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ON FLUCTUATION THEORY FOR SPECTRALLY NEGATIVE LÉVY PROCESSES WITH PARISIAN REFLECTION BELOW, AND APPLICATIONS

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Abstract. As well known, all functionals of a Markov process may be expressed in terms of the generator operator, modulo some analytic work. In the case of spectrally negative Markov processes however, it is conjectured that everything can be expressed in a more direct way using the W scale function which intervenes in the two-sided first passage problem, modulo performing various integrals. This conjecture arises from work on Levy processes [7, 50, 12, 29, 28, 30, 6, 16], where the W scale function has explicit Laplace transform, and is therefore easily computable; furthermore it was found in the papers above that a second scale function Z introduced in [7] (this is an exponential transform (8) of W) greatly simplifies first passage laws, especially for reflected processes.

Z is an harmonic function of the Lévy process (like W), corresponding to exterior boundary conditions $w(x) = e^{\theta x}$ (9), and is also a particular case of a "smooth Gerber-Shiu function" $\mathcal{S}_w$. The concept of Gerber-Shiu function was introduced in [26]; we will use it however here in the more restricted sense of [15], who define this to be a "smooth" harmonic function of the process, which fits the exterior boundary condition $w(x)$ and solves simultaneously the problems (17), (18).

It has been conjectured that similar laws govern other classes of spectrally negative processes, but it is quite difficult to find assumptions which allow proving this for general classes of Markov processes. However, we show below that in the particular case of spectrally negative Lévy processes with Parisian absorption and reflection from below [6, 21, 16], this conjecture holds true, once the appropriate W and Z are identified (this observation seems new).

This paper gathers a collection of first passage formulas for spectrally negative Parisian Lévy processes, expressed in terms of W, Z and $\mathcal{S}_w$, which may serve as an "instruction kit" for computing quantities of interest in applications, for example in risk theory and mathematical finance. To illustrate the usefulness of our list, we construct a new index for the valuation of financial companies modeled by spectrally negative Lévy processes, based on a Dickson–Waters modifications of the de Finetti optimal expected discounted dividends objective. We offer as well an index for the valuation of conglomerates of financial companies.

An implicit question arising is to investigate analog results for other classes of spectrally negative Markovian processes.

1. Introduction

It is a great pleasure to have this opportunity to thank Nikolai Leonenko for all the energy, kindness and patience he put in our mathematical collaborations. He put things into my mind which will remain there, like for example the Kolmogorov–Pearson–Wong diffusions, a topic at the crossroads of probability and analysis.

As well known, the fluctuation/first passage theory of diffusions reduces to the computation of two non-negative monotone functions in the kernel of the associated Sturm-Liouville operator [22], which are related to the first passage times $\tau_b^{\pm}$ (2). A similar situation occurs for spectrally one sided Lévy processes, when one basic function, the so-called W scale function [19], suffices.

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In the joint paper [8], we tried to put these two "solvable models" together and study first passage problems for KPW diffusions with one-sided jumps. The hope was to unify the two theories, and I still hope to go back to this project one day.

The paper below is a modest step in a different direction: it illustrates the fact that the fluctuation theory for spectrally negative Lévy processes with Parisian reflection below [38, 6, 5, 47, 21, 16] is formally identical with the classical one [7, 50, 12, 29, 28, 30, 4, 15], once appropriate scale functions are introduced. Our hope is to stimulate further work on the intriguing question of whether this continues to be true for other classes of spectrally negative Markovian processes. We also hope to provide an easily accessible summary of the general ideas, hoping to make more accessible this evolving literature with a huge potential for applications (mathematical finance, risk, inventory and queueing theory, reliability, etc).

Fluctuation theory (the study of maxima, minima and reflected processes) reduces in the case of spectrally one-sided Lévy processes to the calculation of scale functions. The name scale function in this context and the realization of the importance of the concept should be attributed to Bertoin [19]. The idea was further developed in [7, 50, 12, 15] who illustrated that the answers to a wide variety of first passage problems may be ergonomically expressed in terms of the so called W, Z and $\mathcal{S}_w$ scale functions, informally defined as the harmonic functions of the process corresponding to exterior boundary conditions 0, $e^{\theta x}$ and $w(x)$, respectively. These functions are common to the free process X, to the process reflected below $X^{[0}$ or above $X^{b]}$, and to the doubly reflected process $X^{[0,b]}$, making thus natural to study these processes simultaneously. Subsequently, [29, 28, 30] showed that the Lévy formulas apply also in the much more general context of spectrally negative Markov additive processes, once appropriate **matrix scale** functions are introduced.

Somewhat surprisingly, we show here in section 5, Proposition 1, that the classic formulas apply also to spectrally negative Lévy processes with Parisian absorbtion or reflection below, once one identifies the appropriate scale functions W, Z (this is true at least for the first passage formulas we collected, and will probably be true for others). In principle, similar formulas may hold for other classes of spectrally negative Markov processes, for example Lévy processes with refraction [33, 35, 47] or with a Parisian buffer [16], or KPW diffusions with jumps, and this topic deserves further investigation.

Contents. We start with a review in section 2 of some basics first passage results for spectrally negative Lévy processes, following [7, 28, 15].

We sketch then in section 3, as an appetizer, an interesting financial application worth further study: a dynamic valuation index (22) for the constituents of a conglomerate of financial companies, which generalizes the de Finetti optimal dividends objective (21). Section 4 proceeds with a review of some classic financial optimization problems, solved using scale functions: the de Finetti and Shreve–Lehoczky–Gaver optimization objectives, and the American put option. These motivate their Parisian generalizations in the next section 5, the most important one, which gathers together eight recent first passage results for Parisian spectrally negative Lévy processes [3, 6, 16]. The goal is to provide a concise reference, and to illustrate the fact that these results are formally identical with those for classic spectrally negative Lévy processes, up to identifying the correct scale functions.

In theorem 1. section 6 we illustrate the usefulness of our list of formulas by calculating a static valuation index for financial companies modelled by Parisian spectrally negative Lévy processes, based on their "readiness to pay dividends".

As a second application, we provide in Section 7 a dynamic valuation index for "central branch networks" (conglomerates of financial companies with centralized decision taking).

A typical example of proof is included in Section 8.

2. First passage theory for spectrally negative Lévy processes, via the scale functions

A Lévy process $X(t)_{t\geq 0}$ may be characterized by its Lévy-Khintchine/Laplace exponent, defined by

$$E_0\left[e^{\theta X(t)}\right] = e^{t\kappa(\theta)}.$$

In applied probability we are often confronted with the case of spectrally negative processes with negative jumps only (of course, spectrally positive processes only require a change of sign), and with Lévy-Khintchine decomposition of the form

$$\kappa(\theta) = \frac{\sigma^2}{2}\theta^2 + p\theta + \int_{\mathbb{R}_+\setminus\{0\}} [e^{-\theta y} - 1 + \theta y]\Pi(dy),\ \theta \geq 0. \tag{1}$$

Here the Lévy measure (of $-X$) ‡ satisfies $\Pi(-\infty, 0) = 0$ and $\int_{\mathbb{R}_+\setminus\{0\}}(1\wedge y^2)\Pi(dy) < \infty$, and furthermore the drift (or profit rate) $p = E_0[X(1)] = \kappa'(0_+) > -\infty$ is finite (this requires that the large negative jumps of the process have a finite mean, which is quite sensible, say in modeling of catastrophes) § .

A further particular case to bear in mind is when the Lévy measure has finite mass $\Pi(0,\infty) = \lambda < \infty$. Writing $\Pi(dz) = \lambda F(dz)$ allows decomposing the resulting "Cramér-Lundberg" process as

$$X(t) = x + ct - \sum_{i=1}^{N_\lambda} C_i,\ c = p + \lambda \int_0^\infty zF(dz).$$

where C_i, $i = 1, 2, \ldots$ are i.i.d. nonnegative jumps with distribution $F(dz)$, arriving after exponentially distributed times with mean $1/\lambda$.

First passage theory concerns the first passage times above and below, and the hitting time of a level b, defined by

$$\begin{gathered}\tau_b^+ = \inf\{t \geq 0 : X(t) > b\}, \quad \tau_b^- = \inf\{t \geq 0 : X(t) < b\},\\ \tau^{\{b\}} = \inf\{t \geq 0 : X(t) = b\}\end{gathered} \tag{2}$$

(with $\inf \emptyset = +\infty$). We write sometimes τ for the "ruin time" τ_0^-.

In applications, we are often interested in versions of $X(t)$ which are constrained/regulated at first passage times below or/and above:

$$X^{[0}(t) = X(t) + R_*(t),\ \ X^{b]}(t) = X(t) - R(t).$$

Here,

$$\begin{aligned}&R_*(t) = -(\underline{X}(t) \wedge 0),\ R(t) = R^b(t) = \left(\overline{X}(t) - b\right)_+,\\ &\underline{X}(t) = \inf_{0\leq s\leq t} X(t),\ \overline{X}(t) := \sup_{0\leq s\leq t} X(t),\end{aligned}$$

are the minimal Skorohod regulators constraining $X(t)$ to be nonnegative, and to be smaller than b, respectively ‡ . Also interesting are processes reflected at 0 and refracted

‡Note that even though X has only negative jumps, for convenience we choose the Lévy measure to have mass only on the positive instead of the negative half line.

§$X(t)$ is a Markovian process with infinitesimal generator $\mathcal{G}$, which acts on $f \in C_c^2(\mathbb{R}_+)$ as [54, Thm. 31.5]

$$\mathcal{G}h(x) = \frac{\sigma^2}{2}h''(x) + ph'(x) + \int_{\mathbb{R}_+\setminus\{0\}} [h(x-y) - h(x) + yh'(x)]\Pi(\mathrm{d}y);$$

incidentally, this may be formally written as $\mathcal{G} = \kappa(D)$, where D denotes the differentiation operator.

