \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) 1_{\left(\int_0^t c(X_s) ds \geq (\bar{c} - \delta)t\right)} dt. \end{aligned}$$

Here the second term clearly converges for small  $\delta$ , while the first term converges due to the assumption  $c \geq 0$  and  $f$  bounded, because of the inequality (20), as required.

Also, with the help of (12) it follows,

$$|u(x)| \leq C \exp(\gamma|x|). \quad (21)$$

**4.3. Case 3, locally uniform convergence and centering.** Convergence along with the bound on  $|u(x)|$  in this case follows straightforwardly from the inequality (10), if we only admit that  $-c \equiv \epsilon < \lambda$ . The bound on  $|u(x)|$  then reads,

$$|u(x)| = \left| \int_0^\infty \mathbb{E}_x \exp(\epsilon t) f(X_t) dt \right| \leq C \|f\|_B (\lambda - \epsilon)^{-1} \exp(\gamma|x|). \quad (22)$$

The centering condition holds true due to the same centering assumption on  $f$ : by virtue of Fubini's theorem,

$$\begin{aligned} \int u(x) \mu(dx) &= \int \mu(dx) \int_0^\infty \mathbb{E}_x \exp(+\epsilon t) f(X_t) dt \\ &= \int_0^\infty \int \mu(dx) \mathbb{E}_x \exp(+\epsilon t) f(X_t) dt = 0, \end{aligned}$$

where the fact was used that the measure  $\mu$  is stationary and, hence,  $\int \mu(dx)\mathbb{E}_x f(X_t) = 0$  for each  $t \geq 0$ .

5. PROOF: OTHER PROPERTIES

**5.1. Verification of the equation: simplified version.** As we already know from the previous section, the function  $u$  given by the representation (4) is well-defined, that is, the integral in the right hand side converges for all  $x \in \mathbb{R}^d$ . Let us argue why  $u$  is, indeed, a solution of the Poisson equation. To make explicit the idea, assume for simplicity continuity of both functions  $c$  and  $f$  and suppose that existence of two (classical) derivatives of the function  $u$  is known; later on it will be shown how to drop all these additional assumptions, including classical derivatives instead of Sobolev ones. By the Markov property,

$$u(x) = \int_0^T \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt + \mathbb{E}_x \exp\left(-\int_0^T c(X_s) ds\right) u(X_T),$$

from which,

$$\begin{aligned} -f(x) &= -\lim_{T \rightarrow 0} T^{-1} \int_0^T \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt \\ &= \lim_{T \rightarrow 0} T^{-1} \left( \mathbb{E}_x u(X_T) \exp\left(-\int_0^T c(X_s) ds\right) - u(x) \right) \\ &= \lim_{T \rightarrow 0} \frac{1}{T} \mathbb{E}_x \int_0^T e^{-\int_0^t c(X_s) ds} (Lu - cu)(X_t) dt = Lu(x) - cu(x), \end{aligned}$$

as required. However, as we said earlier, in the sequel we aim to justify the equation without the additional assumption about continuity.

**5.2. Continuity of solution  $u$ .** This continuity of  $u$  will be used in the proof of existence of two Sobolev derivatives in the next subsection. Actually, we shall see a bit more than just continuity: in all three cases 1 – 3 it will be shown that the integral for  $u$  converges to a continuous limit *locally uniformly* with respect to  $x$ . So, similarly to [11] we obtain,

$$u(x) = \lim_{N \rightarrow \infty} u^N(x) := \lim_{N \rightarrow \infty} \mathbb{E}_x \int_0^N \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt,$$

where  $u^N(x)$  is continuous as a solution of the Cauchy problem for a parabolic differential equation, see [6]. So, the limit is also continuous, due to the locally uniform convergence. *Note that neither continuity of  $f$  nor of  $c$  was used in this consideration.*