‡Instead of $X^{b]}(t)$ one may study the "unused capacity" process $Y^b(t) := b - X^{b]}(t) = (\overline{X}(t)\vee b) - X(t)$

at the maximum (or taxed) with coefficient δ, staring from τ_b^+ [33, 35, 47], [4, (1)]:

$$X_\delta(t) = X_\delta^{[0,b[}(t) = X(t) + R_*(t) - \delta R^b(t),\ \delta \le 1.$$

The regulators $R_*(t)$, $R(t)$, defined by having points of increase contained in $\{t \ge 0 : X_\delta(t) = 0\}$ and $\{t \ge 0 : X_\delta(t) = \overline{X_\delta}(t) \vee b\}$ respectively, are more complicated to describe explicitly in this case; for a recursive construction, see [4, Appendix].

Remark 1. *The process $\delta R(t)$ may be interpreted as cumulative tax or dividends paid to some beneficiary.*

2.1. **The W and Z scale function for spectrally negative Lévy processes.** Solving first passage problems involving Markovian processes requires analytic work with their **generator operator**. In the case of Lévy processes, the analytic work may be replaced by the Wiener-Hopf factorization of the Laplace exponent with killing $\kappa(\theta) - q$ (i.e. the identification and separation of its positive and negative roots).

For spectrally negative Lévy process, the factorization involves only one nonnegative root

$$\Phi_q := \sup\{\theta \ge 0 : \kappa(\theta) - q = 0\}, \tag{3}$$

and everything reduces finally to the determination of one family of functions $W_q(x) : \mathbb{R}- > [0, \infty)$, $q \ge 0$, defined on the positive half-line by the Laplace transform:

$$\int_0^\infty \mathrm{e}^{-\theta x} W_q(x)\mathrm{d}x = \frac{1}{\kappa(\theta) - q}, \quad \forall \theta > \Phi_q \tag{4}$$

and taking the value zero on the negative half-line. Note that the singularity Φ_q plays a central role – see for example [19, Thm VII.1], stating that Φ_q is the Lévy exponent of the subordinator τ_x^+, $x \ge 0$.

The answers to many first passage problems may be ergonomically expressed in terms of this function, starting with the classic **gambler's winning problem**. For any $a < b$ and $x \in [a, b]$,

$$\mathbb{E}_x\left(e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_a^-\}}\right) = \frac{W_q(x-a)}{W_q(b-a)} = \int_0^\infty e^{-qt} dP\left\{\tau_b^+ < \min[t, \tau_a^-]\right\}.^{\S} \tag{5}$$

Remark 2. *Note that establishing the equivalence between* (5) *and* (4)*, "up to a constant", is not trivial. One solution when $q = 0$, via excursion theory, is provided in* [19, Thm VII.8] *by using the representation*

$$W(x) = W(\infty) e^{-\int_x^\infty \frac{W'(h)}{W(h)} dh} = W(\infty) e^{-\int_x^\infty \nu(h) dh}, \tag{6}$$

where $\nu(h) = \frac{W'(h)}{W(h)} = n[\overline{\epsilon} > h]$ is the characteristic measure of the Poisson process n of excursion heights (informally, $\nu(h)$ is the rate of excursions starting at the moment τ_h^+ which are bigger than h, and thus cause ruin). Later, [46] *used a Kennedy type martingale, and* [50, (3)] *constructed the scale function using potential theory*

$$W_q(x) = \Phi_q' e^{\Phi_q x} - u_q(-x) = u_q^+(-x) - u_q(-x) = e^{\Phi_q x}[u_q(0) - \frac{u_q^+(x) u_q(-x)}{u_q(0)}], \quad x \ge 0, \tag{7}$$

where Φ_q is the inverse of the Lévy exponent (4) *and u_q is the potential density, which is exponential for nonnegative x, given by $u_q^+(x) = \Phi_q' e^{-\Phi_q x}$, $x \ge 0$.*

A second Z_q scale function introduced recently [15, 28] [‖]

$$Z_q(x, \theta) = e^{\theta x}\left(1 - \left(\kappa(\theta) - q\right)\int_0^x e^{-\theta y} W_q(y) dy\right). \tag{8}$$

[‖] generalizing (10) from [7]

further simplifies the solution of many first passage problems.

For $\Re(\theta)$ large enough to ensure integrability, it holds that

$$Z(x,\theta) = \big(\kappa(\theta) - q\big)\int_0^\infty e^{-\theta y} W(x+y)dy.$$

Thus, the $Z(x,\theta)$ scale function is up to a constant an analytic extension of the Laplace transform of the shifted scale function W (also called Dickson-Hipp transform), and the normalization ensures that $Z(0,\theta) = 1$. $Z_q(x,\theta)$ is a "smooth Gerber-Shiu function" with penalty $w(x) = e^{\theta x}$ in the sense of [15], i.e. the unique "smooth" solution of

$$\begin{cases} (\mathcal{G} - qI)Z_q(x,\theta) = 0, & x \geq 0 \\ Z(x,\theta) = e^{\theta x}, & x \leq 0 \end{cases}, \tag{9}$$

where $\mathcal{G}$ is the Markovian generator of the process $X(t)$ – see [15, (1.12),(5.23), Sec 5] and section 2.2. Indeed, consider the general solution $g(x,\theta) = Z_q(x,\theta) + kW_q(x)$ and note that continuity at 0, i.e. $g(0,\theta) = 1$ implies $k = 0$ when $W_q(0) \neq 0$, and differentiability at 0, i.e. $g'(0,\theta) = \theta$ implies $k = 0$ when $W_q(0) = 0$ (since $W_q'(0) > 0$).

Here are some useful relatives of $Z_q(x,\theta)$ [36]:

$$\begin{aligned} \overline{W}_q(x) &:= \int_0^x W_q(y)\mathrm{d}y, \\ Z_q(x) &:= Z_q(x,0) = 1 + q\overline{W}_q(x) = \frac{\sigma^2}{2}W_q'(y) + cW_q(y) - \int_0^x W_q(y)\overline{\Pi}(x-y)dy, & (10) \\ \overline{Z}_q(x) &:= \int_0^x Z_q(z)\mathrm{d}z = x + q\int_0^x\int_0^z W_q(w)\mathrm{d}w\mathrm{d}z & (11) \end{aligned}$$

(the second definition of $Z_q(x)$ holds since $\mathcal{G}\left(\overline{W}_q\right)(x) = 0$). Note that for $-\infty < x < 0$ we have $Z_q(x,\Phi_q) = e^{\theta x}$, and for $x \leq 0$ $\overline{W}_q(x) = 0$, $Z_q(x) = 1$, $\overline{Z}_q(x) = x$.

The Z function may be also characterized by its respective Laplace transform

$$\widehat{Z}_q(s,\theta) = \frac{1}{\kappa(s) - q}\frac{\kappa(s) - \kappa(\theta)}{s - \theta}$$

(and $\widehat{Z}_q(s) = \frac{1}{\kappa(s)-q}\frac{\kappa(s)}{s}$).

Here are some examples of the utility of the $Z_q(x,\theta)$ function.

Lemma 1. Severity of ruin for a process absorbed or reflected at $b > 0$.
A) The joint Laplace transform of the first hitting time at 0 *and the undershoot is given by* [15], [28, Cor 3], [6, (5)]

$$S_q(x,\theta) := \mathbb{E}_x\left(e^{-q\tau_0^-}e^{\theta X_{\tau_0^-}};\tau_0^- < \tau_b^+\right) = Z_q(x,\theta) - \frac{W_q(x)}{W_q(b)}Z_q(b,\theta),\ \theta \geq 0. \tag{12}$$

B) The joint Laplace transform of the first hitting time at 0 *and the undershoot in the presence of reflection at a barrier $b \geq 0$ is* [15, Prop 5.5], [28, Thm 6]

$$S_q^{b]}(x,\theta) = \mathbb{E}_x^{b]}\left(e^{-q\tau_0^-}e^{\theta X_{\tau_0^-}}\right) = Z_q(x,\theta) - \frac{W_q(x)}{W_q'(b)}Z_q'(b,\theta),\ \theta \geq 0, \tag{13}$$

where $\mathbb{E}^{b]}$ denotes expectation for a process reflected from above at b, using Skorokhod reflection.

Remark 3. *Note the similar structure of* (12) *and* (13)*; formally, switching form absorbtion at b to the measure $\mathbb{E}^{b]}$ involving reflection at b only requires adding derivatives in the b dependent coefficient. This follows easily from the respective boundary conditions $S(b) = 0$, $\left(S^{b]}\right)'(b) = 0$.*

Remark 4. *By using* $\lim_{b\to\infty} \frac{Z_q(b,\theta)}{W_q(b)} = \frac{q-\kappa(\theta)}{\Phi_q-\theta}$, *we recover* $\mathbb{E}_x[e^{-q\tau_0^- + \theta X(\tau_0^-)}] = Z_q(x,\theta) - W_q(x)\frac{\kappa(\theta)-q}{\theta-\Phi_q}$ [6, (7)]. *For* $\theta = 0$ *this is the famous ruin time transform* $\mathbb{E}_x[e^{-q\tau_0^-}] = Z_q(x) - W_q(x)\frac{q}{\Phi_q}$. *Note also the similar transform of the "recovery time"* $\mathbb{E}_x[e^{-q\tau^{\{0\}}}] = \mathbb{E}_x[e^{-q\tau_0^- + \Phi_q X(\tau_0^-)}] = Z_q(x,\Phi_q) - W_q(x)\frac{q}{\Phi_q}$.

Remark 5. *The functions* $Z_q(x)$ *and* $Z_q(x,\theta)$ *appeared first in* [7] *and in* [15, (5.23)], *respectively. In the second paper (first submitted in 2012)* $Z_q(x,\theta)$ *was introduced as a particular case of smooth Gerber-Shiu function associated to an* **exponential payoff** $e^{\theta x}$, *as in Lemma 1, and as a generating function for the smooth Gerber-Shiu functions associated to* **power payoffs**.

Subsequently, the ground-breaking papers [28, 6, 4] *revealed several other first passage laws involving* $Z_q(x,\theta)$.

We turn now to a result which we may be seen as the fundamental first passage law for reflected spectrally negative Lévy processes.

Lemma 2. The Laplace transform of the discounted capital injections/bail-outs until reaching an upper level. *Let* $X(t)$ *denote a spectrally negative Lévy process, let* $R_*(t) = -(0 \wedge \underline{X}(t))$ *denote the regulator at* 0, *let* $X^{[0}(t) = X(t) + R_*(t)$ *denote the process reflected at* 0, *and let* $\mathbb{E}_x^{[0}$ *denote expectation for the process reflected at* 0. *The total capital injections into the reflected process, until the first up-crossing of a level* b, *satisfy* [28, Thm 2]*:*

$$L_*^{[0}(x,b) := \mathbb{E}_x^{[0}[e^{-q\tau_b^+ - \theta R_*(\tau_b^+)}] = \begin{cases} Z_q(x,\theta)Z_q(b,\theta)^{-1} & \theta < \infty \\ \mathbb{E}_x[e^{-q\tau_b^+} I_{\tau_b^+ < \tau}] = W_q(x)W_q(b)^{-1} & \theta = \infty \end{cases}, \tag{14}$$

where W *is the classic scale function* (4).

Remark 6. *This result may be viewed as the fundamental law of spectrally negative Lévy processes, since it implies the smooth two-sided exit formula* (5)*. It is proved in* [28, Thm 2] *as a consequence of a more general result* [28, Thm 13]*, but is essentially equivalent to* (12)*, by using* [28]

$$\mathbb{E}_x^{[0}[e^{-q\tau_b^+ - \theta R_*(\tau_b^+)}] = \mathbb{E}_x[e^{-q\tau + \theta X(\tau_0^-)}; \tau < \tau_b^+]\ \mathbb{E}_0^{[0}[e^{-q\tau_b^+ - \theta R_*(\tau_b^+)}] + W_q(x)W_q(b)^{-1} \tag{15}$$

If the first term is known one gets an equation for the severity of ruin

$$Z(x,\theta)Z(b,\theta)^{-1} = W_q(x)W_q(b)^{-1} + \mathbb{E}_x[e^{\theta X(\tau_0^-)}; \tau_0^- < \tau_b^+]Z(b,\theta)^{-1},$$

with the known solution

$$\mathbb{E}_x\left(e^{-q\tau + \theta X(\tau)}; \tau < \tau_b^+\right) = Z_q(x,\theta) - W_q(x)W_q(b)^{-1}Z_q(b,\theta).$$

And if the severity of ruin is known, one may use (15) *with* $x = 0$ *to solve for*

$$\mathbb{E}_0^{[0}[e^{-q\tau_b^+ - \theta R_*(\tau_b^+)}],$$

provided that $W_q(0) \neq 0$. *When* $W_q(0) = 0$, *one must start with a "perturbation approach", letting* $x \to 0$ *– see for example Section 8.*

Here is a generalization of Lemma 1 B):

Lemma 3. *The* **dividends- penalty law** *for a process reflected at* b *is* [28, Thm 6]*:*

$$S_\vartheta^b(x,\theta) := \mathbb{E}_x^{b]}\left[e^{-q\tau_0^- + \theta X(\tau_0^-) - \vartheta\int_0^{\tau_0^-} e^{-qt} dR(t)}\right] = Z_q(x,\theta) - W_q(x)\frac{Z_q'(b,\theta) + \vartheta Z_q(b,\theta)}{W_q'(b) + \vartheta W_q(b)}. \tag{16}$$

Remark 7. *The function $S^b_\vartheta(x,\theta)$ is q- harmonic, i.e. $(\mathcal{G}-q)S^b_\vartheta(x,\theta)=0$, and satisfies $(S^b_\vartheta)'(b,\theta)+\vartheta S^b_\vartheta(b,\theta)=0$. Minimizing it when $\vartheta\neq 0$, with respect to all possible dividend policies is an example of* **risk sensitive** *dividends optimization. It is expected that the last maximum of the barrier function $G(b)=\frac{Z_q'(b,\theta)+\vartheta Z_q(b,\theta)}{W_q'(b)+\vartheta W_q(b)}$ will play a key role in the answer (note when $\theta=\vartheta=0$, this is tantamount to maximizing the de Finetti barrier function $\frac{W_q(b)}{W_q'(b)}$).*

Extensions to other classes of spectrally negative Markov processes. Formulas with a similar structure in terms of scale functions must hold for other Markovian processes, and some recent papers showed indeed that generalizations of the W, Z functions were still manageable computationally for other types of Markovian processes besides Lévy: for example, see [12, 42, 15] for reflected Lévy processes, and [34, 29] for spectrally negative Markov additive processes, and [6, 16] for spectrally negative Lévy process with Parisian reflection (bailouts). These results increase considerably the arsenal of financial optimization tools.

2.2. **Nonhomogeneous problems and the Gerber–Shiu function $\mathcal{S}_w$.** When $e^{\theta X(\tau_0^-)}$ is replaced by an arbitrary penalty function $w(X(\tau_0^-))$, $w:(-\infty,0]\to\mathbb{R}$ which is "admissible" (satisfies certain integrability condition) in (12), (13), extensions of these formulas still hold if one replaces $Z_q(x,\theta)$ by a "smooth Gerber–Shiu function" $\mathcal{S}_w$ [15]. More precisely, given $0<b<\infty$, $x\in(0,b)$, there exists a unique "smooth GS function" $\mathcal{S}_w$ so that the following hold:

$$S_w(x)=\mathbb{E}_x^{b]}\left[e^{-q\tau_a^-}w\left(X(\tau_a^-)\right)\mathbf{1}_{\{\tau_a^-<\tau_b^+\}}\right]=\mathcal{S}_w(x)-W_q(x)\frac{\mathcal{S}_w(b)}{W_q(b)}, \tag{17}$$

$$S_w^{b]}(x)=\mathbb{E}_x^{b]}\left[e^{-q\tau_a^-}w\left(X(\tau_a^-)\right)\right]=\mathcal{S}_w(x)-W_q(x)\frac{\mathcal{S}_w'(b)}{W_q'(b)}. \tag{18}$$

Stated informally, this amounts to the fact that both these problems admit decompositions involving an identical "non homogeneous solution" $\mathcal{S}_w$.

The "smoothness" required is:

$$\begin{cases}\mathcal{S}_w(0)=w(0),\\ \mathcal{S}_w'(0_+)=w'(0_-), \quad \text{in the case } \sigma^2>0 \text{ or } \Pi([0,1])=\infty.\end{cases} \tag{19}$$

Under these conditions, the function $\mathcal{S}_w$ is unique and may be represented as [15, (5.13)]:

$$\begin{aligned}\mathcal{S}_w(x)&=w(0)Z_q(x)+\frac{\sigma^2}{2}w'(0-)W_q(x)+\\ &\int_0^x W_q(x-y)\int_{z=y}^\infty[w(0)-w(y-z)]\Pi(dz)dy\\ &=w(0)\left(\frac{\sigma^2}{2}W_q'(x)+cW_q(x)\right)+w'(0-)\frac{\sigma^2}{2}W_q(x)-\int_0^x W_q(x-y)w^{(\Pi)}(y)dy,\end{aligned} \tag{20}$$

where $w^{(\Pi)}(y)=\int_{z=y}^\infty[w(y-z)]\Pi(dz)$ is the expected liquidation cost conditioned on a pre-ruin position of y and on a ruin causing jump bigger than y.

Remark 8. *The first two parts of* (20) *may be viewed as boundary fitting terms, and the last part as a "non-local Gerber-Shiu/nonhomogeneous component".*

Lemma 4. *For $w(x)=e^{\theta x}$, the Gerber–Shiu function is $Z(x,\theta)$ and the decomposition* (20) *becomes:*

$$Z_q(x,\theta)=Z_q(x)+\theta\frac{\sigma^2}{2}W_q(x)+\int_0^x W_q(y)\int_{x-y}^\infty[1-e^{\theta(x-y-z)}]\Pi(dz)dy.$$

This maybe easily checked by taking Laplace transforms, since

$$\widehat{W}(s)\frac{\kappa(s)-\kappa(\theta)}{s-\theta} = \widehat{W}(s)\Big(\frac{\kappa(s)}{s} + \theta\frac{\sigma^2}{2} + \frac{\widehat{\pi}(s)-\widehat{\pi}(\theta)}{s-\theta} - \frac{\widehat{\pi}(s)-\widehat{\pi}(0)}{s}\Big)$$

3. An appetizer: a dynamic index for valuation of spectrally negative Lévy subsidiaries, built from the scale functions

In this section we introduce a valuation index for subsidiaries. Note that in classic valuation in risk theory, by the de Finetti objective for example, one fixes a prescribed liquidation point o (usually $o = 0$), in which case the index can be expressed using the **scale function methodology** started in [7] and illustrated in this paper. Here we propose however to further optimize the liquidation point, as is done for example in mathematical finance, for valuing American put options.

Consider a subsidiary with **liquidation value** $w(x)$ and a policy $\pi = (R, R_*, \tau)$ involving some dividend process R, bailout process R_*, and a liquidation/reevaluation stopping time τ. If $dR_*(t)$ consists of a single payoff at the liquidation time τ, then the **modified De-Finetti objective** (26) is

$$V^F(x) = E_x\left[\int_0^\tau e^{-qt}dR(t) + e^{-q\tau}w(X(\tau))\right] \tag{21}$$

Subtracting now from the optimal value obtained by continuing until some stopping time the immediate stopping value yields the following **continuation index** alluded to in the title:

$$\mathcal{I}(x) = \sup_\tau E_x\left[\int_0^\tau e^{-qt}dR(t) + e^{-q\tau}w(X(\tau))\right] - w(x) \tag{22}$$

Remark 9. *This is similar to the concept of Gittins index, in which we modify a value function by subtracting a constant subsidy $\mathcal{I} = \mathcal{I}(x)$ for stopping, and choose this subsidy so that the decisions of whether to continue or stop yield equal payoffs.*

When the stopping time is prescribed by forced stopping, for example at $\tau = \tau_0^-$, an explicit formula is available in terms of the scale functions W, Z and the Gerber-Shiu function $\mathcal{S}_w$ [42, 15]:

$$V^F(x) = \mathcal{S}_w(x) + W_q(x)\frac{1-\mathcal{S}_w'(b)}{W_q'(b)}, \tag{23}$$

With linear liquidation costs for example $w(x) = \begin{cases} kx-K, & x<0 \\ x-K, & x\geq 0\end{cases}$, the smooth Gerber-Shiu function is [42, 15]:

$$\mathcal{S}_w(x) = kZ_{1,q}(x) - KZ_q(x),\ Z_{1,q}(x) = \overline{Z}_q(x) - p\overline{W}_q(x). \tag{24}$$

Optimizing (23) in b is tantamount to optimizing the "barrier function"

$$G(b) = \frac{1-\mathcal{S}_w'(b)}{W_q'(b)}. \tag{25}$$

Furthermore, the "barrier policy" at the last maximum b^* is often optimal among all dividend policies, and, when barrier policies are not optimal, an iterative procedure with starting point b^* may be used to obtain the optimal "multi-bands policy" [55, 15].