**5.3. Two Sobolev derivatives for  $u$  and verification of the equation.** Consider  $\tau := \inf\{t : X_t \notin B\}$  and the following equation

$$Lv - cv = -f \quad \text{in } B, \quad v|_{\partial B} = u$$

in the ball

$$B = B_1(x_0) \equiv \{x \in \mathbb{R}^d : |x - x_0| \leq 1\}.$$

Since we already know that  $u$  is continuous, this boundary condition is well-defined. There is a unique solution  $v \in \bigcap_{p \geq 1} W_p^2$  in  $B$  [8], which by virtue of Itô–Krylov’s formula [5] admits a representation,

$$v(x) = \mathbb{E}_x \left( \int_0^\tau \exp\left(-\int_0^t c(X_s) ds\right) f(X_t) dt + \exp\left(-\int_0^\tau c(X_s) ds\right) u(X_\tau) \right).$$



Due to the strong Markov property, exactly the same representation holds true for  $u(x)$  in the left hand side; so,  $u \equiv v$  on  $B$ , i.e.,

$$u(x) = \mathbb{E}_x \left( \int_0^\tau \exp \left( - \int_0^t c(X_s) ds \right) f(X_t) dt + \exp \left( - \int_0^\tau c(X_s) ds \right) u(X_\tau) \right).$$

Hence,

$$u \in \bigcap_{p \geq 1} W_p^2(B) \subset C(B)$$

(see [8]). This consideration also justifies the equation for  $u$  without the additional assumption about continuity of  $f$  and  $c$ .

## 6. PROOF: UNIQUENESS OF SOLUTION

**6.1. Uniqueness, case 1.** Uniqueness may be shown in a standard manner for the class of functions satisfying the moderate growth established earlier. In all three cases the calculus is the same. For the difference of two solutions  $v = u^1 - u^2$  we have  $Lv - cv = 0$ . So, using moment inequalities and a standard localization procedure, by applying Itô–Krylov’s formula and taking expectations, we get

$$v(x) = u^1(x) - u^2(x) = \mathbb{E}_x \exp \left( - \int_0^t c(X_s) ds \right) v(X_t).$$

We now use a unity split

$$1 = 1 \left( \int_0^t c_1(X_s) ds \geq (\bar{c}_1 - \delta)t \right) + 1 \left( \int_0^t c_1(X_s) ds < (\bar{c}_1 - \delta)t \right)$$

and an exponential estimate

$$\mathbb{P}_x \left( \int_0^t c_1(X_s) ds < (\bar{c}_1 - \delta)t \right) \leq C \exp(\nu|x|) \exp(-\lambda t).$$

Then it follows that  $v(x) \equiv 0$ . Indeed, due to the bound (15), we estimate with any  $t > 0$ ,

$$\begin{aligned} |v(x)| &\leq \mathbb{E}_x \exp \left( - \int_0^t c(X_s) ds \right) |v(X_t)| \\ &= \mathbb{E}_x \exp \left( - \int_0^t c(X_s) ds \right) |v(X_t)| 1 \left( \int_0^t c(X_s) ds \geq (\bar{c} - \delta)t \right) \\ &\quad + \mathbb{E}_x \exp \left( - \int_0^t c(X_s) ds \right) |v(X_t)| 1 \left( \int_0^t c_1(X_s) ds < (\bar{c}_1 - \delta)t \right) \\ &\leq \mathbb{E}_x \exp(-\epsilon(\bar{c}_1 - \delta)t) C \exp(\nu|X_t|) \\ &\quad + \exp(\epsilon\|c_1\|t) (\mathbb{E}_x |v(X_t)|^2)^{1/2} (\mathbb{P}_x \left( \int_0^t c_1(X_s) ds < (\bar{c}_1 - \delta)t \right))^{1/2} \\ &\leq C \exp(-\epsilon(\bar{c}_1 - \delta)t) C \exp(\nu|x|) \\ &\quad + C \exp(\nu'|x|) \exp(-(\lambda/2 - \epsilon\|c_1\|)t) \rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

In the middle of the calculus we have applied Cauchy–Bouniakovsky–Schwarz’ inequality. So, if  $\epsilon > 0$  was chosen small enough, it shows that  $u^1 \equiv u^2$ , which completes the proof of the Theorem 1 in the case 1.