Remark 10. *Several variations of the index* (23) *may be obtained replacing absorption at τ by Parisian absorbtion or reflection, or by adding refraction or other boundary mechanisms, which do not destroy the property of smooth passage upwards.*

Note that these problems do not require separate treatment: the valuation index $\mathcal{I}$ is again of the form (23)*, with appropriate definitions of the scale functions W, Z. Thus, finding two scale functions replaces the work necessary for many first passage problems* [33, 6, 16].

Remark 11. *The optimization of the bailout point o has been less studied, and deserves further attention.*

4. Some applications of the W and Z functions to financial optimization

We turn now to reviewing applications of the W and Z functions to the optimization of several financial objectives involving paying dividends and liquidation expenses, which seem relevant for the problem of evaluating the rentability/efficiency of a subsidiary company.

The De Finetti objective with Dickson-Waters modification consists in maximizing expected discounted dividends until the classic ruin time, amended by a modification [25] penalizing the final liquidation:

$$\begin{aligned} V(x) &= \sup_{\pi} E_x \left[\int_0^{\tau_0^-} e^{-qt} dR_\pi(t) + e^{-q\tau_0^-} w(X(\tau_0^-)) \right] \\ &= V^F(x) + S_w(x). \end{aligned} \tag{26}$$

Here π represents an "admissible" dividend paying policy, $L^\pi(t)$ are the cumulative dividend payments, and $w(x)$ is a bail-out penalty function. The value function must satisfy, possibly in a viscosity sense, the HJB equation [17, (1.21)]:

$$\mathcal{H}(V)(x) := \max[\mathcal{G}_q V(x), 1 - V'(x)] = 0, x \ge 0, \quad V(x) = w(x), x < 0, \tag{27}$$

where $\mathcal{G}_q V(x)$ denotes the discounted infinitesimal generator of the uncontrolled surplus process, associated to the policy of continuing without paying dividends. The second operator $1 - V_F'(x)$ corresponds to the possibility of modifying the surplus by a lump payment. The most important class of policies is that of constant barrier policies π_b, which modify the surplus only when $X(t) > b$, by a lump payment bringing the surplus at b, and than keep it there by Skorokhod reflection, until the next negative jump [§], until the next claim.

Under such a reflecting barrier strategy π_b, the dividend part of the De-Finetti objective has a simple expression in terms of the W scale function:

$$V^F(x,b) = \mathbb{E}_x^{[0,b]} \left[\int_{[0,\tau_0^-]} e^{-qs} dR(s) \right] = W_q(x) W_q'(b)^{-1}, \tag{28}$$

where $\mathbb{E}^{[0,b]}$ denotes the law of the process reflected from above at b, and absorbed at 0 and below.

The penalty part can be expressed in terms of $\mathcal{S}_w(x)$; finally, the modified de Finetti value function is:

$$V(x,b) = \begin{cases} \mathcal{S}_w(x) + W_q(x) \frac{1-\mathcal{S}_w'(b)}{W_q'(b)} & x \le b \\ x - b + V(b,b) & x \ge b \end{cases} \tag{29}$$

where $\mathcal{S}_w$ is the smooth Gerber-Shiu associated to the penalty w [15].

The "barrier function"

$$G_F(b) := \frac{1 - \mathcal{S}_w'(b)}{W_q'(b)}, b \ge 0 \tag{30}$$

plays a central role in financial optimization.

[§] in the absence of a Brownian component, this amounts to paying all the income while at b

The most important cases of bail-out costs $w(x)$ are

(1) exponential $w(x) = e^{\theta x}$, when $\mathcal{S}_w(x) = Z_q(x, \theta)$, and
(2) linear $w(x) = kx + K$. For $x < 0$, the constants $k > 0$ and $K \in \mathbb{R}$ may be viewed as proportional and fixed bail-out costs, respectively. The cases $k \in (0, 1]$ and $k > 1$ correspond to management being held responsible for only part of the deficit at ruin, and to having to pay extra costs at liquidation, respectively. When $K < 0$, late ruin is rewarded; when $K > 0$ early liquidation is rewarded.

    In this case $\mathcal{S}_w(x)$ may be obtained by using $Z_q(x, \theta)$ as generating function in θ, i.e. the coefficients of K, k in $F(x)$ are found by differentiating the $Z(x, \theta)$ scale function 0 and 1 times respectively, with respect to θ. This yields

$$\mathcal{S}_w(x) = k\big(\overline{Z}_q(x) - p\overline{W}_q(x)\big) + KZ_q(x) := kZ_{1,q}(x) + KZ_q(x). \tag{31}$$

4.1. **The Shreve–Lehoczky–Gaver infinite horizon objective, with linear penalties.** Suppose a subsidiary must be bailed out each time its surplus is negative, and assume the penalty costs are linear $w(x) = kx$. The optimization objective of interest combines discounted dividends $R(t)$, and cumulative bailouts $R_*(t)$

$$V^S(x) = \sup_b \mathbb{E}_x^{[0,b]}\left[\int_0^\infty e^{-qt} dR(t) + k\int_0^\infty e^{-qt} dR_*(t)\right]. \tag{32}$$

Since in a diffusion setting this objective has first been considered by Shreve, Lehoczky, and Gaver (SLG) [56] – see also [20, 44] – we will call it the SLG objective.

The expected discounted dividends over an infinite horizon for the doubly reflected process, with expectation denoted $\mathbb{E}^{[0,b]}$, are provided in [12, (4.3)]

$$V^{S,D}(x, b) = \mathbb{E}_x^{[0,b]}\left[\int_0^\infty e^{-qt} dR(t)\right] \quad = \quad Z_q(x)Z_q'(b)^{-1}. \tag{33}$$

The capital injections part of the infinite horizon doubly reflected "SLG objective" is

$$V^{S,w}(x, b) = \mathcal{S}_w(x) - \frac{Z_q(x)}{Z_q'(b)}\mathcal{S}_w'(b) \tag{34}$$

Remark 12. *The Gerber-Shiu function $\mathcal{S}_w(x)$ is common to three distinct nonhomogeneous problems involving a process X:*

(1) *severity of ruin with absorbtion at an upper barrier* (17)
(2) *severity of ruin with reflection at an upper barrier* (18)
(3) *cumulative bailouts at the lower barrier with reflection at an upper barrier, for the doubly reflected process* (34).

Remark 13. *Note that in* (33), (34)*, just as in the relation $\mathbb{E}_x e^{-q\tau_b^+} = Z_q(x)Z_q(b)^{-1}$* [7]*, the second scale function $Z_q(x)$ acts for the process reflected at* 0 *just as first scale function for the process absorbed at* 0.

In particular, with linear costs $w(x) = kx$, (34) becomes:

$$V^{S,w}(x) = -k\mathbb{E}_x^{[0,b]}\left[\int_0^\infty e^{-qt} dR(t)\right] = k\left(Z_{1,q}(x) - Z_q(x)Z_q'(b)^{-1}Z_{1,q}'(b)\right),$$

and the optimal dividend distribution is always of **constant barrier** type [12, (4.4)].

4.2. **Optimal dividend barrier strategies. The modified De Finetti dividend barrier function and the optimality of barrier strategies**. The last global maximum b^* achieving the max in

$$\max_b V(x, b) = \mathcal{S}_w(x) + W_q(x)\frac{1 - \mathcal{S}_w'(b)}{W_q'(b)}$$

plays a central role in the optimal dividends distribution policy (even when this is not of single barrier type). To determine b^*, it suffices to study the barrier influence function $G_F(b) = \frac{1-\mathcal{S}'_w(b)}{W'_q(b)}$ (30). For example with linear costs $w(x) = kx + K$, this becomes

$$G_F(b) = \frac{1 - kZ'_{1,q}(b) - KZ'_q(b)}{W'_q(b)}, \tag{35}$$

with $Z_{1,q}(b)$ defined in (31); [42, Lem. 4.1, Lem. 4.2] show that this barrier function does attain a global maximum $b^* \in [0, \infty)$, and that it is increasing-decreasing if $W'_q(b)$ is log-convex.

Example 1. *With the SLG objective, the value function is*

$$\max_b V^S(x, b) = \mathcal{S}_w(x) + Z_q(x)\frac{1 - \mathcal{S}'_w(b)}{Z'_q(b)}$$

and the BF is $\frac{1-\mathcal{S}'_w(b)}{Z'_q(b)}$. *With linear bailout costs $w(x) = kx$, this becomes*

$$G_S(b) = \frac{1 - kZ'_{1,q}(b)}{Z'_q(b)} = \frac{1 - k(Z_q(b) - pW_q(b))}{qW_q(b)}. \tag{36}$$

After further removing a multiple of $W_q(b)$ from the numerator, we arrive at the equivalent optimization of $\widetilde{G}(b) = \frac{1-kZ_q(b)}{qW_q(b)}$ [12, (5.4)].

Remark 14. *The barrier functions and their largest maxima b^* are easy to compute and central for solving numerically all barrier optimization problems, but determining whether the single barrier strategy at b^* is optimal is in many cases an open problem. However, the condition* $G(0) \geq 0$ *yields simple criteria, just as in the case of the modified de Finetti objective; this motivated us to investigate also the optimality condition* $G'(0) \leq 0$ *– see* [9, 10] *and section 6.*

5. A list of first passage laws for Lévy processes with Poissonian (Parisian) detection of insolvency

A useful type of models developed recently [6, 5, 16] assume that insolvency is only **observed periodically**, at an increasing sequence of Poisson observation times $\mathcal{T}_r = \{t_i, i = 1, 2, \dots\}$, the arrival times of an independent Poisson process of rate r, with $r > 0$ fixed [§] .

The analog concepts for first passage times are the stopping times

$$T_b^+ = \inf\{t_i : X(t_i) > b\}, \quad T_a^- = \inf\{t_i > 0 : X(t_i) < a\} \tag{37}$$

We write sometimes T_0^- for T_0^-. Under Parisian observation times, first passage is recorded only when the most recent excursion below a/above b has exceeded an exponential rv $\mathcal{E}_r$ of rate r.

Remark 15. *We will refer to stopping at T_0^- as Parisian absorbtion. A* **spectrally negative Lévy processes with Parisian reflection** *below* 0 *may be defined by pushing the process up to* 0 *each time it is below* 0 *at an observation time T_i. In both cases, this will not be made explicit in the notation; classic and Parisian absorbtion and reflection will be denoted in the same way (note that the first is a limit of the second).*

[§]The concept of periodic observation may be extended to the Sparre Andersen (non Lévy) case, using geometrically distributed intervention times at the times of claims. This deserves further investigation.

Remark 16. *Parisian detection below* 0 *is related to the "time spent in the red"*

$$T_{<0} := \int_0^\infty I_{\{X(t)<0\}} dt,$$

a fundamental risk measure studied by [48, 43].

Indeed, the probability of Parisian ruin not being observed (and of recovering without bailout) when $p > 0, q = 0$ *is* [37, Cor 1,Thm 1], [6, (11)]

$$P_x[T_0^- = \infty] = P_x[T_{<0} < \mathcal{E}(r)] = E_x\left[e^{-rT_{<0}}\right] = p\frac{\Phi_r}{r} Z(x, \Phi_r), \tag{38}$$

where Φ_r *is the inverse of the Laplace exponent . This could be viewed as a* **state dependent** *extension of the profit parameter p, to measure the profitability of risk processes. See* [40, 41] *for further information about the relation between Laplace transforms of occupation times and the fluctuation theory of SNLP observed at independent Poisson arrival times.*

The following proposition list some basic first passage results for processes with Parisian detection of ruin, reflected or absorbed, following [6, 21, 16]. Note that these results coincide with the ones with classic, "hard" detection of ruin, and imply them when $r \to \infty$. They also suggest that the known first passage results with hard ruin for SNMAPs [34, 29, 28, 3] might generalize to the Parisian case, provided that properly defined scale matrix functions are introduced, and written in correct order. To facilitate further work, we provide for each result below, besides the Lévy Parisian reference, also the corresponding non-Parisian SNMAP reference from [29, 28, 3], and also references from [12, 6, 4, 5] for problems where the SNMAP case is not yet resolved.

Proposition 1. First passage results for processes with Parisian detection, followed by reflection or absorbtion. *Let X be a spectrally negative Lévy process with Parisian detection below* 0*, and fix* $b > 0$*. Assuming* $x \in [0, b]$ *and* q*,* $r > 0$*,* $0 \le \theta < \infty$*, the following hold:*

(1) *The* **capital injections/bailouts law for a Parisian reflected process, until** τ_b^+ [16, Cor 3.1 ii)], [28, Thm 2]. *Let* $X^{[0}(t)$ *denote the SNMAP process reflected at* 0*, let* $R_*(t) = -(0 \wedge \underline{X}(t))$ *denote its regulator at* 0*, so that* $X^{[0}(t) = X(t) + R_*(t)$*, and let* $\mathbb{E}_x^{[0}$ *denote expectation for the process with Parisian reflection at* 0*. Then:*

$$B^b(x, \theta) := \mathbb{E}_x^{[0}[e^{-q\tau_b^+ - \theta R_*(\tau_b^+)}] = \begin{cases} Z_{q,r}(x, \theta) Z_{q,r}(b, \theta)^{-1} & \theta < \infty \\ E^{[0}[e^{-q\tau_b^+}; \tau_b^+ < T_0^-] = W_{q,r}(x) W_{q,r}(b)^{-1} & \theta = \infty \end{cases}, \tag{39}$$

where

$$Z_{q,r}(x, \theta) = \frac{r}{q + r - \kappa(\theta)} Z_q(x, \theta) + \frac{q - \kappa(\theta)}{q + r - \kappa(\theta)} Z_q(x, \Phi_{q+r}), \tag{40}$$

with $\theta = \Phi_{q+r}$ *interpreted in the limiting sense, and where*

$$W_{q,r}(x) := Z_q(x, \Phi_{q+r})$$

[3], [6, (12)] [§] . *When* $r \to \infty$*,* $Z_{q,\infty}(x, \theta) = Z_q(x, \theta)$*,* $W_{q,\infty}(x) = W_q(x)$ *and* (39) *reduces to classic results* [28].

[§]The notation $W_{q,r}(x) := Z_q(x, \Phi_{q+r})$ has been chosen to emphasize that this replaces, for processes with with Parisian ruin, the W_q scale function in the classic "gambler's winning" problem $\mathbb{E}_x[e^{-q\tau_b^+}; \tau_b^+ < \tau_0^-] = \frac{W_q(x)}{W_q(b)}$.

(2) *The* **severity of Parisian ruin with absorbtion at** τ_b^+*,*

$$S^b(x,\theta) = \mathbb{E}_x\left[e^{\theta X(T_0^-)}; 1_{T_0^- < \tau_b^+ \wedge \mathcal{E}_q}\right]$$

is [6, (15)] [28, Cor 3]*:*

$$S^b(x,\theta) = Z_{q,r}(x,\theta) - W_{q,r}(x) W_{q,r}(b)^{-1} Z_{q,r}(b,\theta) = Z_{q,r}(x,\theta) - L_{*,0}(x,b) Z_{q,r}(b,\theta).$$

(3) *Let* $U_q^{|a,b|}(x,B) = \mathbb{E}_x\Big(\int_0^{\tau_a^- \wedge \tau_b^+} e^{-qt} 1_{\{X(t)\in B\}} \mathrm{d}t\Big)$*, denote the* q**-resolvent of a doubly absorbed spectrally negative Lévy process with Parisian ruin** [21, Thm 2]*, for any Borel set* $B \in [a,b]$*. Then,*

$$U_q^{|a,b|}(x,B) = \int_a^b 1_{\{y\in B\}}\Big[\frac{W_{q,r}(x-a)W_{q,r}(b-y)}{W_{q,r}(b-a)} - W_{q,r}(x-y)\Big]\mathrm{d}y \quad a<y<b \tag{41}$$

(the analog classic result is Theorem 8.7 of [36]*).*

Remark 17. *It is natural to conjecture that the resolvents for (partly) reflected processes will also be of the same form as the classic ones* [49, Thm. 1]*,* [30, Thm 2, Cor.2]*.*

(4) *The* **dividends-penalty law for a process reflected at** b**, with Parisian ruin** [28, Thm 6]*, is:*

$$\begin{aligned} S^b(x,\theta,\vartheta) &:= \mathbb{E}_x^{b]}\left[e^{-\vartheta R(T_0^-)+\theta X(T_0^-)}; T_0^- < \mathcal{E}_q\right] \\ &= Z_{q,r}(b,\theta) - W_{q,r}(b)\frac{Z'_{q,r}(b,\theta)+\vartheta Z_{q,r}(b,\theta)}{W'_{q,r}(b)+\vartheta W_{q,r}(b)} \\ &= \big[Z_q(x,\theta) - Z_q(x,\Phi_{q+r})H(b,\Phi_{q+r})^{-1}H(b,\theta)\big] r(r+q-\kappa(\theta))^{-1}, \end{aligned} \tag{42}$$

where $H(b,\theta) = \vartheta Z_q(b,\theta) + Z_q'(b,\theta) = (\theta+\vartheta)Z_q(b,\theta) - (\kappa(\theta)-q)W_q(b)$ ¶ *. The second, rather complicated formula, is* [6, (23)]*.*

Note that when $r\to\infty$*,* (42) *recovers the classic* [7, 46],[28, Thm 6], [6, (25)]*, by using* $Z_q(b,\Phi_{q+r}) \to W_q(b)$*:*

$$\begin{aligned} S_\vartheta^b(x,\theta) &= \mathbb{E}_x^{b]}[e^{-\vartheta R(\tau_0^-)+\theta X(\tau_0^-)}; \tau_0^- < \infty] = \mathbb{E}_{Y(0)=b-x}[e^{-\vartheta R(t_b)-\theta(Y(t_b)-b)}] \\ &= Z_q(x,\theta) - W_q(x)\left(W_q'(b)+\vartheta W_q(b)\right)^{-1}\left(Z_q'(b,\theta)+\vartheta Z_q(b,\theta)\right)^{\ddagger} \end{aligned}$$

where $Y_x(t) = \overline{X}(t) - X(t)$ *is the draw-down process/ reflection from the maximum.*

At first sight, (42) *and the classic version* (42) *look different; however, a little algebra will convince us that* (42) *may also be written as* (42)*, with* Z_q*,* W_q *updated to their Parisian versions* $Z_{q,r}$*,* $W_{q,r}$*.*

When $\vartheta = 0$ § (43) *yields the* **severity of ruin for a regulated process** [27]*:*

$$\mathbb{E}_x^{b]}[e^{\theta X(\tau)}; \tau_0^- < \mathcal{E}_q] = Z(x,\theta) - W(x)W'(b_+)^{-1}Z'(b,\theta) = Z(x,\theta) - V^F(x,b)Z'(b,\theta). \tag{43}$$

¶The structure of this formula reflects the fact that Φ_{q+r} is a removable singularity

§When $\vartheta = 0 = x$, and X is Lévy with bounded variation, the **joint law of an excursion with reflection at an upper barrier, and of the final overshoot** is

$$\mathbb{E}_0^{b]}[e^{\theta X(\tau)}; \tau_0^- < \mathcal{E}_q] = 1 + \frac{1}{c}W_+'(b)^{-1}\left(W(b)\kappa(\theta) - \theta Z(b,\theta)\right)$$

and $\mathbb{E}_0^{b]}[e^{-q\tau}; \tau_0^- < \infty] = 1 - \widetilde{q}\frac{W(b)}{W_+'(b)}$, $\widetilde{q} = \frac{q}{c}$. When $b\to\infty$, we recover the Laplace transform of an upward excursion $\widehat{\rho} = \widehat{\rho}_\delta = \mathbb{E}_0[e^{-q\tau}] = 1 - \frac{\widetilde{q}}{\Phi_q} = \frac{\lambda}{c}\widehat{\overline{F}}(\Phi(q))$.