**6.2. Uniqueness, case 2.** For the difference of two solutions  $v = u^1 - u^2$  we have  $Lv - cv = 0$ . So, using moment inequalities and a standard localization procedure, by applying Itô–Krylov’s formula and taking expectations, we get

$$v(x) = u^1(x) - u^2(x) = \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) v(X_t).$$

We now use a unity split

$$1 = 1\left(\int_0^t c(X_s) ds \geq (\bar{c} - \delta)t\right) + 1\left(\int_0^t c(X_s) ds < (\bar{c} - \delta)t\right)$$

and an exponential estimate

$$\mathbb{P}_x\left(\int_0^t c(X_s) ds < (\bar{c} - \delta)t\right) \leq C \exp(\nu|x|) \exp(-\lambda t).$$

Then it follows that  $v(x) \equiv 0$ . Indeed, recall that in the case 2,  $c \geq 0$ . Hence, due to the bound (20), we estimate, with any  $t > 0$ ,

$$\begin{aligned} |v(x)| &\leq \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) |v(X_t)| \\ &= \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) |v(X_t)| 1\left(\int_0^t c(X_s) ds \geq (\bar{c} - \delta)t\right) \\ &\quad + \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) |v(X_t)| 1\left(\int_0^t c(X_s) ds < (\bar{c} - \delta)t\right) \\ &\leq \mathbb{E}_x \exp(-(\bar{c} - \delta)t) C \exp(\nu|X_t|) \\ &\quad + (\mathbb{E}_x |v(X_t)|^2)^{1/2} (\mathbb{P}_x\left(\int_0^t c(X_s) ds < (\bar{c} - \delta)t\right))^{1/2} \\ &\leq C \exp(-(\bar{c} - \delta)t) C \exp(\nu|x|) \\ &\quad + C \exp(\nu'|x|) \exp(-\lambda/2)t \rightarrow 0, \quad t \rightarrow \infty. \end{aligned}$$

So,  $u^1 \equiv u^2$ , which completes the proof of the Theorem 1 in the case 2.

**6.3. Uniqueness, case 3.** For the difference of two (centered) solutions  $v = u^1 - u^2$  we have  $Lv - cv = 0$ . So, using moment inequalities and a standard localization procedure, by applying Itô–Krylov’s formula and taking expectations, we get

$$v(x) = u^1(x) - u^2(x) = \mathbb{E}_x \exp\left(-\int_0^t c(X_s) ds\right) v(X_t) \equiv \exp(+\epsilon t) \mathbb{E}_x v(X_t).$$

Recall the bound  $|v(x)| \leq C \exp(\gamma|x|)$ , where  $\gamma > 0$  can be chosen arbitrarily small (and, of course, respectively,  $C$  depends on  $\gamma$ ). By using the bounds (10) and (12) and taking  $0 < \delta < \bar{c}$ , and due to the centering property of  $v$ , we estimate,

$$\begin{aligned} |v(x)| &\leq \exp(+\epsilon t) |\mathbb{E}_x v(X_t)| = \exp(+\epsilon t) \left| \int v(y) Q_t(x, dy) \right| \\ &= \exp(+\epsilon t) \left| \int v(y) (Q_t(x, dy) - \mu(dy)) \right| \\ &\leq \exp(+\epsilon t) \left( \int v^2(y) (Q_t(x, dy) + \mu(dy)) \right)^{1/2} \left( \int |Q_t(x, dy) - \mu(dy)| \right)^{1/2} \\ &\leq \exp(+\epsilon t) C \exp(\gamma|x|) \exp(-t\lambda/2) \rightarrow 0, \quad t \rightarrow \infty, \end{aligned}$$

if  $\epsilon > 0$  is small enough. We used, in particular, Cauchy–Bouniakovsky–Schwarz’ inequality and the fact that  $\mu$  integrates exponentials  $\exp(\gamma|x|)$  with small  $\gamma$ . So,  $u^1 \equiv u^2$ , which completes the proof of the Theorem 1 in the case 3.

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- Current address:* University of Leeds, UK; National Research University Higher School of Economics, and Institute for Information Transmission Problems, Moscow, Russia  
*E-mail address:* [a.veretennikov @ leeds.ac.uk](mailto:a.veretennikov@leeds.ac.uk)

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