Remark 18. *When $x = b$, we may factor the transform* (42)

$$\mathbb{E}_b^{b]}\left[e^{\theta X(T_0^-)-\vartheta R(T_0^-)}; T_0^- < \mathcal{E}_q\right]$$

as:

$$\Omega(\Omega+\vartheta)^{-1}\Bigg(Z_q(b,\theta) - \Omega^{-1}\Big(\theta Z_q(b,\theta) + (q-\kappa(\theta))\, W_q(b)\Big)\Bigg) r(r+q-\kappa(\theta))^{-1}, \quad (44)$$

$$\begin{aligned}\Omega &= V^F(b,b)^{-1} = W'_{q,r}(b) W_{q,r}(b)^{-1} = Z'_q(b,\Phi_{q+r}) Z_q(b,\Phi_{q+r})^{-1}\\ &= \Phi_{q+r} - rW_q(b) Z_q(b,\Phi_{q+r})^{-1}.\end{aligned}$$

Indeed,

$$\begin{aligned}&Z_q(b,\theta) - Z_q(b,\Phi_{q+r})\left((\Phi_{q+r}+\vartheta) Z_q(b,\Phi_{q+r}) - rW_q(b)\right)^{-1} H(b,\theta)\\ &= Z_q(b,\theta) - \left(\vartheta + \Phi_{q+r} - rW_q(b) Z_q(b,\Phi_{q+r})^{-1}\right)^{-1} H(b,\theta)\\ &= Z_q(b,\theta) - (\vartheta+\Omega)^{-1} H(b,\theta),\end{aligned}$$

and (44) *follows by simple algebra. By* (44)*, $R(T_0^-)$ and $X(T_0^-)$ are independent when starting from b, and the former has an exponential distribution with parameter Ω* [6, (23),(26)].

(5) *The* **expected discounted dividends (upper regulation at b) until T_0^-** [6, (27)] *are :*

$$V^F(x,b) = \mathbb{E}_x^{b]}\left[\int_0^{T_0^-} e^{-qt} dR(t)\right] = W_{q,r}(x) W'_{q,r}(b)^{-1}. \quad (45)$$

(6) *The* **expected discounted dividends with reflection at 0 at Parisian times, until the total bail-outs surpass an exponential variable $\mathcal{E}_\xi$** [4, (15)] *are*

$$V^S(x,b,\theta) = \mathbb{E}_x^{[0,b]}\left[\int_0^\infty e^{-qs} 1_{[R_*(s)<\mathcal{E}_\theta]} dR(s)\right] = Z_{q,r}(x,\theta) Z'_{q,r}(b,\theta)^{-1} \quad (46)$$

When $\theta = 0$, this becomes [16, Cor 3.3] [12, (4.3)]*:*

$$V^S(x,b) = \mathbb{E}_x^{[0,b]}\left[\int_0^\infty e^{-qt} dR(t)\right] \quad = \quad Z_{q,r}(x) Z'_{q,r}(b)^{-1}, \quad (47)$$

where $Z_{q,r}(x) = Z_{q,r}(x,0)$.

(7) *The* **expected total discounted bailouts at Parisian times up to τ_b^+** *are given for $0 \le x \le b$ by* [16, Cor 3.2 ii)]*:*

$$V_*^F(x,b) := \mathbb{E}_x^{[0}\left[\int_0^{\tau_b^+} e^{-qt} \mathrm{d}R_*(t)\right] = Z_{q,r}(x) Z_{q,r}(b)^{-1} \mathcal{S}(b) - \mathcal{S}(x). \quad (48)$$

where

$$\mathcal{S}(x) = \mathcal{S}_{q,r}(x) = \frac{r}{q+r}\Big(\overline{Z}_q(x) + \frac{\kappa'(0_+)}{q}\Big). \quad (49)$$

(8) *The* **total discounted bailouts at Parisian times over an infinite horizon**, *with reflection at b are* [16, Cor 3.4]*:*

$$V_*^S(x,b) = \mathbb{E}_x^{[0,b]}\left[\int_0^\infty e^{-qt} \mathrm{d}R_*(t)\right] \quad = \quad Z_{q,r}(x) Z'_{q,r}(b)^{-1} \mathcal{S}'(b) - \mathcal{S}(x). \quad (50)$$

Remark 19. *Similar results hold for processes $X_\delta^{b[}(t)$ with δ-refraction at a fixed point b [33, 35, 52, 47]. The scale functions are:*

$$w_q^{b[}(x) = W_q(x) + \delta \int_b^x \mathbb{W}_q(x-y)W_q'(y)dy, \tag{51}$$

$$z_q^{b[}(x,\theta) = Z_q(x,\theta) + \delta \int_b^x \mathbb{W}_q(x-y)Z_q'(y,\theta)dy. \tag{52}$$

For example, by [35, Cor. 2], it holds that

$$E_x\left[e^{-rT_{<0}}\right] = P_x[T_0^- = \infty] = (p-\delta)\frac{\Phi_r}{r-\delta\Phi_r} z^{b[}(x,\Phi_r),\ 0 \le \delta \le p. \tag{53}$$

Remark 20. *Some of the results above have been extended to processes $X_\delta^{[0[}(t)$ with classic reflection at 0 and refraction at the maximum [4, (3)]. Thus, (39) holds with $Z_q(x,\theta)$ replaced by $Z_q^{\frac{1}{1-\delta}}(x,\theta)$ [4, Thm 3.1]. The proof uses the probabilistic interpretation $\mathbb{E}_x^{[0}[e^{-q\tau_b^+ - \theta R_*(\tau_b^+)}] = P[\tau_b^+ < \mathcal{E}_q \wedge K_\theta]$, where K_θ is the first time when the total bail-out exceeds an independent exponential rv. $\mathcal{E}_\theta$. Finally, [6, (22)] extend this to the case when τ_b^+ is replaced by T_b^+.*

Remark 21. *The proof of the results above typically requires in the finite variation case only applying the strong Markov property; however, in the infinite variation case, the same problems require a perturbation approach– see for example the proof of Proposition 1 (A), in section 8, or the use of the beautiful Ito excursion theory.*

6. Acceptance-rejection of Lévy subsidiary companies observed at Poissonian times, based on readiness to pay dividends

Even in the one-dimensional case, the final choice of an acceptance-rejection principle is not at all obvious. A first intuition is that an acceptable subsidiary must satisfy the classic positive profit condition

$$p := E_0[X(1)] > 0 \tag{54}$$

or its extension involving linear liquidation/bailout costs [42]. However, these equations only exploit the mean of the process involved and ignore its current state. To remediate this deficiency, one could turn to model and state-dependent formulas like (38). To be of practical use, an acceptance-rejection index should have a complexity similar to that of the expressions above, and also intervene in some important optimization problem.

Note that the profitability/viability condition of [42] is equivalent to

$$G(0) \ge 0,$$

where G is the barrier influence function, and interesting variations may be obtained by replacing absorbtion at 0 with reflection or Parisian reflection, which change the scale functions. The simplicity of all the resulting formulas comes from the fact that the scale functions are only evaluated at 0. This suggested an acceptance-rejection criteria introduced in [9, 10], based on the readiness of subsidiaries to pay dividends at $b = 0$.

Definition 1. *A subsidiary will be called* **efficient** *if the barrier $b = 0$ is locally optimal for paying dividends over some interval $b \in [0,\epsilon), \epsilon > 0$, i.e. if it holds that*

$$G'(0) \le 0.$$

The motivation of this condition is that companies satisfying it are functional even in the absence of cash reserves, and can contribute cash-flows to the central branch without having to wait first until their reserves build out; efficiency is thus translated as **readiness to pay dividends**. This criterion turns out to be a useful complement of the **viability** concept $G(0) \ge 0$ (which at its turn generalizes the classic $p \ge 0$). **An additional bonus** is that un-efficient subsidiaries may be turned into efficient ones by choosing an extra killing q_i' to render the barrier $b^* = 0$ locally optimal; this means that the central branch will terminate subsidiaries deemed un-efficient by stopping bailouts after times $\mathcal{E}_{q_i'}$ with exponential law of parameters q_i'; $(q_i')^{-1}$ will be referred to as "patience" parameters. The killing rate q_i' will be 0 for subsidiaries deemed efficient.

We illustrate now the application of the efficiency concept for spectrally negative Lévy processes, under the SLG objective.

We assume $\sigma = 0$, $\Pi(0, \infty) = \lambda < \infty$, and bail-outs at classic ruin times. This optimal dividend problem is fully analyzed in [12, Thm. 3], and in particular [12, Lem. 2] show that the optimal SLG constant barrier is $b^* = 0$ iff

$$k \leq 1 + \frac{q}{\lambda} \Leftrightarrow q = \lambda(k-1). \tag{55}$$

Note this is a simple application of the optimality of 0 for the barrier function $\widetilde{G}(b) = \frac{1-kZ_q(b)}{qW_q(b)}$. Indeed,

$$q\widetilde{G}'(b) = -\frac{kqW_q^2(b) + W_q'(b)(1-kZ_q(b))}{W_q^2(b)}$$
$$\widetilde{G}'(0) \leq 0 \Leftrightarrow kq/c^2 + (1-k)(q+\lambda)/c^2 \geq 0 \Leftrightarrow q + \lambda \geq k\lambda$$

Remark 22. *The efficiency criterion* (55) *does not take into account the law of X, beyond the total mass of its Lévy measure. However, it does have the interesting feature of making possible to turn partially efficient subsidiaries into efficient ones, by introducing extra killing q_i. This follows from the fact that the function $k(q)$ which solves the equation $G'(0) = 0$ is increasing. This encouraging feature motivated us to remedy this by using the SLG objective in more sophisticated environments including periodic observations, refraction, considered here, and also by using the De Finetti objective – see* [10].

The next result provides a nontrivial efficiency criteria under the SLG infinite horizon cumulative dividends-bailouts objective with Parisian reflection

Theorem 1. *a) The SLG value function with Parisian reflection and linear bailout costs kx is:*

$$V_{SLG}(x) = Z_{q,r}(x)Z_{q,r}'(b)^{-1} - k\left(Z_{q,r}(x)Z_{q,r}'(b)^{-1}\mathcal{S}'(b) - \mathcal{S}(x)\right)$$
$$= k\mathcal{S}(x) + Z_{q,r}(x)\frac{1-k\mathcal{S}'(b)}{Z_{q,r}'(b)}$$

b) The barrier $b = 0$ is a local maximum iff the influence function $G(b) := \frac{1-k\mathcal{S}'(b)}{Z_{q,r}'(b)}$ satisfies

$$G'(0) \leq 0 \Leftrightarrow k\Big(\mathcal{S}'(0)Z_{q,r}''(0) - \mathcal{S}''(0)Z_{q,r}'(0)\Big) \leq Z_{q,r}''(0)$$
$$\Leftrightarrow k\frac{r}{q+r}\Big(Z_{q,r}''(0) - qW_q(0_+)Z_{q,r}'(0)\Big) \leq Z_{q,r}''(0)$$
$$\Leftrightarrow k \leq (1+\frac{q}{r})\frac{\Phi_{q+r} - rW_q(0_+)}{\Phi_{q+r} - (r+q)W_q(0_+)}. \tag{56}$$

In the finite variation case [§] (56) *holds iff*

$$k \leq k(q,r) := (1+\frac{q}{r})\frac{\Phi_{q+r} - r/c}{\Phi_{q+r} - (r+q)/c}$$

[§]in the infinite variation case, the first equation still holds, but the efficiency index does not reflect the distribution, since Φ_{q+r} cancels

Remark 23. *It may be checked that $k(q,r)$ increases in q from $k(0,r)=1$ to infinity and thus an inefficient subsidiary with high transaction cost $k>k(q,r)$ may be turned into efficient by increasing the killing q sufficiently. More precisely, solving the inequality* (57) *yields $k<1+q/r$, or*

$$k\geq 1+q/r,\ \ q+r<\kappa\left(\frac{(k-1)(q+r)}{c(k-1-q/r)}\right)\Leftrightarrow \Phi_{q+r}<\frac{(k-1)(q+r)}{c(k-1-q/r)},$$

and inefficient subsidiaries may be made efficient by choosing an extra killing rate q_i' obtained by letting $q_i:=q+q_i'$ solve the equality $\Phi_{q+r}=\frac{(k-1)(q+r)}{c(k-1-q/r)}$. For example, for the Cramér-Lundberg with exponential claims and Laplace exponent $\kappa(s)=s\left(s-\frac{\lambda}{\mu+s}\right)$, this yields a cubic equation in q. The solution

$$q=(k-1)\frac{\sqrt{(r+\lambda+c\mu)^2+4r\lambda(k-1-c\mu/r)}-(r+\lambda+c\mu)}{2(k-1-c\mu/r)}$$

is increasing in k, with a removable singularity at $k=1+c\mu/r$. The corresponding maximum of the barrier influence function is:

$$G(0)=\frac{1-k\frac{r}{q+r}}{Z_{q,r}'(0)}=\frac{1-k\frac{r}{q+r}}{\frac{q}{q+r}\Phi_{q+r}}=\frac{q+r-kr}{q\Phi_{q+r}}$$

Proof of theorem 1. The SLG objective with Parisian bail-outs and its derivative is are:

$$G(b)=\frac{1+k\mathcal{S}'(b)}{Z_{q,r}'(b)},\ G'(b)=\frac{kZ_{q,r}'(b)\mathcal{S}''(b)-(1+k\mathcal{S}'(b))Z_{q,r}''(b)}{(Z_{q,r}'(b))^2}.$$

Efficiency is equivalent to

$$G'(0_+)\leq 0\Longrightarrow k\left(Z_{q,r}'(0)\mathcal{S}''(0)-\mathcal{S}'(0))Z_{q,r}''(0)\right)\leq Z_{q,r}''(0). \tag{57}$$

Using

$$\begin{aligned}
Z_q'(x,\theta)&=\theta Z_q(x,\theta)+(q-\kappa(\theta))W_q(x),\\
Z_q''(x,\theta)&=\theta Z_q'(x,\theta)+(q-\kappa(\theta))W_q'(x)\\
&=\theta^2 Z_q(x,\theta)+\theta(q-\kappa(\theta))W_q(x)+(q-\kappa(\theta))W_q'(x),\\
Z_{q,r}'(x)&=\frac{q}{q+r}\Big(\Phi_{q+r}Z_q(x,\Phi_{q+r})-rW_q(x)\Big)+\frac{r}{q+r}qW_q(x)=\frac{q}{q+r}\Phi_{q+r}Z_q(x,\Phi_{q+r}),\\
Z_{q,r}''(x)&=\frac{q}{q+r}\Phi_{q+r}\Big(\Phi_{q+r}Z_q(x,\Phi_{q+r})-rW_q(x)\Big)
\end{aligned} \tag{58}$$

we find

$$\begin{aligned}
Z_{q,r}'(b)&=\frac{q}{q+r}\Phi_{q+r}Z_q(b,\Phi_{q+r}),Z_{q,r}''(b)=\frac{q}{q+r}\Phi_{q+r}\left(\Phi_{q+r}Z_q(b,\Phi_{q+r})-rW_q(b)\right),\\
\mathcal{S}'(b)&=-\frac{r}{r+q}Z_q(b),\mathcal{S}''(b)=-\frac{rq}{r+q}W_q(b),
\end{aligned}$$

and finally

$$k\left(\Phi_{q+r}-(r+q)W_q(0_+)\right)\leq(1+\frac{q}{r})\left(\Phi_{q+r}-rW_q(0_+)\right).$$

The result follows by noting that the coefficient of k is always positive ∎

Remark 24. *In the finite variation case, with $r\to\infty$,* (57) *becomes $k\Big(W_q'(0)-\frac{q}{c^2}\Big)\leq W_q'(0)\Leftrightarrow k\leq 1+\frac{q}{\lambda}$, which fits* [12, (5.6)] *(also,*

$$\lim_{r\to\infty}\frac{\Phi_{q+r}-r/c}{\Phi_{q+r}-(r+q)/c}=1+\lim_{r\to\infty}\frac{q/c}{\Phi_{q+r}-(r+q)/c}=1+\frac{q}{\lambda}.$$

7. Heuristic valuation of CB networks, using claims line dividend policies

We consider here one of the simplest risk networks, involving a parent company/central branch, and several subsidiary branches [9, 10, 2].

Definition 2. A central branch (CB) risk network *is formed from:*

(1) *Several spectrally negative subsidiaries $X_i(t)$, $i = 1, \ldots, I$, which must be kept above certain prescribed levels o_i by bail-outs from a central branch (CB) $X_0(t)$, or be liquidated when they go below o_i.*
(2) *The reserve of the CB is a spectrally negative process denoted by $X_0(t)$ in the absence of subsidiaries, and by $X(t)$ after subtracting the bailouts. The ruin time*
$$\tau = \tau_0^- = \inf\{t \geq 0 : X(t) < 0\}$$
causes the ruin of the whole network and leads to a severe penalty.
(3) *The CB must also cover a certain proportion $\overline{\alpha}_i = 1 - \alpha_i$ of each claim $C_{i,j}$ of subsidiary i, leaving the subsidiary to pay only $\alpha_i C_{i,j}$, where $\alpha_i \in [0, 1]$ are called proportional reinsurance retention levels.*

A natural approach for **evaluating financial companies**, going back to de Finetti [24] and Modigliani and Miller [45] is to consider the optimal expected discounted cumulative dividends/optimal consumption until ruin [§] – see [39] for further references on this venerable approach.

If the liquidation time τ is also optimized
$$\mathcal{I}^F(\boldsymbol{u}) := \sup_{\pi=(R_0,R_1,\ldots,R_I,\tau)} E_{\boldsymbol{u}} \int_0^\tau e^{-qt} \left(\sum_{i=0}^I dR_i(t) \right), \tag{59}$$
the result $\mathcal{I}^F(\boldsymbol{u})$ is a Gittins type valuation index.

We will propose here a heuristic multi-dimensional valuation index, based on a specific dividends policy, which, remarkably, was found to be exact in [11], if the retention levels are small enough .

We recall first from [14, 13] that when $I = 1$ and
$$c_0 \leq c_1 \frac{\overline{\alpha}_1}{\alpha_1},$$
i.e. if the angle of the vector $\boldsymbol{\alpha} = (\alpha_1, \overline{\alpha}_1)$ with the u_1 axis is bigger than that of $\boldsymbol{c} = (c_1, c_0)$, then the lower cone
$$\mathcal{C} := \{0 \leq u_0 \leq u_1 \frac{\overline{\alpha}_1}{\alpha_1}\}$$
contains $\boldsymbol{c}$ and is invariant with respect to the stochastic flow, i.e. that starting with initial capital $(u_1, u_0) \in \mathcal{C}$, the process (X_1, X_0) will stay there. In particular, in the lower cone $\mathcal{C}$ ruin can only happen for the CB/reinsurer X_0,and the ruin probability is a classic one-dimensional ultimate ruin probability
$$\Psi(u_1, u_0) = \Psi(\alpha_1 \frac{u_0}{\overline{\alpha}_1}, u_0) := \Psi_0(u_0), \forall u_0$$
see for example [53, 1]. Furthermore, the lower cone is also invariant with respect to the **optimal** discounted dividends policy [11].

Turning now to several dimensions, it is easy to check that:

[§]More generally, τ could be replaced by other stopping times, like the drawdown (Azema-Yor) stopping time $\tau_\xi := \inf\{t \geq 0 : X(t) \leq \xi \sup_{0 \leq s \leq t} X(s)\}$, where $\xi \in (0, 1)$ is a fixed constant.

Lemma 5. *The stochastic flow leaves invariant the cone*

$$\mathcal{C} := \{0 \le u_0 \le u_i \frac{\overline{\alpha}_i}{\alpha_i},\ \overline{\alpha}_i = 1 - \alpha_i,\ i = 1, \ldots, I\},$$

provided the "(extra) cheap reinsurance" condition

$$c_0 \le c_i \frac{\overline{\alpha}_i}{\alpha_i}, i = 1, \ldots, I \tag{60}$$

is satisfied.

The boundary edge

$$u_1 \frac{1 - \alpha_1}{\alpha_1} = \cdots = u_i \frac{1 - \alpha_i}{\alpha_i} = u_0,\ i = 1, \ldots, I, \tag{61}$$

to be called "**claims line**", plays a prominent role in two recent papers, [18] [§] and [11], who solved the optimal dividends problem in the (extra) cheap reinsurance two-dimensional case $c_1 \frac{1-\alpha_1}{\alpha_1} > c_0$. The last paper showed that:

(1) Starting from the claims line, the optimal policy is to stay on this line by cashing the excess income of the subsidiary as dividends.
(2) Starting from points away from the claims line, in the cheap reinsurance case, the optimal policy is to reach the claims line by one lump sum payment.
(3) In the extra cheap reinsurance case, the optimal policy is more complicated, when starting in a certain egg-shaped subset of the non-invariant cone (where parts of the premia are cashed, following a "shortest path", in some sense).

The first two findings prompt us to introduce multi-dimensional "claims line" policies for (extra) cheap reinsurance networks, under which the network follows this line in the absence of claims, by **subsidiaries cashing part of their premia as dividends**. Subsequently, whenever the CB or one subsidiary drop below, **all the other subsidiaries** reduce their reserves by **lump sum dividend** taking, bringing back the process on the claims line.

Remark 25. *These strategies may not be optimal; however, by postulating that the subsidiary processes are just linear functions of the CB process, they greatly simplify the problem, and the value of the network expected dividends decomposes as a sum of one-dimensional quantities– see next Lemma.*

Lemma 6. *For a general CB network, and a fixed admissible dividends process $R_0(t)$, the de Finetti value function for the equilibrium policy associated to $\pi = (R_0, \tau)$ is:*

$$V_\pi^F(x) = E_x \left[\int_0^\tau e^{-qt} \Big[dR_0(t) + \widetilde{c}dt - \gamma dX_0(t) - \sum_{i=1}^{I} (\gamma \frac{\overline{\alpha}_i}{\alpha_i} - 1) dX_i(t) \Big] \right],$$

where

$$\gamma = \sum_{i=1}^{I} \frac{\alpha_i}{\overline{\alpha}_i},\ \widetilde{c} = \gamma \sum_{i=1}^{I} c_i \frac{\overline{\alpha}_i}{\alpha_i}.$$

Optimizing dividends reduces thus to a one-dimensional problem.

§who computed an explicit value function maximizing an expected exponential utility at a fixed terminal time for multi-dimensional reinsurance model under the "cheap reinsurance" assumption that the drifts point along the line $c_1 \frac{1-\alpha_1}{\alpha_1} = \cdots = c_I \frac{1-\alpha_I}{\alpha_I}$.

8. Proof of Proposition 1.I)

By the Markov property, we may decompose $g(x,b) := \mathbb{E}_x e^{-q\tau_b^+ - \theta R_*(\tau_b^+)}$, $\theta > q+r$, in three parts:

$$
\begin{aligned}
&g(x,b) = \mathbb{E}_x[e^{-q\tau_b^+}; \tau_b^+ < \tau_0^-] + \mathbb{E}_x\left[e^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}[e^{-q\tau_0^+}; \tau_0^+ < e_r]; \tau_0^- < \tau_b^+\right] g(0,b)\\
&+ \mathbb{E}_x\left[e^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}[e^{-qe_r+\theta X_{e_r}}; e_r < \tau_0^+]; \tau_0^- < \tau_b^+\right] g(0,b)\\
&= \frac{W_q(x)}{W_q(b)} + g(0,b)\left\{\mathbb{E}_x[e^{-q\tau_0^- + \Phi_{q+r}X_{\tau_0^-}}; \tau_0^- < \tau_b^+] + r\int_0^\infty e^{-\theta u}\right.\\
&\left.\left(W_{q+r}(u)\mathbb{E}_x[e^{-q\tau_0^- + \Phi_{q+r}X_{\tau_0^-}}; \tau_0^- < \tau_b^+] - \mathbb{E}_x[e^{-q\tau_0^-}W_{q+r}(X_{\tau_0^-}+u); \tau_0^- < \tau_b^+]\right) du\right\}.
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
&\int_0^\infty e^{-\theta u} E_x[e^{-q\tau_0^-}W_{q+r}(X_{\tau_0^-}+u)1_{\tau_0^- < \tau_b^+}]du\\
&= E_x 1_{\tau_0^- < \tau_b^+} e^{-q\tau_0^- + \theta X_{\tau_0^-}} \int_0^\infty e^{-\theta(X_{\tau_0^-}+u)} W_{q+r}(X_{\tau_0^-}+u)du\\
&= E_x 1_{\tau_0^- < \tau_b^+} e^{-q\tau_0^- + \theta X_{\tau_0^-}} \int_0^\infty e^{-\theta v} W_{q+r}(v)dv\\
&= \frac{1}{\psi(\theta) - q - r}\left(Z_q(x,\theta) - W_q(x)\frac{Z_q(b,\theta)}{W_q(b)}\right),
\end{aligned}
$$

we find

$$
\begin{aligned}
g(x,b) &= \left\{\frac{r}{\psi(\theta)-q-r}\left(Z_q(x,\Phi_{q+r}) - Z_q(x,\theta) - W_q(x)\frac{Z_q(b,\Phi_{q+r}) - Z_q(b,\theta)}{W_q(b)}\right)\right.\\
&\qquad\left. + Z_q(x,\Phi_{q+r}) - W_q(x)\frac{Z_q(b,\Phi_{q+r})}{W_q(b)}\right\} g(0,b)\\
&\quad + \frac{W_q(x)}{W_q(b)}\\
&= \left\{Z_{q,r}(x,\theta) - W_q(x)\frac{Z_{q,r}(b,\theta)}{W_q(b)}\right\} g(0,b) + \frac{W_q(x)}{W_q(b)}.
\end{aligned}
$$

Now in the finite variation case we may substitute $x = 0$, and, using $W_q(0) > 0$, conclude that $g(0,b) = \frac{1}{Z_{q,r}(b,\theta)}$, which yields the result.

In the infinite variation case, we may use a perturbation approach. For $b > x > 0$, we have

$$
\begin{aligned}
g(0,b) &= \mathbb{E}[e^{-q\tau_x^+}; \tau_x^+ < e_r]g(x,b) + \mathbb{E}[e^{-qe_r+\theta X_{e_r}}; e_r < \tau_x^+, X_{e_r} < 0]g(0,b)\\
&\quad + \int_0^x \mathbb{E}[e^{-qe_r}; e_r < \tau_x^+, X_{e_r} \in dy]g(y,b)dy = e^{-\Phi_{q+r}x}g(x,b) + I_2(x)g(0,b) + I_3(x),
\end{aligned} \tag{62}
$$

$$
\begin{aligned}
I_2(x) &= r\int_{-\infty}^{0}\left(e^{-\Phi_{q+r}x}W_{q+r}(x-y)-W_{q+r}(-y)\right)e^{\theta y}dy\\
&= r\int_0^{\infty} e^{-\Phi_{q+r}x-\theta y}W_{q+r}(x+y)dy-\frac{r}{\psi(\theta)-q-r}\\
&= r\int_x^{\infty} e^{-\Phi_{q+r}x-\theta(z-x)}W_{q+r}(z)dz-\frac{r}{\psi(\theta)-q-r}\\
&= \frac{r}{\psi(\theta)-q-r}(e^{-\Phi_{q+r}x+\theta x}-1)-r\int_0^x e^{-\Phi_{q+r}x-\theta(z-x)}W_{q+r}(z)dz\\
&= \frac{r}{\psi(\theta)-q-r}(e^{-\Phi_{q+r}x+\theta x}-1)+o(W_q(x)).
\end{aligned}
$$

We can check that

$$
\begin{aligned}
&e^{-\Phi_{q+r}x}(Z_q(x,\Phi_{q+r})-Z_q(x,\theta))\\
&= e^{-\Phi_{q+r}x}\left[e^{\Phi_{q+r}x}(1-r\int_0^x e^{-\Phi_{q+r}y}W_r(y)dy)-e^{\theta x}(1-r\int_0^x e^{-\theta y}W_r(y)dy)\right]\\
&= 1-e^{-\Phi_{q+r}x+\theta x}+o(W_q(x)),
\end{aligned}
$$

$$
Z_q(x,\Phi_{q+r}) = e^{\Phi_{q+r}x}\left(1-q\int_0^x e^{-\Phi_{q+r}y}W_q(y)dy\right)=e^{\Phi_{q+r}x}+o(W_q(x)),\ \text{and}
$$

$$
I_3(x)\le \int_0^x E[e^{-qe_r};e_r<\tau_x^+,X_{e_r}\in dy]dy = r\int_0^x e^{-\Phi_{q+r}x}W_{q+r}(x-y)dy=o(W_q(x)).
$$

Solving now (62) for $g(0,b)$ and letting $x\to 0+$, we find again

$$
\begin{aligned}
g(0,b) &= \lim_{x\to 0+}\frac{e^{-\Phi_{q+r}x}\frac{W_q(x)}{W_q(b)}}{e^{-\Phi_{q+r}x}W_q(x)\frac{Z_q(b,\Phi_{q+r})}{W_q(b)}+re^{-\Phi_{q+r}x}W_q(x)\frac{Z_q(b,\Phi_{q+r})-Z_q(b,\theta)}{(\psi(\theta)-q-r)W_q(b)}+o(W_q(x))}\\
&= \frac{\psi(\theta)-q-r}{(\psi(\theta)-q)Z_q(b,\Phi_{q+r})-rZ_q(b,\theta)}=\frac{1}{Z_{q,r}(b,\theta)}.
\end{aligned}
$$

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